Magnetic Polarisability with the Background Field Method

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The magnetic polarisability is a fundamental property of a system of charged particles.

- Describes the response to an external magnetic field
- Provides a description of hadron structure
- Experimentally measured in Compton scattering experiments

What is the magnetic polarisability of the proton and neutron?
**Status**

The graph shows the magnetic polarisability $\beta^p$ of the proton as a function of the squared mass $m^2$ in units of GeV$^2$. The data points from various experiments are plotted:

- Experiment: MacGibbon
- Experiment: Pasquini
- Experiment: PDG
- Experiment: McGovern
- Experiment: Beane
- Experiment: Blanpied
- Experiment: Olmos de León

The trend indicates a decrease in $\beta^p$ with increasing $m^2$. The graph provides a visual comparison of the experimental results with the background field method.
Magnetic Polarisability with the Background Field Method

\[ \beta^p \] (10^{-4} \text{ fm}^3) vs. \( m^2_\pi \) (GeV^2)

- Lattice: Chang: 1506.05518
- Experiment

\[ m_\pi^2 \] (GeV^2) from 0.0 to 0.6
Energy & Two-point Correlation Function

- Directly calculate hadron energies in an external magnetic field
- The energy of a baryon in an external magnetic field is

\[ E(B) = M + \vec{\mu} \cdot \vec{B} + \frac{q e B}{2 M} - \frac{4 \pi}{2} \beta B^2 + \mathcal{O}(B^3) \]

- Evaluate two point correlation functions \( G(\vec{p}, t) \propto \sum_{\alpha} e^{-E_{\alpha} t} \)

Two point correlation function quark-flow diagram for a baryon
Background Field Method

- Introduce a background field on the lattice

  \[
  \text{Continuum: } D_\mu \rightarrow D_\mu^{\text{QCD}} + i q e A_\mu \\
  \text{Lattice: } U_\mu (x) \rightarrow e^{i q e A_\mu (x)} U_\mu (x)
  \]

- Choose \( \vec{A} \) appropriately to generate constant magnetic field \( \vec{B} = +B \hat{z} \)

- Periodic spatial boundary conditions impose a quantisation for a uniform field

  \[
  a^2 q e B^2 = \frac{2 \pi k}{N_x N_y}
  \]

- \( k_d = 0, 1, 2, \ldots \) for the field strength experienced by the \( d \) quark
Wilson Term Mass Renormalisation

- Wilson term causes unphysical quark mass renormalisation in background magnetic field
- In free-field limit this change is

\[ m_{[w]}(B) = m(0) + \frac{a}{2} |q_e B| \]

- Observe through investigation of QCD-free (connected) neutral pion energies
- First discussed by Bali et al. 1510.03899, 1707.05600
Wilson Term Mass Renormalisation

![Graph showing the relationship between aE(B) and eBa² for different particles (π⁺, π⁺, π⁺, π⁺).]
Clover term

\[ aE(B) = 0.00 + 0.02u + 0.04d \]
Clover term

- Careful examination of the clover term
  
  $$a c_{cl} \sum_{\mu < \nu} \sigma_{\mu \nu} F_{\mu \nu}$$

- reveals it cancels the Landau shift induced by the Wilson term in the free-field limit

- This condition is modified by inclusion of QCD
  - Allow QCD and electromagnetic field strengths to have different clover coefficients

  $$c_{cl} \rightarrow C_{SW} F_{\mu \nu}^{QCD} + C_{EM} F_{\mu \nu}^{EM}$$

- and set $C_{EM}$ such that Wilson Landau shift is cancelled

  $$C_{EM} = C_{EM}^{Tree}$$

- 1910.14244
Quark Operators

- Standard lattice QCD interpolators are inefficient at isolating energy eigenstates in a background magnetic field
- Quarks are charged!
  - Quarks experience Landau type effects
  - QCD causes quarks to hadronise for composite Landau energy
- Competing effects, introduce a quark projection operator that includes QCD and QED

Figure: **Left:** Mode for the lowest quantised magnetic field strength $k_d = 1$. **Right:** Two degenerate eigenmodes of second quantised field strength $k_d = 2$. 
SU(3) \times U(1) Projection Operator

- Two-dimensional lattice Laplacian operator

\[ \Delta_{\vec{x},\vec{x}'} = 4 \delta_{\vec{x},\vec{x}'} - \sum_{\mu=1,2} U_\mu (\vec{x}) \delta_{\vec{x}+\vec{\mu},\vec{x}'} + U_\mu^\dagger (\vec{x} - \vec{\mu}) \delta_{\vec{x}-\vec{\mu},\vec{x}'} , \]

- Use low-lying eigenmodes of the 2D Laplacian \((\psi_{i\vec{B}})\) to project the propagator

\[ P_n (\vec{x}, t; \vec{x}', t') = \sum_{i=1}^{n} \langle \vec{x}, t \mid \psi_{i\vec{B}} \rangle \langle \psi_{i\vec{B}} \mid \vec{x}', t' \rangle \delta_{zz'} \delta_{tt'} \]

- Projected propagator is

\[ S_n(\vec{x}, t; \vec{0}, 0) = \sum_{\vec{x}'} P_n(\vec{x}, t; \vec{x}', t) S(\vec{x}', t; \vec{0}, 0) \]

- Also project hadronic level Landau effects for proton- using lattice Landau levels
Recall the energy-field relation for an external magnetic field

\[ E(B) = M + \vec{\mu} \cdot \vec{B} + \frac{|qe B|}{2M} - \frac{4\pi}{2} \beta B^2 + \mathcal{O}(B^3) \]

Construct energy difference \( E(B) - m \) for aligned and anti-aligned spin-field orientations and combine

\[
\left( \frac{G(+s, +B) + G(-s, -B)}{G(+s, 0) + G(-s, 0)} \right) \left( \frac{G(+s, -B) + G(-s, +B)}{G(+s, 0) + G(-s, 0)} \right) = e^{-(2 \delta E) t}
\]

Extract effective energy shift in standard manner

Hence determine \( \beta \) using

\[
\delta E(B, t) = + \frac{|qe B|}{2M} - \frac{4\pi}{2} \beta B^2 + \mathcal{O}(B^4)
\]
Proton Energy Shift

![Proton Energy Shift Graph]

\[ \delta E(B) (\text{GeV}) \]

- \( k_B = 1 \), \( \chi_{dof}^2 = 1.0 \) len =6
- \( k_B = 3 \), \( \chi_{dof}^2 = 0.15 \) len =6
- \( k_B = 2 \), \( \chi_{dof}^2 = 0.43 \) len =6
Polarisability Fit

- Fit to these energy shifts \( \delta E(B, t) \)

\[
\delta E(B, t) - \frac{|qeB|}{2M} = -\frac{4\pi}{2} \beta B^2 = c_2 k^2
\]

- where \( k \) is the field quanta from background magnetic field quantisation condition
Polarisability Fit

\[ \kappa = 0.13727 \]

- \[ q = 1 \text{ constrained} \]
- \[ c^2 k^2 \]
- \[ \chi^2_{dof} = 1.099 \]
## Ensemble Details

<table>
<thead>
<tr>
<th>$\kappa_{ud}$</th>
<th>$m_\pi$ (MeV)</th>
<th>Number of Sources</th>
<th>Number of configurations</th>
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<tbody>
<tr>
<td>0.13700</td>
<td>702</td>
<td>5</td>
<td>399</td>
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<td>0.13727</td>
<td>570</td>
<td>4</td>
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<td>0.13754</td>
<td>411</td>
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</tr>
<tr>
<td>0.13770</td>
<td>296</td>
<td>7</td>
<td>400</td>
</tr>
</tbody>
</table>

- Lattice Volume: $32^3 \times 64$
- 2 + 1 flavour dynamical-fermion QCD
- Physical lattice spacing $a = 0.0907$ fm
- Electroquenched - “sea” quarks experience no background magnetic field
Lattice Results

![Graph showing lattice results for magnetic polarisability](image-url)
Making Contact With Experiment

- Use a chiral effective field theory analysis to
  1. Account for finite volume effects
  2. Model sea-quark-loop contributions to $\beta$ using techniques of partially quenched $\chi PT$
Making Contact With Experiment

![Graph showing the magnetic polarisability with the Background Field Method](image)

- $\beta^p (10^{-4} \text{ fm}^3)$
- $m^2_{\pi} (\text{ GeV}^2)$

Lattice

Experiment
Making Contact With Experiment

\[ \beta^p (10^{-4} \text{fm}^3) \]

\[ m^2 (\text{GeV}^2) \]

- Full-QCD Infinite Volume
- Experiment
Making Contact With Experiment

- Use a chiral effective field theory analysis to
  1. Account for finite volume effects
  2. Model sea-quark-loop contributions to $\beta$ using techniques of partially quenched $\chi PT$
  3. Perform a chiral extrapolation to the physical point

- Use the techniques of
Making Contact With Experiment

\[ \beta_p (10^{-4} \text{ fm}^3) \]

\[ m_{\pi}^2 (\text{ GeV}^2) \]

- Full-QCD Infinite Volume
- \( \chi \text{EFT} \) Extrapolation
- Experiment
Neutron Energy Shift

- Identical process for neutron correlation functions
- A different fit function used
  \[ \delta E(B, t) = \frac{-4 \pi}{2} \beta B^2 = c_2 k^2 \]
- No Hadronic U1 Landau wavefunction projection
Neutron Energy Shift

\[ \delta E(B) \ (\text{GeV}) \]

\[ k_d = 1, \ \chi^2_{dof} = 1.0, \text{ len } = 6 \]

\[ k_d = 2, \ \chi^2_{dof} = 0.42, \text{ len } = 5 \]

\[ k_d = 3, \ \chi^2_{dof} = 0.59, \text{ len } = 4 \]
Polarisability Fit

\[ \delta E_{\beta} \text{ (GeV)} \]

\[ k_d \]

\[ \kappa = 0.13727 \]

\[ c_2 k^2, X_{dof}^2 = 0.8754 \]
Neutron Extrapolation

\[ \beta_n \left(10^{-4} \text{ fm}\right) \]

\[ m_{\pi}^2 \left(\text{ GeV}^2\right) \]

- **Full-QCD Infinite Volume**
- **\( \chi \text{EFT Extrapolation} \)**
- **Experiment**
\[ \beta^p - \beta^n = 0.80 (28) (4) \times 10^{-4} \text{ fm}^3 \]
Summary

- Removed the additive mass renormalisation due to the Wilson term in a background magnetic field.
- Calculated the magnetic polarisability of the proton and neutron using lattice QCD and background field method.
- Specialised projection techniques have been used to account Landau effects
  - Enabling energy shift plateaus
- Chiral effective field theory analysis has been performed to connect lattice results to experiment.
- Techniques are applicable to further elements of the hadronic spectrum.
Neutron $\beta^n$
Table: Magnetic polarisability values for the neutron and proton at each quark mass. The numbers in parentheses describe statistical (systematic) uncertainties. The value at $m_\pi = 0.140$ GeV is the result of our chiral extrapolation.

<table>
<thead>
<tr>
<th>$m_\pi$ (GeV)</th>
<th>$\beta^n \left( \text{fm}^3 \times 10^{-4} \right)$</th>
<th>$n \chi_{dof}^2$</th>
<th>$\beta^p \left( \text{fm}^3 \times 10^{-4} \right)$</th>
<th>$p \chi_{dof}^2$</th>
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</thead>
<tbody>
<tr>
<td>0.702</td>
<td>1.91(12)</td>
<td>0.85</td>
<td>1.91(19)</td>
<td>0.96</td>
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<tr>
<td>0.570</td>
<td>1.68(10)</td>
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<td>1.89(18)</td>
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<tr>
<td>0.411</td>
<td>1.58(29)</td>
<td>0.74</td>
<td>2.03(21)</td>
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</tr>
<tr>
<td>0.296</td>
<td>1.42(37)</td>
<td>0.91</td>
<td>2.08(22)</td>
<td>0.33</td>
</tr>
<tr>
<td>0.140</td>
<td>2.06(26)(20)</td>
<td></td>
<td>2.79(22)(18)</td>
<td></td>
</tr>
</tbody>
</table>
Figure: Magnetic dipole polarizability $\beta_{M1}$ of the nucleon. Figure from 2006.16124