Maiani-Testa meets the inverse problem

Mattia Bruno
in collaboration with Maxwell T. Hansen

APLAT 2020
August 6th, 2020

The problem

"It is of considerable interest to identify the physical quantities, if any, which can be extracted directly from euclidean correlation functions, avoiding analytic continuation" [Maiani, Testa '90]

Maiani-Testa I

$$\langle \widetilde{\pi}_{\boldsymbol{q}_1}(t_1)\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)J(0)\rangle \overset{t_1\to\infty}{\simeq}$$

$$[2\omega_{\boldsymbol{q}_1}]^{-1}\sqrt{Z_\pi}e^{-\omega_{\boldsymbol{q}_1}t_1}e^{-\omega_{\boldsymbol{q}_2}t_2}\langle \pi, \boldsymbol{q}_1|\widetilde{\pi}_{\boldsymbol{q}_2}(0)\, e^{-(\hat{H}-\omega_{\boldsymbol{q}_1}-\omega_{\boldsymbol{q}_2})t_2}\, J(0)|0\rangle$$

$\hat{H}$ physical hamiltonian, J scalar current, $\widetilde{\pi}$ pion fields, $\omega_{\boldsymbol{q}_i} = \sqrt{M_\pi^2 + q_i^2}$

0. set $\boldsymbol{q}_1 = -\boldsymbol{q}_2 = \boldsymbol{q}$

1. insert complete set of states ($d\Phi_n$: n-particle phase space)

$\sum_n \int d\Phi_n \langle \pi, \boldsymbol{q}|\widetilde{\pi}_{-\boldsymbol{q}}(0)|n, \mathrm{out}\rangle e^{-(E-2\omega_{\boldsymbol{q}})t_2}\mathcal{F}_n$

$\mathcal{F}_n = \langle n, \mathrm{out}|J(0)|0\rangle$: time-like $1 \to n$ form factor

2. identify off-shell contributions

$\widetilde{\pi}_{-\boldsymbol{q}} \to$ pole at $\eta = q^2 - M_\pi^2$ (virtuality)

$\langle \pi, \boldsymbol{q}|\widetilde{\pi}_{-\boldsymbol{q}}(0)|n, \mathrm{out}\rangle = \mathrm{discon} \times \delta_{2n} + \sqrt{Z_\pi}\boldsymbol{\mathcal{M}}^*_{2n}(\eta)[\eta + i\epsilon]^{-1}$

$\lim_{\eta\to 0}\boldsymbol{\mathcal{M}}_{2n}(\eta) = \mathcal{M}_{2n}$ on-shell $2 \to n$ scattering matrix

Maiani-Testa II

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0) | n, \mathrm{out} \rangle = \mathrm{discon} \times \delta_{2n} + \sqrt{Z_\pi} \boldsymbol{\mathcal{M}}^*_{2n}(\eta)[\eta + i\epsilon]^{-1}$$

1. disconnected part isolates $\mathcal{F}_2$, complex?

$$Z_\pi^{-1/2}[2\omega_{\boldsymbol{q}}] \langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}}(0) e^{-(\hat{H} - 2\omega_{\boldsymbol{q}})t_2} J(0) | 0 \rangle = \mathcal{F}_2[4\omega^2_{\boldsymbol{q}}] + 2\omega_{\boldsymbol{q}} \frac{1}{2} \sum_n \int d\Phi_n \frac{e^{-(E - 2\omega_{\boldsymbol{q}})t_2}}{\eta + i\epsilon} \boldsymbol{\mathcal{M}}^*_{2n}(\eta)\, \mathcal{F}_n$$

2. Maiani and Testa clever observation: separate the absorptive part

$$2\omega_{\boldsymbol{q}} \tfrac{1}{2} \textstyle\sum_n \int d\Phi_n (-2\pi i)\delta(\eta) e^{-(E-2\omega_{\boldsymbol{q}})t_2} \boldsymbol{\mathcal{M}}^*_{2n} \mathcal{F}_n = -i\mathrm{Im}\,[\mathcal{F}_2]$$

3. $\mathcal{F}_2 - i\mathrm{Im}\,[\mathcal{F}_2] = \mathrm{Re}\,[\mathcal{F}_2] \to$ real ✓, time-like ✓

Let's turn to principal value part $\mathcal{P}\frac{1}{\eta}$

Maiani-Testa III

$$2\omega_{\boldsymbol{q}} \frac{1}{2} \sum_n \int d\Phi_n \mathcal{P} \frac{1}{\eta} e^{-(E-2\omega_{\boldsymbol{q}})t_2} \boldsymbol{\mathcal{M}}^*_{2n} \mathcal{F}_n \qquad [\eta(E) = E(E - 2\omega_{\boldsymbol{q}})]$$

a. integration $E \in [2M_\pi, \infty)$
b. $\boldsymbol{\mathcal{M}}$ on-shell only at pole

$\rightarrow$ for large t_2, $E \approx 2M_\pi$ dominates integral

Conclusion: physical scattering only at $\boldsymbol{q} = 0$ [Maiani, Testa, '90]

$$\langle \pi, \mathbf{0} | \widetilde{\pi}_{\mathbf{0}}(t_2) J(0) | 0 \rangle \overset{t_2 \gg 0}{\rightarrow} \mathcal{F}_2(4M_\pi^2) \left[1 + a \sqrt{\frac{M_\pi}{\pi t_2}} + O(t_2^{-3/2}) \right]$$

at threshold $\mathcal{F}_2$+scatt.length a $\rightarrow$ No-go theorem for $\boldsymbol{q} \neq 0$

Maiani-Testa: at threshold there is no inverse problem!

analytic control over inverse problem thanks to t_2!

$t_2 \Delta E \ll 1$, with ΔE level spacing

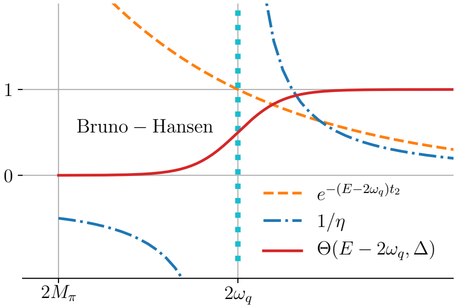
for $n = 2\pi$ $d\Phi_2 \rightarrow dE \sqrt{E^2/4 - M_\pi^2}$ regulates pole $1/\eta(E)$

Our proposal

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0) \qquad e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle$ [Maiani-Testa '90]

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle$ [Bruno-Hansen, in prep.]

$$\sum_n \int d\Phi_n \mathcal{P}\frac{1}{\eta} \Theta(E - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(E-2\omega_{\boldsymbol{q}})t_2} \boldsymbol{\mathcal{M}}^*_{2n} \mathcal{F}_n$$



smooth Θ, smearing width Δ

tames growing exponentials in $2M_\pi < E < 2\omega_{\boldsymbol{q}}$

$(\Delta > 0) + (\mathcal{P}\frac{1}{\eta})$ regulate pole

Generalized Maiani-Testa

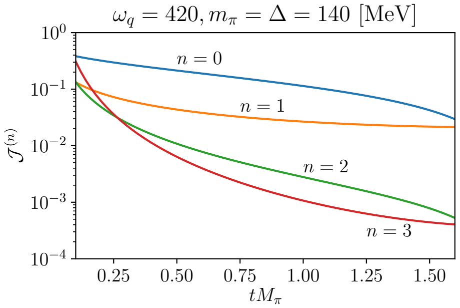
$$\langle \pi, \mathbf{0} | \widetilde{\pi}_{\mathbf{0}}(0) e^{-(\hat{H}-2\omega_{\mathbf{0}})t_2} J(0) | 0 \rangle$$ [Maiani-Testa '90]

$$\stackrel{t_2 \gg 0}{\rightarrow} \mathcal{F}_2(4M_\pi^2) \left[1 + a \sqrt{\frac{M_\pi}{\pi t_2}} + O(t_2^{-3/2}) \right]$$

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}} \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0) | 0 \rangle$$ [Bruno-Hansen, in prep.]

$$\rightarrow \mathrm{Re}\left[\mathcal{F}_2(4\omega_{\boldsymbol{q}}^2)\right] + \sum_{n=0} g_n \mathcal{J}^{(n)}(t_2, \omega_{\boldsymbol{q}}, \Delta)$$

$\mathcal{J}^{(n)}$ pure analytic functions

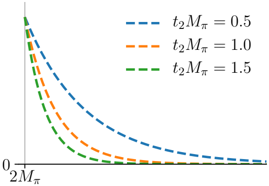


$\boldsymbol{q} = 0$ and large t_2:
reproduce $1/\sqrt{t_2}$

sum all intermediate channels
$\rightarrow g_0 \simeq \mathrm{Im}\left[\mathcal{F}_2\right]$

$g_{n>0}$ off-shell

Inverse problem

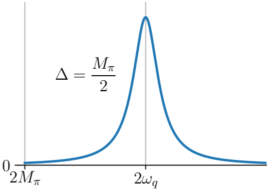


[Maiani-Testa '90]

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$

physical scattering at $\boldsymbol{q}_1 = \boldsymbol{q}_2 = 0$

exponentials mimick "half" $\delta(E - 2M_\pi)$

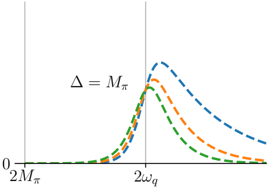


[Bulava-Hansen '18]

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \delta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta)\, J(0)|0\rangle$

physical scattering at $E = 2\omega_{\boldsymbol{q}}$

ordered double-limit $\lim_{\Delta\to 0} \lim_{V\to\infty}$



[Bruno-Hansen, in prep.]

$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$

physical scattering at pole $E = 2\omega_{\boldsymbol{q}}$

physical scattering at fixed Δ

Practical implementations

Finite but large volume, say $M_\pi L \simeq 5$

1. exact reconstruction

large basis of operators $\rightarrow$ GEVP

$O(30)$ energy levels possible [HadSpec]

$\pi\pi$ $I=1$, P-wave, in context of $(g-2)_\mu$

4/5 levels at physical pions [MB, Izubuchi, Meyer, Lehner '18]

2. approx. reconstruction

Backus-Gilbert the Θ is possible [Hansen-Lupo-Tantalo '19]

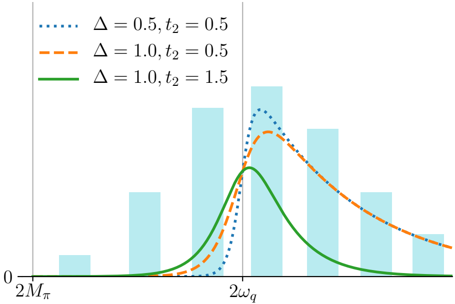
less severe inverse problem than δ

likely more suited for higher-energies

Finite Volume errors

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0) \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0)|0\rangle = \int d\omega K(\omega, t_2) \rho_L(\omega)$$

Spectral-function $\rho_L(\omega) = \sum_n \delta(\omega - E_n) c_n$



$\Delta \approx M_\pi$ we expect
$O(e^{-M_\pi L})$ FV errors

Large $t_2 \simeq$ narrow δ-function

Maiani-Testa: $t_2 \Delta E \ll 1$, ΔE level spacing
window in t_2 where method $O(e^{-M_\pi L})$

Conclusions

Generalization of the Maiani-Testa result

- away from threshold
- time-like form factor (real and imaginary)
- resummed all intermediate channels
- understood the connection with the inverse problem

Next steps

1. extension to $2 \to 2$ processes
2. improve understanding of FV errors
3. numerical tests

Thanks for your attention