Compton amplitude
via
the Feynman-Hellman theorem
K. Utku Can
CSSM, The University of Adelaide
Asia-Pacific Symposium for Lattice Field Theory (APLAT 2020), 4-7 August 2020

with QCDSF — UKQCD — CSSM:
A. Hannaford-Gunn (Adelaide), R. Horsley (Edinburgh), Y. Nakamura (RIKEN, Kobe), H. Perlt (Leipzig),
P. E. L. Rakow (Liverpool), G. Schierholz (DESY, Hamburg), K. Y. Somfleth (Adelaide), H. Stüben (Hamburg),
R. D. Young (Adelaide), and J. M. Zanotti (Adelaide)

Motivation

Physics (just to name a few):

- Always: Understanding the dynamics of the strong interactions
- LHC physics needs PDF input for cross-section estimates
- Identifying power corrections/higher-twist effects in PDFs
- New physics searches need interference ($\gamma^*/Z$) structure functions

Technical:

- Operator mixing/renormalization issues in OPE approach in LQCD
- 4pt functions costly, Feynman-Hellmann approach needs 2pt functions only
Outline

- Forward Compton Amplitude & the Nucleon Structure Functions
- Feynman-Hellmann Theorem & the Compton Amplitude
- Moments of the Nucleon Structure Functions
- Scaling and Power Corrections
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Forward Compton Amplitude

\[ T_{\mu\nu}(p, q) = i \int d^4z e^{iq \cdot z} \rho_{ss'} \langle p, s' | \mathcal{T} \{ J_\mu(z) J_\nu(0) \} | p, s \rangle \text{, spin avg. } \rho_{ss'} = \frac{1}{2} \delta_{ss'} \]

\[ = \left( -g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \mathcal{F}_1(\omega, Q^2) + \left( p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left( p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) \frac{\mathcal{F}_2(\omega, Q^2)}{p \cdot q} \]

\[ \omega = \frac{2p \cdot q}{Q^2} \]

Forward Compton Amplitude & the Nucleon Structure Functions

\[ \omega = 2 \frac{p \cdot q}{Q^2} \]

Compton Structure Functions (SF)

DIS Crosss Section ~ Hadron Tensor

Forward Compton Amplitude ~ Compton Tensor
Nucleon Structure Functions

\[ T_{\mu\nu}(p, q) = i \int d^4z e^{iq\cdot z} \rho_{ss'} \langle p, s' \vert \mathcal{T} \{ J_\mu(z) J_\nu(0) \} \vert p, s \rangle, \text{ spin avg. } \rho_{ss'} = \frac{1}{2} \delta_{ss'} \]

\[ \omega = \frac{2p \cdot q}{Q^2} \]

\[ \mathcal{F}_1(\omega, Q^2) = \mathcal{F}_2(\omega, Q^2) \]

Consider:
\[ \mu = \nu = 3 \text{ and } p_z = q_z = 0 \]
\[ T_{33}(p, q) = \mathcal{F}_1(\omega, Q^2) \]

Optical theorem relates the Compton SF to DIS SF:
\[ \text{Im } \mathcal{F}_1(\omega, Q^2) = 2\pi F_1(x, Q^2) \]

so we can write down the dispersion relation:
\[ \mathcal{F}_1(\omega, Q^2) = \frac{2\omega^2}{\pi} \int_1^\infty d\omega' \frac{\text{Im } \mathcal{F}_1(\omega', Q^2)}{\omega'(\omega^2 - \omega'^2 - i\epsilon)} \]

\[ = 4\omega^2 \int_0^1 dx \frac{x F_1(x, Q^2)}{1 - x^2\omega^2 - i\epsilon} \]

CA in unphysical region
related to the inelastic structure function
Nucleon Structure Functions

\[ \omega = \frac{2p \cdot q}{Q^2} \]

Forward Compton Amplitude & the Nucleon Structure Functions

Compton amplitude with \( \mu = \nu = 3 \) and \( p_z = q_z = 0 \)

\[ T_{33}(p, q) = \mathcal{F}_1(\omega, Q^2) \]

Compton SF

\[ \mathcal{F}_1(\omega, Q^2) = 4\omega^2 \int_0^1 dx \frac{xF_1(x, Q^2)}{1 - x^2\omega^2} \]

we are at the unphysical \(|\omega| < 1\) region, no need for \( i\epsilon \)

Taylor expand \([1-(x\omega)^2]^{-1}\)

\[ = \sum_{n=1}^{\infty} 2\omega^{2n}M_{2n}^{(1)}(Q^2) \]

moments of the nucleon structure function \( F_1(x, Q^2) \)

\[ \text{subtracted dispersion relation} \]

Once we have Compton amplitude data, we can extract the moments!

\[ T_{33}(p, q) = \sum_{n=1}^{\infty} 2\omega^{2n}M_{2n}^{(1)}(Q^2) \]
Shape of the Compton Amplitude

Structure function (leading twist)

Compton Amplitude

\[ T_{33}(p, q) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2) \]

Moments of the DIS Structure Functions

NNPDF3.1 NNLO

100 sets

\[ Q^2 = 9 \text{ GeV}^2 \]

(DIS region)
FH Theorem at 1\textsuperscript{st} order

in Quantum Mechanics:

\[ \frac{\partial E_\lambda}{\partial \lambda} = \langle \phi_\lambda | \frac{\partial H_\lambda}{\partial \lambda} | \phi_\lambda \rangle \]

- \( H_\lambda \): perturbed Hamiltonian of the system
- \( E_\lambda \): energy eigenvalue of the perturbed system
- \( \phi_\lambda \): eigenfunction of the perturbed system

- expectation value of the perturbation of a system is related to the shift in the energy eigenvalue

in Lattice QCD: energy shifts in the presence of a weak external field

\[ S \rightarrow S(\lambda) = S + \lambda \int d^4x \mathcal{O}(x) \]

\[ E_\lambda \rightarrow \text{spectroscopy, 2-pt function} \]

\[ \langle 0 | \mathcal{O} | 0 \rangle \rightarrow \text{determine 3-pt} \]

\[ \bar{q}(x) \Gamma_\mu q(x), \Gamma_\mu \in \{1, \gamma_\mu, \gamma_5 \gamma_\mu, \ldots\} \]

Applications:
- \( \sigma \)-terms
- Form factors
**Compton Amplitude from FHT at 2\textsuperscript{nd} order**

\[
T_{\mu\mu}(p, q) = \int d^4 x e^{i q \cdot x} \langle N(p) \mid \mathcal{T} \{ J_\mu(x) J_\mu(0) \} \mid N(p) \rangle
\]

4-pt function

**Action modification**

\[
S \rightarrow S(\lambda) = S + \lambda \int d^4 x (e^{i q \cdot x} + e^{-i q \cdot x}) J_\mu(x)
\]

local EM current

\[
J_\mu(x) = \sum_q e_q \bar{q}(x) \gamma_\mu q(x)
\]

**Determine the Compton Amplitude from second order energy shifts!**
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Valence u/d quarks with modified action, $S(\lambda)$
- Local EM current insertion, $J_\mu(x) = Z_V \bar{q}(x) \gamma_\mu q(x)$ with $Z_V = 0.8611(84)$
- Feynman-Hellmann implementation at the valence quark level
- 4 Distinct field strengths, $\lambda = [\pm 0.0125, \pm 0.025]$
- 5 different current momenta in the range, $3 \lesssim Q^2 \lesssim 7 \text{ GeV}^2$
- $\mathcal{O}(10^4)$ measurements for each pair of $Q^2$ and $\lambda$
- Access to a range of $\omega$ values for several $(p, q)$ pairs
  - An inversion for each $q$ and $\lambda$, varying $p$ is relatively cheap
- Connected 2-pt correlators calculated only, no disconnected
- Jacobi-smeared sources and sinks, rms $r \sim 0.5$ fm
Strategy | Energy shifts

Extract energy shifts for each $\lambda$

Get the 2nd order derivative

Ratio of perturbed to unperturbed 2-pt functions

$$R_\lambda^e(p, t) \equiv \frac{G^{(2)}_{+\lambda}(p, t)G^{(2)}_{-\lambda}(p, t)}{(G^{(2)}(p, t))^2}$$

$$t \gg 0 \quad \longrightarrow A_\lambda(p)e^{-2\Delta E_{N_\lambda}^e(p)t}$$

Slope of the curve

$$\Delta E_{N_\lambda}^e(p) = \lambda^2 \left( \frac{\partial^2 E_N(p)}{\partial \lambda^2} \right)_{\lambda=0} + \mathcal{O}(\lambda^4)$$
Moments of the Nucleon Structure Functions

kuc et al. (CSSM/QCDSF/UKQCD) arXiv:2007.01523 [hep-lat]

\[
\frac{\partial^2 E_N}{\partial^2 \lambda} \bigg|_{\lambda=0} = - \frac{T_{33}(p, q)}{E_N(p)} = - \frac{F_1(\omega, Q^2)}{E_N(p)}
\]

\[
\omega = 1/x = \frac{2p \cdot q}{Q^2}
\]

Strategy | Structure Functions

\[ q = (4,1,0) \ 2\pi/L, \ Q^2 = 4.66 \text{ GeV}^2 \]

Remember our kinematic choices

\[ \mu = \nu = 3 \ \text{and} \ p_z = q_z = 0 \]

\[ T_{33}(p, q) = F_1(\omega, Q^2) \]

\[ m_s \sim 470 \text{ MeV} \]

32^3x64, 2+1 flavor
 Moments | Fit

\[ m_x \sim 470 \text{ MeV} \]

Moments of the Nucleon Structure Functions

\[ \bar{F}_1(\omega, Q^2) = 4(\omega^2 M_2^{(1)}(Q^2)) + \omega^4 M_4^{(1)}(Q^2) + \cdots + \omega^{2n} M_{2n}^{(1)}(Q^2) + \cdots \]

Enforce monotonic decreasing of moments for \textit{u} and \textit{d} only, not necessarily true for \textit{u} – \textit{d}

\[ M_2^{(1)}(Q^2) \geq M_4^{(1)}(Q^2) \geq \cdots \geq M_{2n}^{(1)}(Q^2) \geq \cdots \geq 0 \]

We truncate at \( n = 6 \)

No dependence to truncation order for \( 3 \leq n \leq 10 \)

Bayesian approach by MCMC method

Sample the moments from Uniform priors

\[ M_2^{(1)}(Q^2) \sim \mathcal{U}(0, 1) \]

\[ M_{2n}^{(1)}(Q^2) \sim \mathcal{U}(0, M_{2n-2}^{(1)}(Q^2)) \]

Remember:

\[ T_{33}(p, q) = \sum_{n=1}^{\infty} 2\omega^{2n} M_{2n}^{(1)}(Q^2) \]

\[ T_{33}(p, q) = \mathcal{F}_1(\omega, Q^2) \]

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\( M_2^{(1)}(Q^2) \) is tricky to impose monotonic decreasing and positivity bound

Multivariate Likelihood function, \( \exp(-\chi^2/2) \)

\[ \chi^2 = \sum_{i,j} \left[ \bar{F}_{1,i} - \bar{F}_{1}^{\text{obs}}(\omega_i) \right] C^{-1}_{ij} \left[ \bar{F}_{1,j} - \bar{F}_{1}^{\text{obs}}(\omega_j) \right] \]

covariance matrix
Scaling and Power Corrections

kuc et al. (CSSM/QCDSF/UKQCD) arXiv:2007.01523 [hep-lat]

Scaling

Unique ability to study the $Q^2$ dependence of the moments!

Possible for the first time in a lattice simulation!

- Global PDF-fit cuts $\sim 10 \text{GeV}^2$
- Credible scaling region $\sim 16 \text{GeV}^2$
- Need $Q^2 > 10 \text{GeV}^2$ data to reliably extract moments and report at $\mu = 2 \text{GeV}$

$m_u \sim 470 \text{MeV}$

$32^3 \times 64$, 2+1 flavor

$M^{(1)}_{2n, uu-dd}(Q^2)$

Lowest isovector ($u-d$) moment of nucleon structure function $F_1$

$Q^2 [\text{GeV}^2]$
Power Corrections

Compton amplitude includes all possible power corrections!

- Power corrections below $\sim 3 \text{ GeV}^2$ ?
- naïve modelling via
  - $M_{2n}^{(1)}(Q^2) = M_{2n}^{(1)} + C_{2n}/Q^2$
- Need more statistics and lower $Q^2$ data

Preliminary data points from $48^3 \times 96$ configurations

$M_{2,uu-dd}^{(1)}(Q^2)$

Lowest isovector $(u - d)$ moment of nucleon structure function $F_1$

$Q^2 [GeV^2]$ vs $M_{2,uu-dd}^{(1)}(Q^2)$

- Different $m_\pi$
- $m_\pi \sim 410$ MeV

Qualitative comparison
No systematics yet

Callan-Gross Relation \( \frac{\omega}{2} \mathcal{F}_2(\omega, Q^2) - \mathcal{F}_1(\omega, Q^2) = 0 \), if no power corrections

\( m_u \sim 470 \text{ MeV} \)

32\times64, 2+1 flavor

Needs \( T_{44} \)

\( \mathcal{F}_2 \sim (T_{44} - T_{33}) \)

u-quark only

\( Q^2 = 4.66 \text{ GeV}^2 \)

\( Q^2 = 2.74 \text{ GeV}^2 \)
Summary & Outlook

- A new versatile approach!
- Systematic investigation of power corrections, higher-twist effects and scaling now within reach
- (Possibly) overcomes the operator mixing/renormalization issues
- Can be extended to:
  - mixed currents, interference terms
  - spin-dependent structure functions
  - GPDs (work in progress...)
- Back to PDFs: Inverse Problem!
  
  see arXiv:2001.05366 for an attempt:
Thank you

further questions/comments → kadirutku.can@adelaide.edu.au