Radial Lattice Quantization of 3D $\phi^4$ Field Theory

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based on arXiv:2006.15636

Asia-Pacific Symposium for Lattice Field Theory
August 6 2020
Outline

1. Motivation and Goals
2. Lattice Construction
3. Restored Symmetries on the Critical Surface
4. Ricci Improved Two Point Correlator
5. Outlook
Radial Quantization for Numerical Calculation

Radial Quantization

- Conformal Map to Cylinder

\[ ds^2_{\mathbb{R}^d} = dr^2 + r^2 d\Omega^2, \quad r = R_0 e^{t/R} \]
\[ = \left( \frac{r}{R} \right)^2 (dt^2 + R^2 d\Omega^2) = \Omega^2_W ds^2_{\mathbb{R} \times S^{d-1}} \]

- Two Point Conformal Correlator

\[ \langle 0 | \phi(\vec{x})\phi(0) | 0 \rangle_{\mathbb{R}^d} = |\vec{x}|^{-2\Delta} \phi \]
\[ \langle 0 | \phi(t, \hat{n})\phi(0, \hat{z}) | 0 \rangle_{\mathbb{R} \times S^{d-1}} \xrightarrow{t \gg R} e^{-\Delta \phi t/R} \]

- Correlation length: \( \xi^{-1} = \Delta \phi / R \).
- Finite volume exponentially large in \( T \).

"Whether this will provide a useful numerical approach to critical exponents remains to be seen". "...[i]t is necessary to approximate the continuum by a sequence of lattices ... for \( d \neq 2 \) however, the space is curved and only a finite number of regular lattices my be embedded in the space." — J. Cardy [J. Phys. A 18, L757 (1985)], [Domb and Lebowitz]
Constructing Classical Simplicial Action

Regge Calculus

- Topology: target Manifold approximated by a sequence of simplicial complexes.
- Geometry: Simplicial complex taken to be piecewise flat, defines metric distances.

FEM/DEC

- Finite Elements: classic action given in terms of geometric weights by linearly interpolating fields over each simplex.
- Discrete Exterior Calculus: Boundary operator and Voronoï dual on simplicial complex naturally realize exterior derivative and Hodge dual for form fields — gauge fields, Kahler-Dirac fermions.
Simplicial Lattice for $\mathbb{R} \times S^2$

Figure: $L = 3$ refined icosahedron

- Coarsest refinement is the Icosahedron, $N_V = 12$.
- “Refined” by subdividing each icosahedral face with regular lattice, $N_V = 10L^2 + 2$.
- Project outwards onto circumscribing sphere. Lengths given by secant distances.
- Refined icosahedron lattice copied uniformly along temporal direction.
Classical Lattice Action for $\phi^4$ Theory

- **Continuum Scalar Action**

$$S_{\text{cont}} = \frac{1}{2} \int_M d^d x \sqrt{g} \left[ g^{\mu \nu} \partial_\mu \phi(x) \partial_\nu \phi(x) + \xi_0 \text{Ric} \phi^2(x) \right]$$

$$+ m^2 \phi^2(x) + \lambda \phi^4(x)$$

- **FEM Action**

$$S_{\text{FEM}} = \frac{a_t}{2} \left[ \sum \frac{l_{xy}^*}{l_{xy}} \left( \phi_{t,x} - \phi_{t,y} \right)^2 + \frac{g_x}{4R^2} \phi_{t,x}^2 + \sqrt{g_x} \left[ \left( \phi_{t,x} - \phi_{t+1,x} \right)^2 + m^2 \phi_{t,x}^2 + \lambda \phi_{t,x}^4 \right] \right]$$

- **Rescale to Dimensionless fields and couplings**

$$S = \frac{1}{2} \left[ \sum \frac{l_{xy}^*}{l_{xy}} \left( \phi_{t,x} - \phi_{t,y} \right)^2 + \frac{a^2}{4R^2} \sqrt{g_x} \phi_{t,x}^2 + \sqrt{g_x} \left[ \frac{a^2}{a_t^2} \left( \phi_{t,x} - \phi_{t+1,x} \right)^2 + m_0^2 \phi_{t,x}^2 + \lambda_0 \phi_{t,x}^4 \right] \right]$$

- **Ricci Curvature Term** irrelevant at the interacting fixed point. We initially omit it from the action in order to study the dominant scaling toward the continuum limit. Later we reintroduce for improved action.
Quantum Corrections to Classical FEM Action

- Local variations in the effective cutoff amplified by UV divergences.
- UV divergences are translation invariant and universal.
- After subtracting divergence, finite piece generates a locally scheme dependent mass.
- Local scheme dependence can be exactly canceled in perturbation theory.

**QFE Conjecture**

Once the local scheme dependence is canceled in perturbation theory, “QFE” lattice action tuned to the critical surface converges to the nonperturbative CFT as the cutoff is removed.
UV Divergent Diagrams

**1-loop Fit:**
- \( a = 0.2397(1) \)
- \( b = 0.349(6) \)
- \( c = -0.522(3) \)

**2-loop Fit:**
- \( a = 0.1260(1) \)
- \( b = 0.00235(1) \)
Binder Cumulant Study

Binder Cumulant

Search for critical surface using 4th-order Binder Cumulant,

\[ U_4(m_0^2, \lambda_0) = \frac{3}{2} \left[ 1 - \frac{\langle M^4 \rangle}{3 \langle M^2 \rangle^2} \right], \quad M = \sum_{x, t} \sqrt{g_x} \phi_{t,x}, \]

which is 0 in disordered/Gaussian phase and 1 in the ordered phase.
Recovery of Rotational Symmetry at Criticality

2-point Correlator

- Project onto spherical harmonics

\[ C_{lm}(t_1 - t_2) = \langle \phi_{t_1}^*, \phi_{t_2}, lm \rangle \]

\[ \phi_{t, lm} = \sum_x \sqrt{g_x} \phi_{t, x} Y_{lm}[x] \]

- Each level “l” should be \(2l + 1\) degenerate.
- Break to icosahedral irreps on finite lattice.
- Study “normalized error” in rotational symmetry at \(l = 3\).

\[ 1 - \frac{C_{31}(t)}{C_{30}(t)} \]
Towards the Continuum Limit

Conformal Dimensions

- 2-point correlator projected onto \( l \)th partial wave
- Ground state at large times is \( l \)th descendent of \( \phi \),
  \[ C_l(t) \xrightarrow{\text{large } t} A_l e^{\mu_{\phi,l} t} \]
  where \( \mu_{\phi,l} = c_R \frac{a}{R} \Delta_{\phi,l} \)
- \( c_R \) is the renormalized speed of light.

Ricci Scaling

From fitting, find leading scaling approximately \( \mathcal{O}(a^{0.4}) \), consistent with Ricci term

\[ a^{\Delta_{\epsilon} - 1} = a^{0.4126} \]. We fit,

\[ \Delta_{\phi,l}(a) = c_R \left( \Delta_{\phi,l} + l + A_l a^{\Delta_{\epsilon} - 1} + B_l a \right) \]

with \( \Delta_{\phi} \approx 0.5181 \) and \( \Delta_{\epsilon} \approx 1.4126 \) fixed from bootstrap.
Ricci Term in Perturbation Theory

Ratio Method

For small \( a^2 / R^2 \) the Ricci term is a perturbative correction.

\[
\delta S = - \frac{a^2}{8R^2} \sqrt{g_x} \phi_{t,x}^2
\]

To first order in \( a^2 / R^2 \) this shifts the scaling dimension,

\[
\delta \Delta \phi, l = - \frac{a^2}{8R^2} \left\langle \Delta \phi, l | \sqrt{g_x} \phi_{t,x}^2 | \Delta \phi, l \right\rangle_c
\]

Determine (nonperturbative) form factor through standard ratio method.

\[
\left\langle \Delta \phi, l | \sqrt{g_x} \phi_{t,x}^2 | \Delta \phi, l \right\rangle_c \approx \frac{\left\langle \phi_{t_1,lm} \sqrt{g_x} \phi_{t,x}^2 \phi_{t_2,lm} \right\rangle}{\phi_{t_1,lm} \phi_{t_2,lm}}
\]

We find for \( l = 0, 1 \),

\[
\delta \Delta \phi, l=0 = -0.3777(38)a^{\Delta \epsilon - 1}, \quad \delta \Delta \phi, l=1 = -0.083(10)a^{\Delta \epsilon - 1}
\]
Ricci Improvement for $\Delta_{\phi}, l=0$
Conclusions

Conclusion

- We have established that once the leading Ricci scaling towards the continuum limit is taken into account, one can achieve accurate extraction of scaling dimensions from exponential decay of correlation functions on $\mathbb{R} \times S^2$.
- Renormalization of the speed of light and Ricci term must be studied carefully at high precision.

Ongoing work for 3D Ising CFT

- Parallel code to generate higher statistics
- Higher statistical precision to study renormalization effects and subleading lattice artifacts.
- 4-point function to study the central charge and other OPE coefficients.
Outlook

Other Developments

- $\phi^4$ Field Theory on $S^2$ [Phys.Rev.D 98 (2018) 1, 014502]
- Dirac and Domain Wall Fermions [Phys.Rev.D 95 (2017) 11, 114510]
- Kahler-Dirac Fermions [arXiv:1810.10626]
- “Simplical Lattice Field Theory on de Sitter and anti de Sitter Manifolds,” Richard Brower, Friday at 16:00 (JST)

Some Future Prospects

- $\mathbb{R} \times S^3$ for the 4d Conformal Window
- Lattice Radial Quantization at finite density/charge to study large charge CFTs
- Gapped theories on spherical spatial volumes