Chiral symmetry and taste symmetry on the eigenvalue spectrum of staggered Dirac operators

Hwancheol Jeong
(SWME Collaboration)

Lattice Gauge Theory Research Center
Seoul National University

APLAT 2020
August 4, 2020

Project collaborators

- **Hwancheol Jeong**, Seungyeob Jwa, Jeehun Kim, Nam Soo Kim, Sunghee Kim, Sunkyu Lee, Weonjong Lee, Youngjo Lee, Jeonghwan Pak
  [Seoul National University]
- Jon Bailey
  [Yonsei University]
- Chulwoo Jung
  [Brookhaven National Laboratory]
- Jangho Kim
  [Goethe University Frankfurt am Main]

Table of Contents

Chirality operator

Staggered bilinears

Continuum	Lattice (stag)
$\overline{\psi}^{f}_{\alpha}(x)\,[(\gamma_S)_{\alpha\beta}\otimes(\xi_F)_{ff'}]\,\psi^{f'}_{\beta}(x)$	$\overline{\chi}(x_A)[\gamma_S\otimes\xi_T]_{AB}\,\chi(x_B)$ $\equiv\overline{\chi}(x_A)\overline{(\gamma_S\otimes\xi_T)}_{AB}\,U(x_A,x_B)\,\chi(x_B)$ ($x_A = 2x + A$ with $A_\mu\in\{0,1\}$)

- $\gamma_S \in \{1,\, \gamma_\mu,\, \gamma_{\mu\nu},\, \gamma_{\mu 5},\, \gamma_5\}$: Dirac spin matrix, $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$
- $\xi_F, \xi_T \in \{1,\, \xi_\mu,\, \xi_{\mu\nu},\, \xi_{\mu 5},\, \xi_5\}$: **flavor**/**taste** matrix, $\{\xi_\mu,\xi_\nu\}=2\delta_{\mu\nu}$
- $\overline{(\gamma_S\otimes\xi_T)}_{AB} = \dfrac{1}{4}\mathrm{Tr}\Big(\gamma_A^\dagger\gamma_S\gamma_B\gamma_T^\dagger\Big)$
- $U(x_A,x_B)\equiv\mathbb{P}_{\mathrm{SU(3)}}\left[\sum_{p\in\mathscr{C}} V(x_A,x_{p_1})V(x_{p_1},x_{p_2})\cdots V(x_{p_n},x_B)\right]$

Chirality operator

Name	Operator	Symmetry	Memo
distance parity	$\Gamma_\varepsilon \equiv [\gamma_5 \otimes \xi_5]$	$[\mathrm{U_A}(1)]^{\mathrm{Latt}}_{\mathrm{stag}}$	conserved
chirality	$\Gamma_5 \equiv [\gamma_5 \otimes \mathbb{1}]$	anomalous $\mathrm{U_A}(1)$	approx.
shift	$\Xi_5 \equiv [\mathbb{1} \otimes \xi_5]$	SU(4) taste	approx.

- Ward identity:

$$[\gamma_5 \otimes \xi_5] = [\gamma_5 \otimes \mathbb{1}][\mathbb{1} \otimes \xi_5] = [\mathbb{1} \otimes \xi_5][\gamma_5 \otimes \mathbb{1}] \tag{1}$$

- Satisfy the same recursion relations as γ_5:

$$[\gamma_5 \otimes \mathbb{1}]^{2n+1} = [\gamma_5 \otimes \mathbb{1}], \quad [\gamma_5 \otimes \mathbb{1}]^{2n} = [\mathbb{1} \otimes \mathbb{1}] \tag{2}$$

$$[\mathbb{1} \otimes \xi_5]^{2n+1} = [\mathbb{1} \otimes \xi_5], \quad [\mathbb{1} \otimes \xi_5]^{2n} = [\mathbb{1} \otimes \mathbb{1}] \tag{3}$$

Golterman operator (old)

- Staggered bilinears by **M. Golterman**
  [M. Golterman, Nucl.Phys.B 273 (1986) 663]

$$\mathscr{O}_{S\times T}(x) = \sum_A \boldsymbol{\rho_{S\times T}(A)}\, \overline{\chi}(x_A)\, \boldsymbol{M_{S\times T}}\, \chi(x_A) \qquad (4)$$

  where $\boldsymbol{\rho_{S\times T}(A)}$ is a phase factor, and

$$\boldsymbol{M_{S\times T}}\, \chi(x_A) = \prod_\mu \left[(1 - |\boldsymbol{S_\mu} - \boldsymbol{T_\mu}|) + |\boldsymbol{S_\mu} - \boldsymbol{T_\mu}|\, \boldsymbol{\widetilde{D}_\mu} \right] \chi(x_A) \qquad (5)$$

$$\boldsymbol{\widetilde{D}_\mu}\, \chi(x_A) = \frac{1}{2} \left[V_\mu(x_A)\, \chi(x_A + \hat{\mu}) \boldsymbol{+} V_\mu^\dagger(x_A - \hat{\mu})\, \chi(x_A - \hat{\mu}) \right] \qquad (6)$$

- $\boldsymbol{S_\mu}, \boldsymbol{T_\mu} \in \{0,1\}$: $\boldsymbol{\gamma_S} = \gamma_0^{S_0}\gamma_1^{S_1}\gamma_2^{S_2}\gamma_3^{S_3}$, $\boldsymbol{\gamma_T} = \gamma_0^{T_0}\gamma_1^{T_1}\gamma_2^{T_2}\gamma_3^{T_3}$
- **True irreducible**

- $\rho_{\gamma_5\otimes\mathbb{1}}(A) = \frac{1}{4}\,\mathrm{Tr}\left(\gamma_A^\dagger \gamma_5 \gamma_B \mathbb{1}^\dagger\right)\Big|_{B=\overline{A}} = (-1)^{A_1+A_3}$
- $M_{\gamma_5\otimes\mathbb{1}}\, \chi(x_A) = \widetilde{D}_3\, \widetilde{D}_2\, \widetilde{D}_1\, \widetilde{D}_0\, \chi(x_A)$

  : 16 terms contribute, include points **outside the hypercube**

Golterman vs. ours

- Golterman's operator ($[\gamma_5 \otimes \mathbb{1}]_{\text{Golt}}$) vs. ours ($[\gamma_5 \otimes \mathbb{1}]_{\text{ours}}$)

	$[\gamma_5 \otimes \mathbb{1}]_{\text{Golt}}$	$[\gamma_5 \otimes \mathbb{1}]_{\text{ours}}$
Ward identities $[\gamma_5 \otimes \xi_5] = [\gamma_5 \otimes \mathbb{1}][\mathbb{1} \otimes \xi_5]$ $[\gamma_5 \otimes \mathbb{1}] = [\gamma_5 \otimes \xi_5][\mathbb{1} \otimes \xi_5]$ $[\mathbb{1} \otimes \xi_5] = [\gamma_5 \otimes \xi_5][\gamma_5 \otimes \mathbb{1}]$	×	○
recursion relations $[\gamma_5 \otimes \mathbb{1}]^{2n+1} = [\gamma_5 \otimes \mathbb{1}]$ $[\gamma_5 \otimes \mathbb{1}]^{2n} = [\mathbb{1} \otimes \mathbb{1}]$ $\left[\frac{1}{2}(1 \pm \gamma_5) \otimes \mathbb{1}\right]^n = \left[\frac{1}{2}(1 \pm \gamma_5) \otimes \mathbb{1}\right]$ $\left[\frac{1}{2}(1 + \gamma_5) \otimes \mathbb{1}\right]\left[\frac{1}{2}(1 - \gamma_5) \otimes \mathbb{1}\right] = 0$	×	○

Dirac eigenmodes of staggered quarks

Simulation details

- Gauge field and quarks

gluon action	tree level Symanzik
tadpole improvement	yes
β	5.0
geometry	20^4
a	0.077(1) fm
$1/a$	2.6 GeV
valence quarks	HYP staggered fermions
N_f	0 (quenched QCD)

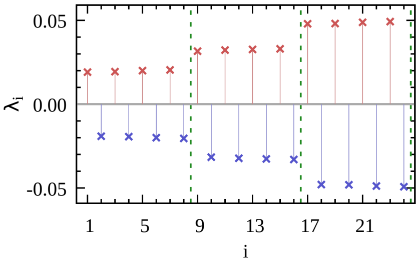
[HPQCD and UKQCD, Phys.Rev. D72 (2005) 054501]

- Eigenvalue calculation

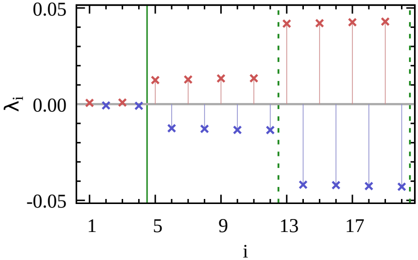
  : Lanczos (+ implicit restart & Chebyshev polynomial) [CPS library]

Dirac eigenmodes of staggered quarks

- Eigenvalue spectrum of massless **staggered Dirac operator**



(a) $Q = 0$ **(no zero modes)**

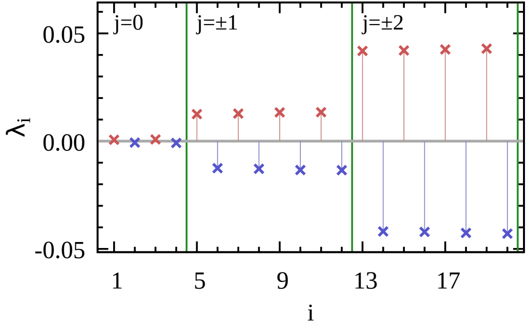


(b) $Q = -1$ **(four zero modes)**

- (serial notation) $\lambda_{2n} = -\lambda_{2n-1}$: $[\mathrm{U_A}(1)]^{\mathrm{Latt}}_{\mathrm{stag}}$ symmetry
- Four-fold (near-)degeneracy: **quartet**

  ⇒ (approximate) SU(4) **taste** symmetry

Quartet notation



i	j	m
1	0	1
2	0	2
3	0	3
4	0	4
5	+1	1
7	+1	2
9	+1	3
11	+1	4
6	−1	1
8	−1	2
10	−1	3
12	−1	4

- i : serial notation ($\lambda_{2n} = -\lambda_{2n-1}$)
- (j, m) : **quartet** notation ($\lambda_{-j,m} = -\lambda_{+j,m}$)
  - j : quartet index, $j = 0, \pm 1, \pm 2, \cdots$
  - m : taste index, $m \in \{1, 2, 3, 4\}$

Chiral Ward identity

Chiral Ward identity

- D_s: massless staggered Dirac operator
- $D_s^\dagger = -D_s \quad \Rightarrow \quad D_s \left| f_\lambda^s \right\rangle = i\lambda \left| f_\lambda^s \right\rangle$
- $[\gamma_5 \otimes \xi_5] D_s = -D_s [\gamma_5 \otimes \xi_5] \quad \Rightarrow \quad [\gamma_5 \otimes \xi_5] \left| f_{+\lambda}^s \right\rangle = e^{+i\theta} \left| f_{-\lambda}^s \right\rangle$
- $[\gamma_5 \otimes \xi_5] = [\gamma_5 \otimes \mathbb{1}][\mathbb{1} \otimes \xi_5] = [\mathbb{1} \otimes \xi_5][\gamma_5 \otimes \mathbb{1}]$
- $\Rightarrow$ **chiral Ward identities** for staggered fermions:

$$[\gamma_5 \otimes \mathbb{1}] | f_{+\lambda}^s \rangle = e^{+i\theta} [\mathbb{1} \otimes \xi_5] | f_{-\lambda}^s \rangle \tag{7}$$

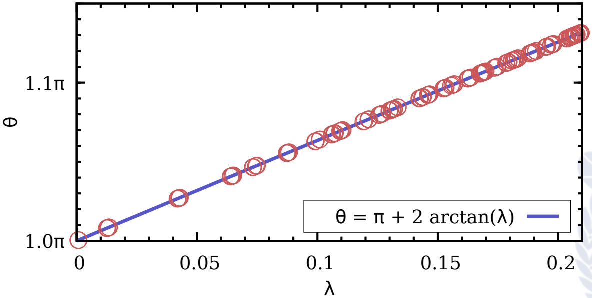
$$[\gamma_5 \otimes \mathbb{1}] | f_{-\lambda}^s \rangle = e^{-i\theta} [\mathbb{1} \otimes \xi_5] | f_{+\lambda}^s \rangle \tag{8}$$

Phase of Γ_ε transformation

- In theory, θ is random.
- In practice, a typical even-odd preconditioning of eigenvalue calculation induces a constrained phase:

$$e^{i\theta} = \langle f^s_{-\lambda} | \Gamma_\epsilon | f^s_{+\lambda} \rangle = -\frac{1+i\lambda}{1-i\lambda} = e^{i(\pi+2\beta)} \quad \text{where } \beta \equiv \arctan(\lambda)$$

- $\Rightarrow \boldsymbol{\theta = \pi + 2\beta}$



Leakage with chiral Ward identity

- **Leakage**
  - $\Gamma_5(\alpha,\beta) = \langle f^s_\alpha | [\gamma_5 \otimes \mathbb{1}] | f^s_\beta \rangle$
  - $\Xi_5(\alpha,\beta) = \langle f^s_\alpha | [\mathbb{1} \otimes \xi_5] | f^s_\beta \rangle$

- Rewriting chiral Ward identities,

$$\Gamma_5(\alpha,\pm\beta) = e^{\pm i\theta_\beta}\, \Xi_5(\alpha,\mp\beta) \tag{9}$$

$$\Gamma_5(\pm\alpha,\beta) = e^{\mp i\theta_\alpha}\, \Xi_5(\mp\alpha,\beta) \tag{10}$$

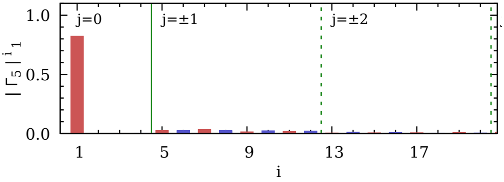
- Taking norm and combining all,

$$\begin{aligned} |\Gamma_5(\alpha,\beta)| &= |\Xi_5(\alpha,-\beta)| = |\Xi_5(-\alpha,\beta)| = |\Gamma_5(-\alpha,-\beta)| \\ &= |\Gamma_5(\beta,\alpha)| = |\Xi_5(\beta,-\alpha)| = |\Xi_5(-\beta,\alpha)| = |\Gamma_5(-\beta,-\alpha)| \end{aligned} \tag{11}$$

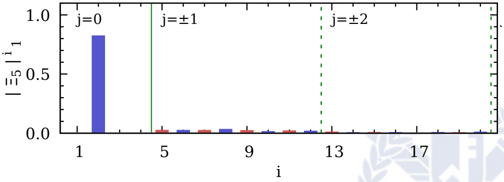
Chiral Ward identities: diagonal

- Diagonal WI

leakage	value
$\lvert\Gamma_5(\lambda_1,\lambda_1)\rvert$	0.8238257
$\lvert\Xi_5(\lambda_2,\lambda_1)\rvert$	0.8238257
$\lvert\Xi_5(\lambda_1,\lambda_2)\rvert$	0.8238257
$\lvert\Gamma_5(\lambda_2,\lambda_2)\rvert$	0.8238257



(a) $\lvert\Gamma_5(\lambda_i,\lambda_1)\rvert$

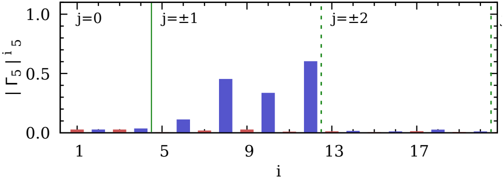


(b) $\lvert\Xi_5(\lambda_i,\lambda_1)\rvert$

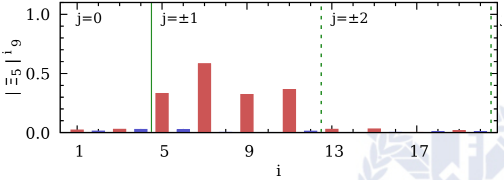
Chiral Ward identities: off-diagonal

- Off-diagonal WI

leakage	value
$\lvert\Gamma_5(\lambda_{10},\lambda_5)\rvert$	0.3344229
$\lvert\Gamma_5(\lambda_5,\lambda_{10})\rvert$	0.3344229
$\lvert\Xi_5(\lambda_6,\lambda_{10})\rvert$	0.3344229
$\lvert\Xi_5(\lambda_{10},\lambda_6)\rvert$	0.3344229
$\lvert\Xi_5(\lambda_9,\lambda_5)\rvert$	0.3344229
$\lvert\Xi_5(\lambda_5,\lambda_9)\rvert$	0.3344229
$\lvert\Gamma_5(\lambda_6,\lambda_9)\rvert$	0.3344229
$\lvert\Gamma_5(\lambda_9,\lambda_6)\rvert$	0.3344229



(a) $\lvert\Gamma_5(\lambda_i,\lambda_5)\rvert$



(b) $\lvert\Xi_5(\lambda_i,\lambda_9)\rvert$

Leakage pattern for chirality operator

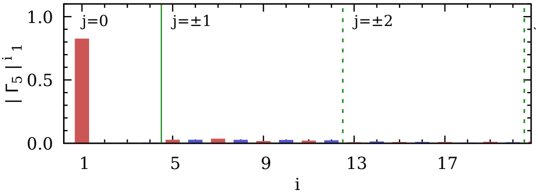
Leakage pattern for zero modes

- **Leakage patterns** for would-be **zero** mode $\lambda_1 (= \boldsymbol{\lambda_{j=0,m=1}})$

◆ $|\Gamma_5|^i_1 = |\Gamma_5(\lambda_i, \lambda_1)|$

◆ $\lambda_1 \longrightarrow \lambda_1$

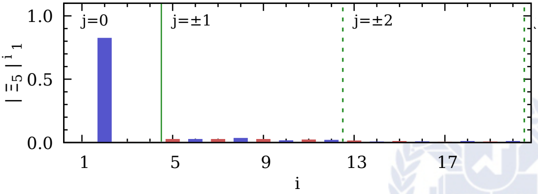
$\lambda_{j=0,m=1}$
$\longrightarrow \lambda_{j=0,m=1}$



◆ $|\Xi_5|^i_1 = |\Xi_5(\lambda_i, \lambda_1)|$

◆ $\lambda_1 \longrightarrow \lambda_2$

$\lambda_{j=0,m=1}$
$\longrightarrow \lambda_{j=0,m=2}$



Leakage pattern for non-zero modes (I)

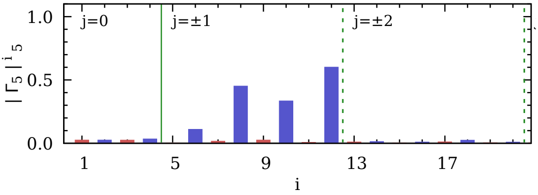
- **Leakage patterns** for **non-zero** mode $\lambda_5 (= \boldsymbol{\lambda_{j=+1,m=1}})$

◆ $|\Gamma_5|_5^i = |\Gamma_5(\lambda_i, \lambda_5)|$

◆ $\lambda_5 \longrightarrow \lambda_6, \lambda_8, \lambda_{10}, \lambda_{12}$

$\lambda_{j=+1,m=1}$

$\longrightarrow \lambda_{j=-1,m=1,2,3,4}$

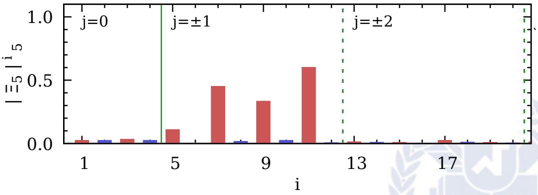


◆ $|\Xi_5|_5^i = |\Xi_5(\lambda_i, \lambda_5)|$

◆ $\lambda_5 \longrightarrow \lambda_5, \lambda_7, \lambda_9, \lambda_{11}$

$\lambda_{j=+1,m=1}$

$\longrightarrow \lambda_{j=+1,m=1,2,3,4}$



Leakage pattern for non-zero modes (II)

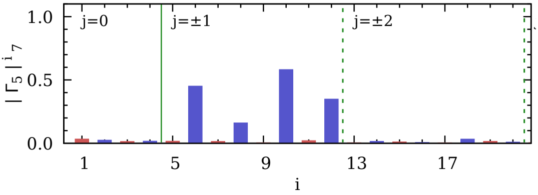
- **Leakage patterns** for **non-zero** mode $\lambda_7 (= \boldsymbol{\lambda_{j=+1,m=2}})$

◆ $|\Gamma_5|^i_7 = |\Gamma_5(\lambda_i, \lambda_7)|$

◆ $\lambda_7 \longrightarrow \lambda_6, \lambda_8, \lambda_{10}, \lambda_{12}$

$\boldsymbol{\lambda_{j=+1,m=2}}$

$\longrightarrow \boldsymbol{\lambda_{j=-1,m=1,2,3,4}}$

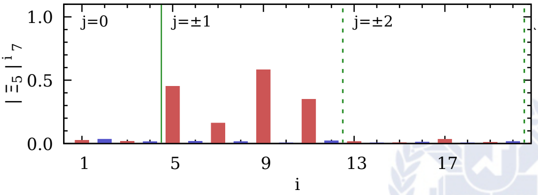


◆ $|\Xi_5|^i_7 = |\Xi_5(\lambda_i, \lambda_7)|$

◆ $\lambda_7 \longrightarrow \lambda_5, \lambda_7, \lambda_9, \lambda_{11}$

$\boldsymbol{\lambda_{j=+1,m=2}}$

$\longrightarrow \boldsymbol{\lambda_{j=+1,m=1,2,3,4}}$



Summary of leakage pattern

- **Leakage patterns** for zero modes and non-zero modes

leakage	**zero mode**	**non-zero mode**
$\lvert\Gamma_5\rvert_{j,m}^{j',m'}$	no leakage (chirality)	to parity partner **quartet**
	$j' = j = 0,\ m' = m$	$j' = -j \neq 0,\ m' = 1,2,3,4$
$\lvert\Xi_5\rvert_{j,m}^{j',m'}$	to parity partner	to its own **quartet**
	$j' = j = 0,\ m' \neq m$	$j' = j \neq 0,\ m' = 1,2,3,4$

- Ward identity: $\lvert\Gamma_5\rvert_{j,m}^{+j',m'} = \lvert\Xi_5\rvert_{j,m}^{-j',m'}$

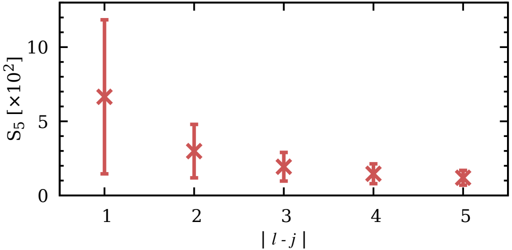
⇒ Evidence of **chiral symmetry** & **taste symmetry**

Leakage by taste symmetry breaking

- $S_5(\ell, j) \equiv \frac{1}{16} \sum_{m,m'} |\Xi_5|^{\ell,m'}_{j,m}$

  : leakage from one quartet(j) to another quartet($\ell \neq j$)

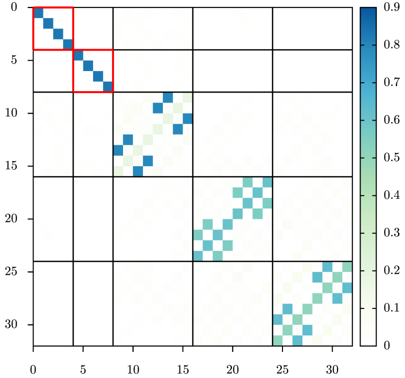
Continuum	Lattice (stag)
$S_5 = 0$	$S_5 > 0$ by **taste symmetry breaking**



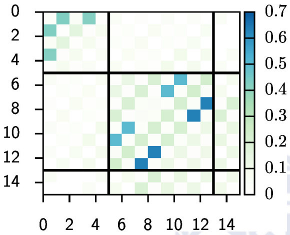
- $S_5 \lesssim 10\%$
- $S_5 \to 0$
- $\Rightarrow$ Leakage to other quartets is a **random noise**.

Machine learning of leakage pattern

- Matrix elements $|\Gamma_5|^i_k = |\Gamma_5(\lambda_i, \lambda_k)| = |\langle f^s_{\lambda_i} | [\gamma_5 \otimes \mathbb{1}] | f^s_{\lambda_k} \rangle|$



- Identify a quartet group by training the leakage pattern



⇒ **99.4(23)% correct!**

[Talk by **Sunkyu Lee** (next)]

Nonperturbative renormalization of chirality

Renormalization factor for chirality

- Renormalization factor κ_P for chirality measurement

$$4 \times Q = -\boldsymbol{\kappa_P} \times \sum_{\lambda \in S_0} \langle f_\lambda^s | [\gamma_5 \otimes \mathbb{1}] | f_\lambda^s \rangle \tag{12}$$

where S_0 is the set of zero modes

- $\kappa_P = \dfrac{Z_{P\times S}(\mu)}{Z_{P\times P}(\mu)}$ where

$$\mathcal{O}_S = \bar{\chi}\, [\gamma_5 \otimes \mathbb{1}]\, \chi \tag{13}$$

$$\mathcal{O}_P = \bar{\chi}\, [\gamma_5 \otimes \xi_5]\, \chi \tag{14}$$

$$[\mathcal{O}_S]_R(\mu) = Z_{P\times S}(\mu)\, [\mathcal{O}_S]_B \tag{15}$$

$$[\mathcal{O}_P]_R(\mu) = Z_{P\times P}(\mu)\, [\mathcal{O}_P]_B \tag{16}$$

- $\Rightarrow \boldsymbol{\kappa_P = -\dfrac{4Q}{C_0}}$ where $C_0 = \sum_{\lambda \in S_0} \boldsymbol{\Gamma_5}(\lambda, \lambda)$: independent of μ

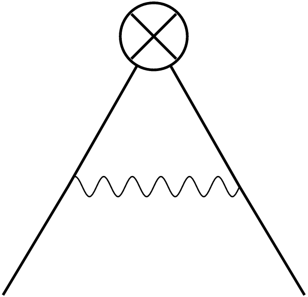
Renormalization factor κ_P

- $\kappa_P = -\frac{4Q}{C_0}$ where $C_0 = \sum_{\lambda \in S_0} \Gamma_5(\lambda, \lambda)$

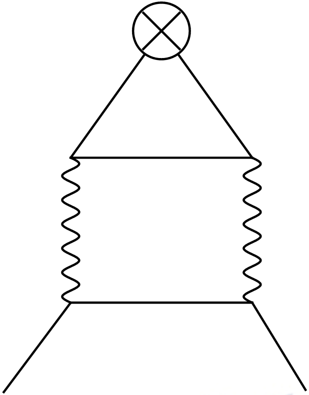
topological charge	number of samples	κ_P
$\|Q\| = 1$	72	1.26(13)
$\|Q\| = 2$	68	1.22(3)
$\|Q\| = 3$	45	1.23(2)
weighted average	241	**1.23(2)**

- Computationally, our method is much cheaper than typical NPR methods such as RI-MOM or RI-SMOM.

RI-MOM and RI-SMOM for $Z_{P\times S}$



- Connected
- **Cheap**



- Disconnected
- **Very expensive**

Conclusion

Conclusion

- A new chirality operator Γ_5 and a new shift operator Ξ_5 respect the **recursion relations** as γ_5.
- Γ_5 and Ξ_5 are related to each other through the **Ward identity** of the conserved $[\mathrm{U_A}(1)]^{\mathrm{Latt}}_{\mathrm{stag}}$ symmetry.
- **Leakage patterns** of Γ_5 and Ξ_5 allow us to distinguish zero modes from non-zero modes, and determine the topological charge reliably.
- We have also obtained the renormalization factor ratio $\boldsymbol{\kappa_P}$ from the chirality measurement.
- [arXiv:2005.10596]

Thank you for listening.

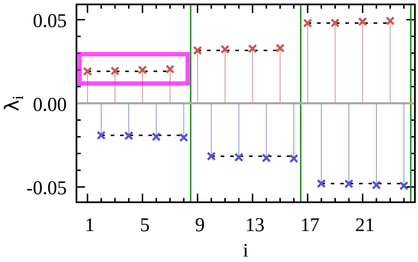
Backup

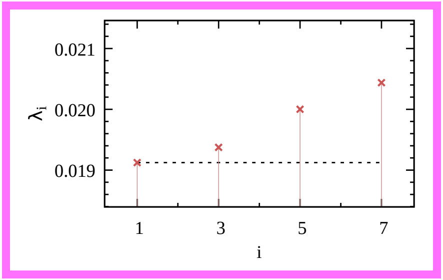
Eigenvalues of Dirac operator

Continuum	Lattice (stag)
D	D_s
$D^\dagger = -D$ $\Rightarrow D\lvert f_\lambda\rangle = i\lambda\lvert f_\lambda\rangle$	$D_s^\dagger = -D_s$ $\Rightarrow D_s\lvert f^s_\lambda\rangle = i\lambda\lvert f^s_\lambda\rangle$
$\Gamma_{5F} D = -D\Gamma_{5F}$ $(\ \Gamma_{5F} \equiv [\gamma_5 \otimes \xi_F]\ )$ $\Rightarrow D(\Gamma_{5F}\lvert f_{+\lambda}\rangle) = -i\lambda(\Gamma_{5F}\lvert f_{+\lambda}\rangle)$ $\Rightarrow \Gamma_{5F}\lvert f_{+\lambda}\rangle = e^{+i\theta}\lvert f_{-\lambda}\rangle$ for $\lambda \neq 0$	$\Gamma_\varepsilon D_s = -D_s\Gamma_\varepsilon$ $(\ \Gamma_\varepsilon \equiv [\gamma_5 \otimes \xi_5]\ )$ $\Rightarrow D_s(\Gamma_\varepsilon\lvert f^s_{+\lambda}\rangle) = -i\lambda(\Gamma_\varepsilon\lvert f^s_{+\lambda}\rangle)$ $\Rightarrow \Gamma_\varepsilon\lvert f^s_{+\lambda}\rangle = e^{+i\theta}\lvert f^s_{-\lambda}\rangle$ for $\lambda \neq 0$

Eigenvalue spectrum of D_s ($Q = 0$)

- Eigenvalue spectrum of $\boldsymbol{D_s}$ ($Q = 0$, **no zero modes**)



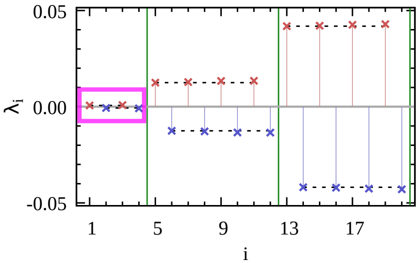


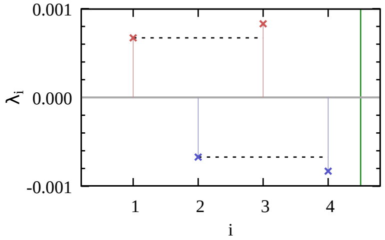
- (serial notation) $\lambda_{2n} = -\lambda_{2n-1}$: $[\mathrm{U_A}(1)]^{\mathrm{Latt}}_{\mathrm{stag}}$ symmetry
- Four-fold (near-)degeneracy: **quartet**

  $\Rightarrow$ (approximate) SU(4) **taste** symmetry

Eigenvalue spectrum of D_s ($Q = -1$)

- Eigenvalue spectrum of $\boldsymbol{D_s}$ ($Q = -1$)

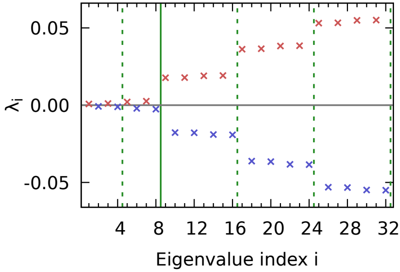




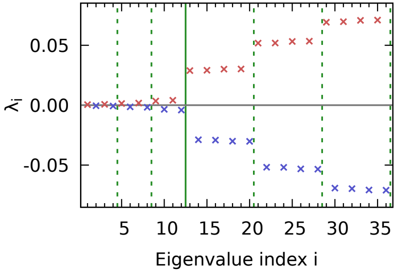
- Four (would-be) **zero modes**: $\lim_{a\to 0} \lambda_i = 0$
- Respect SU(4) **taste** symmetry (quartet)
- Respect $[\mathrm{U_A}(1)]^{\mathrm{Latt}}_{\mathrm{stag}}$ symmetry ($\lambda_{2n} = -\lambda_{2n-1}$)
- For would-be zero modes, two symmetries are fully overlapped.

  $\Rightarrow$ **Zero** modes and **non-zero** modes can be distinguished.

Eigenvalue spectrum ($Q = -2, -3$)



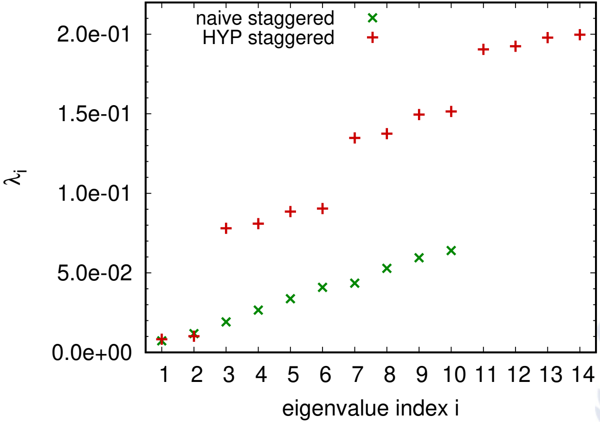
(a) $Q = -2$



(b) $Q = -3$

Eigenvalue spectrum : naive vs HYP

- 12^4 quenched lattice at $a \simeq 0.125\,\text{fm}$



Lanczos

- Lanczos (iteration) : calculate eigenvalues & eigenvectors of a Hermitian matrix

  [Phase 1] Hermitian → tri-diagonal : Lanczos algorithm

  [Phase 2] tri-diagonal → diagonal : QR iteration (with Givens rot.)

- Real benefit of Lanczos is that eigenvalues of a submatrix of the tridiagonal matrix generated by Lanczos approximate eigenvalues of the full matrix.

$$\begin{pmatrix} \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times \end{pmatrix} \xrightarrow[A\to T]{\text{Phase 1}} \begin{pmatrix} \times & \times & & \\ \times & \times & \times & \\ & \times & \times & \times \\ & & \times & \times \end{pmatrix} \xrightarrow[T\to D]{\text{Phase 2}} \begin{pmatrix} \times & & & \\ & \times & & \\ & & \times & \\ & & & \times \end{pmatrix}$$

Implicitly restarted Lanczos

- Larger submatrix
  - Better approximation of eigenvalues

  - Larger memory to keep eigenvectors
  - More computational cost for Lanczos and diagonalization
- ⇒ Implicit restart

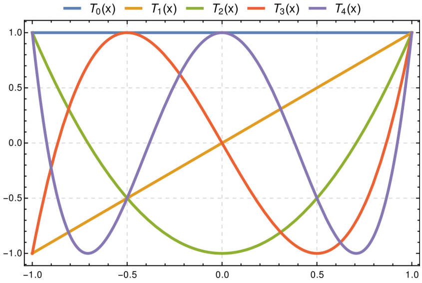
  : Restart Lanczos from a smaller submatrix after deflating unnecessary eigenvectors
- As if we have started with a shifted initial vector $(D - \lambda \mathbb{I})\, v_1$

Chebyshev polynomial

- Eigenvalues of a submatrix converge to the eigenvalues of the largest or the smallest, or who has a lower density.

- Less dense, converge faster

- $\Rightarrow$ Chebyshev polynomial $T_n(x)$



- In $[-1, 1]$, soft and bounded
- Diverge outside
- Using $T_n(D)$ instead of D improves the convergence and excludes unnecessary region

Application of Lanczos

- Hermitian form : $iD_s, D_s^\dagger D_s, \dots$
- We use $D_s^\dagger D_s$.
  - positive semi-definite ⇒ eigenvalues are non-negative ($\lambda^2 \geq 0$)
  - even-odd splitting ⇒ fast convergence, less memory
- If $|g^s_{\lambda^2}\rangle$ is an eigenvector of $D_s^\dagger D_s$,

$$D_s^\dagger D_s \,|g^s_{\lambda^2}\rangle = \lambda^2 \,|g^s_{\lambda^2}\rangle \tag{17}$$

$$|g^s_{\lambda^2}\rangle = c_1 \,|f^s_{+\lambda}\rangle + c_2 \,|f^s_{-\lambda}\rangle \tag{18}$$

- $|f^s_{\pm\lambda}\rangle$ are obtained by projections

$$P_\pm \equiv (D_s \pm i\lambda) \tag{19}$$

Chirality and index theorem

- (topological charge) $\boldsymbol{Q} = \frac{1}{32\pi^2} \int d^4x \, \varepsilon_{\mu\nu\rho\sigma} \, \mathrm{Tr}\left[F_{\mu\nu}(x) F_{\rho\sigma}(x) \right]$
- Atiyah-Singer **index theorem**

Continuum	Lattice (stag)
$Q = \frac{1}{N_f}(n_- - n_+)$ ▶ $N_f = 4$ ▶ $n_\pm$: number of right-handed(+) and left-handed(−) zero modes	?

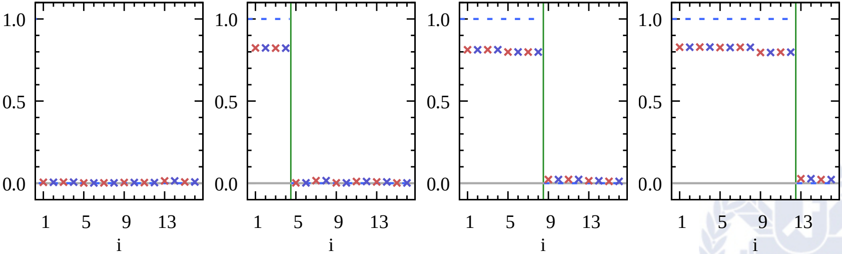
- Chirality operator

Continuum	Lattice (stag)
$\langle f_\lambda \| [\gamma_5 \otimes \mathbb{1}] \| f_\lambda \rangle$	$\langle f^s_\lambda \| [\gamma_5 \otimes \mathbb{1}] \| f^s_\lambda \rangle$

Chirality measurement

- **Chirality**

	Continuum	Lattice (stag)
operator	$\langle f_\lambda \| [\gamma_5 \otimes \mathbb{1}] \| f_\lambda \rangle$	$\langle f_\lambda^s \| [\gamma_5 \otimes \mathbb{1}] \| f_\lambda^s \rangle$
$\lambda = 0$	± 1	$\sim \pm 0.8 \, (< \pm 1)$
$\lambda \neq 0$	0	~ 0

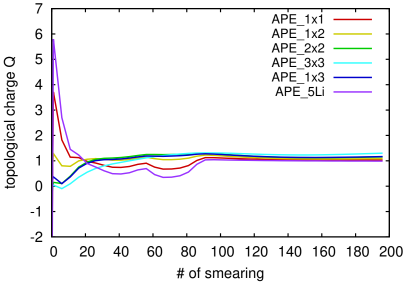


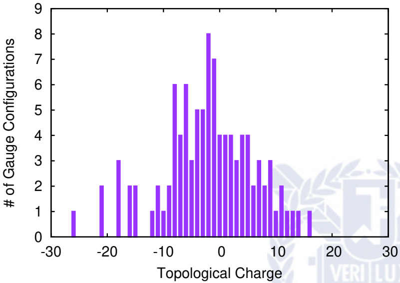
(a) $Q = 0$ (b) $Q = -1$ (c) $Q = -2$ (d) $Q = -3$

Topological charge

- $Q = \dfrac{1}{32\pi^2} \int d^4x\, \varepsilon_{\mu\nu\rho\sigma}\, \mathrm{Tr}\left[F_{\mu\nu}(x)\, F_{\rho\sigma}(x) \right]$
- APE smearing
- 5 Loop improved operator

[Forcrand, Perez, Stamatescu, Nucl.Phys. B499 (1997)]





Proof for renormalization factor κ_P

$$
\begin{aligned}
N_f Q &= \int d^4x\, m_R\, [\bar{q}_f\, \Gamma_5\, q_f]_R && (20) \\
&= Z_m\, m_B\, Z_{P\times S} \int d^4x\, [\bar{q}_f\, \Gamma_5\, q_f]_B && (21) \\
&= Z_m\, Z_{P\times S}\, m_B \left(-\sum_\lambda \frac{\langle f_\lambda^s | \Gamma_5 | f_\lambda^s \rangle}{i\lambda + m_B} \right) && (22) \\
&= -\frac{Z_{P\times S}}{Z_{P\times P}} \sum_{\lambda \in S_0} \langle f_\lambda^s | \Gamma_5 | f_\lambda^s \rangle\,, && (23)
\end{aligned}
$$

since $m_R[\bar{q}_f\, \Gamma_\varepsilon\, q_f]_R = m_B[\bar{q}_f\, \Gamma_\varepsilon\, q_f]_B$.