Double-winding Wilson loops towards flux tube interaction in SU(N) lattice gauge theory

S. Kato (Oyama National College of Technology)

Collaborations:

A. Shibata (KEK), K.-I. Kondo (Chiba University)

Asia-Pacific Symposium for Lattice Field Theory (APLAT2020) @online, 7 Aug 2020
1. Introduction

What is double-winding Wilson loop operator?

In 2015, J. Greensite et al. introduced a following ``double-winding'' Wilson loop operator in lattice gauge theory: ([1] Phys.Rev.D91,054509(2015))

\[ W(C') \equiv \text{tr}\left[ \prod_{\ell \in C} U_{\ell} \right], \quad (C = C_1 \times C_2) \]

This is a path-ordered product of (gauge) link variables along a closed contour C which is composed of two loops C_1 and C_2.

\[ <\text{shifted double-winding}> W_S \quad <\text{coplanar double-winding}> W_C \]
Why do we consider such operator?

They considered such operator to examine possible mechanisms for quark confinement.

They studied the $L_1$-dependence of a coplanar double-winding Wilson loop average, $\langle W_C \rangle$, with the other lengths $L$, $L_2$, and $\delta L (=0)$ being fixed:

Difference-of-areas law: $\langle W_C \rangle \simeq \exp[-\sigma |S_1 - S_2|] = \exp[-\sigma L_2 (L - L_1)]$

Sum-of-areas law: $\langle W_C \rangle \simeq \exp[-\sigma' (S_1 + S_2)] = \exp[-\sigma' L_2 (L + L_1)]$
They performed the numerical simulations for SU(2) LGT on $20^4$-lattice at $\beta = 2.4$, and studied $L_1$-dependence of $\ln < W_C >$ with the other lengths being fixed, $L = 10$, $L_2 = 1 \sim 10$ and $\delta L = 0$.

They also studied $L_1$-dependence of $\ln < W_C >$ from center vortex d.o.f and abelian d.o.f.


• They also studied $L_1$-dependence of $\ln < W_C >$ from center vortex d.o.f and abelian d.o.f.
Motivation

In this way, the study of double-winding Wilson loops itself is interesting because it can be used to test the confinement mechanism in QCD. J. Greensite et al. showed difference-of-areas law for "coplanar" double-winding Wilson loop average in SU(2) LGT.

How about SU(3) LGT? How about Large N?

We study double-winding Wilson loops in SU(N) lattice gauge theory by using both strong coupling expansions and numerical simulations.

1) We examine how the area law falloff of a "coplanar" double-winding Wilson loop average, $W_C$, depends on the number of color $N$, which may contain information on the possible mechanism of quark confinement, e.g., magnetic monopole, center vortex, etc.

2) We evaluate "shifted" double-winding Wilson loop average, $W_S$, by changing the distance of a transverse direction, which may contain an information about interactions between two color flux tubes.
2. A “coplanar” double-winding Wilson loop in SU(N) LGT

■ strong coupling expansion

One of a set of plaquettes tiling the areas $S_1$ and $S_2$ which gives the non-trivial contribution to a coplanar double-winding Wilson loop average:

$$S_g = \sum_{n, \mu < \nu} \frac{1}{g^2} \left\{ \begin{array}{c} \hat{\nu} \\ \hat{\mu} \end{array} \right\}$$

$$\approx p_N \left( \frac{1}{g^2 N} \right)^{(N-2)S_2 + (S_1 - S_2)}$$
Another set of plaquettes tiling the areas $S_1$ and $S_2$ which gives the non-trivial contribution to a coplanar double-winding Wilson loop average:

\[
q_N = - \frac{N^{2S_2}}{2} \left\{ \left[ \frac{1}{N(N-1)} \right]^{S_2-1} - \left[ \frac{1}{N(N+1)} \right]^{S_2-1} \right\} \quad (S_2 \geq 1)
\]

\[
\approx q_N \left( \frac{1}{g^2 N} \right)^{2S_2 + (S_1 - S_2)} = q_N \left( \frac{1}{g^2 N} \right)^{S_1 + S_2}
\]
SU(2) : Difference-of-areas law [reconfirmed and improved]

\[
\langle W_C \rangle = 2p_2 \left( \frac{1}{2g^2} \right)^{S_1-S_2} + 2q_2 \left( \frac{1}{2g^2} \right)^{S_1+S_2} + \cdots \\
p_2 = -2, \quad q_2 = -\frac{4S_2}{2} \left\{ \left[ \frac{1}{2} \right]^{S_2-1} - \left[ \frac{1}{6} \right]^{S_2-1} \right\}
\]

\[S_1 \equiv L \cdot L_2, \quad S_2 \equiv L_1 \cdot L_2\]

L_1-dependence of \( \langle W_C \rangle \) from the S.C.E in SU(2) LGT: (L=10, \( L_2=1 \) and \( 1/g^2N=2.5/8 \))

\[\begin{array}{c}
\text{L}_1 \\
\text{dependence of } \langle W_C \rangle \text{ from the S.C.E in SU(2) LGT: } \\
(L=10, L_2=1 \text{ and } 1/g^2N=2.5/8)
\end{array}\]
SU(3) : max-of-areas law [New]

\[ \langle W_C \rangle = p_3 \left( \frac{1}{3g^2} \right)^{S_1} + q_3 \left( \frac{1}{3g^2} \right)^{S_1+S_2} + \cdots \]

\[ p_3 = -3, \quad q_3 = -\frac{9S_2}{2} \left\{ \left[ \frac{1}{6} \right]^{S_2-1} - \left[ \frac{1}{12} \right]^{S_2-1} \right\} \]

\[ S_1 \equiv L \cdot L_2 \]
\[ S_2 \equiv L_1 \cdot L_2 \]

L₁-dependence of \( \langle W_C \rangle \) from the S.C.E in SU(3) LGT:
\( (L=10, L_2=1 \text{ and } 1/g^2N=6.0/18) \)
SU(N) (N≥4) : sum-of-areas law [New]

\[
\langle W_C \rangle = p_N \left( \frac{1}{g^2 N} \right)^{(N-2)S_2 + S_1 - S_2} + q_N \left( \frac{1}{g^2 N} \right)^{S_1 + S_2} + \cdots
\]

For N≥4, we find that the second term in above equation gives the dominant contribution in the strong coupling expansion for \( \langle W_C \rangle \), since the inequality holds,

\[ S_1 + S_2 \leq (N-2)S_2 + S_1 - S_2 \]

for N≥4.

Thus we conclude that the sum-of-areas law of a coplanar double-winding Wilson loop is allowed for N≥4.
numerical simulation

SU(2) : Difference-of-areas law [reconfirmed]

Lattice set up:
- standard Wilson action
- $24^4$-lattice, $\beta=2.5$
- 100 configurations

$L_1$-dependence of $-\langle W_C \rangle$ from the numerical simulation in SU(2) LGT:
(L=10, $L_2=3$ and $\beta=2.5$)

(S.C.E)
SU(3) : max-of-areas law [New]

Lattice set up:
• standard Wilson action
• $24^4$-lattice, $\beta=6.2$
• 200 configurations
• APE smearing method ($N'=12, \alpha=0.1$)

$L_1$-dependence of $\langle W_C \rangle$ from the numerical simulation in SU(3) LGT:
$(L=10, L_2=4,6,8$ and $\beta=6.2)$
3. A “shifted” double-winding Wilson loop in SU(N) LGT

The setting up of a shifted double-winding Wilson loop operator ($W_s$):

\[ S_1 \equiv L \cdot L_2 \]
\[ S_2 \equiv L_1 \cdot L_2 \]
strong coupling expansion

The diagrams which may give a leading contribution in the S.C.E are given by a set of plaquettes tiling as follows:

\[
\langle W_S \rangle \equiv \langle W(C_1 \times C_2) \rangle_{R \neq 0} = N \left( \frac{1}{g^2 N} \right)^{S_1 + S_2} 
+ \left( \frac{1}{g^2 N} \right)^{2R(L_1 + L_2)} \times \left\{ \begin{array}{c} p_N \left( \frac{1}{g^2 N} \right)^{(N-2)S_2 + S_1 - S_2} \\ + q_N \left( \frac{1}{g^2 N} \right)^{S_1 + S_2} \end{array} \right\} + \cdots
\]
As is explained in [1], the shifted double-winding Wilson loop ($W_s$) at a fixed time can be interpreted as a tetra-quark system consisting of two static quarks and two static antiquarks.

The pairs of quark-antiquarks are connected by a pair of color flux tubes, as seen in the bottom panel.

We study how interactions between the two color flux tubes change, when the distance $R$ is varied.
\[ \langle W_S \rangle = N \left( \frac{1}{g^2 N} \right)^{S_1+S_2} \]

\[ + \left( \frac{1}{g^2 N} \right)^{2R(L_1+L_2)} \times \left\{ p_N \left( \frac{1}{g^2 N} \right)^{(N-2)S_2+S_1-S_2} + q_N \left( \frac{1}{g^2 N} \right)^{S_1+S_2} \right\} + \cdots \]

We find that the second term dominates for \( R < R_C \), and the first term dominates for \( R > R_C \).

This means that the left diagram dominates for \( R < R_C \), and the right diagram dominates for \( R > R_C \).

Therefore, the dominant diagram switches from left to right at a certain value \( R_C \) as \( R \) increases, just like the minimal surface spanned by a soap film.
\[ \langle W_S \rangle \]

\[
= 4 \left( \frac{1}{2g^2} \right)^{S_1+S_2} + 2p_2 \left( \frac{1}{2g^2} \right)^{S_1-S_2+2R(L_1+L_2)} + 2q_2 \left( \frac{1}{2g^2} \right)^{S_1+S_2+2R(L_1+L_2)} + \cdots 
\]

\( R_C \equiv L_1/(1+L_1/T) \)

Results from S.C.E & numerical simulation [New]

SU(2): R-dependence of a shifted double-winding Wilson loop average (Ws)

\[
\langle W_S \rangle
\]

S.C.E

(1/2g^2=2.5/8, L=5, L_2=1, L_1=3)

Numerical: 100 conf.

(\(\beta=2.5, L=5, L_2=2, L_1=3\))
**SU(3)**: R-dependence of a shifted double-winding Wilson loop average ($W_s$)

\[ \langle W_s \rangle = 3 \left( \frac{1}{3g^2} \right)^{S_1+S_2} + p_3 \left( \frac{1}{3g^2} \right)^{S_1+2R(L_1+L_2)} + q_3 \left( \frac{1}{3g^2} \right)^{S_1+S_2+2R(L_1+L_2)} + \cdots \]

---

**S.C.E**

\(\frac{1}{3g^2}=6.0/18, \ L=5, \ L_2=1, \ L_1=3\)

**Numerical**: 200 conf.

\(\beta=6.2, \ L=8, \ L_2=8, L_1=1 \sim 6\)

---

2020/8/7
4. Conclusion and outlook

We have studied the double-winding Wilson loops in SU(N) lattice gauge theory by using both strong coupling expansion and numerical simulation.

(1) We have examined how the area law falloff of a `coplanar" double-winding Wilson loop average depends on the number of color N, by changing the size of minimal area \( S_2 \) of loop \( C_2 \).
We have reconfirmed the difference-of-areas law for \( N=2 \), and have found new results that `max-of-areas law" for \( N=3 \) and sum-of-areas law for \( N \geq 4 \). These results are consistent with Matsudo-Kondo (Phys.Rev.D96,105011(2017) ).

(2) We have evaluated a `shifted" double-winding Wilson loop average by changing the distance of a transverse direction, and have found that their long distance behavior doesn't depend on the number of color N, but the short distance behavior depends on \( N \).

outlook;
- Extract an information about interactions between two color flux tubes form `shifted" double-winding Wilson loop.
- Search an explicit expression for the Abelian operator which reproduce full double-winding Wilson loops.

Thank you!