Quantum Links for U(1) Gauge Theory on Qubits and Reduction to Z2 Gauge Theory and Toric Code

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This work is a part of QuLat collaboration https://qulat.sites.uiowa.edu/

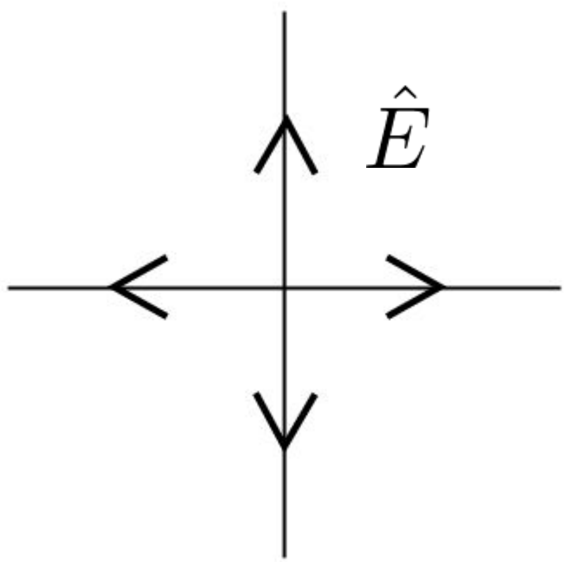
Overview

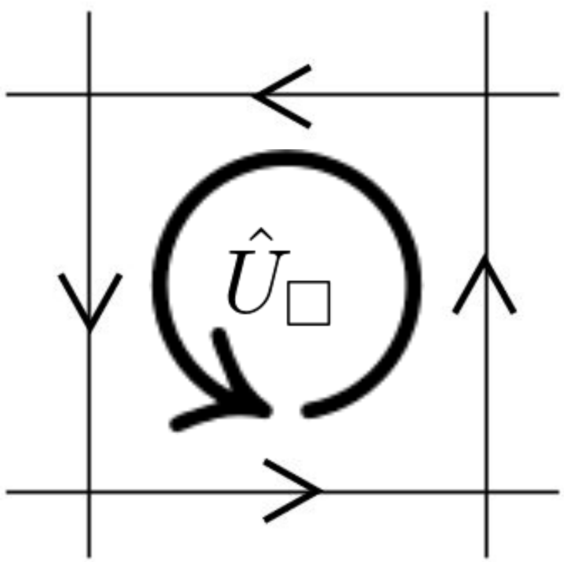
- We know the Hamiltonian of the quantum link model [1]
  - we can simulate e.g. real-time evolution, exponential resources tho.
  - But we can put it on a quantum computer/simulator
- We study the 2+1D QED toy model on a triangular lattice.
  - D-theory [2] rewrites the model in the spin basis
  - construct a quantum circuit and test it with the IBM Q device.
  - Right now investigating the similarity to the Z2 GT.

[1] R. Brower, S. Chandrasekharan and U. J. Wiese, "QCD as a quantum link model," Phys. Rev. D60,094502 (1999)
[2] R. Brower, S. Chandrasekharan, S. Riederer and U. J. Wiese, Nucl. Phys. B693, 149 (2004)

Hamiltonian Formalism of 2+1D Quantum Link Model

$$\overbrace{\hat{H} = \frac{g^2}{2}\sum_{x,\mu} Tr[\hat{E}_L^2(x,\mu) + \hat{E}_R^2(x,\mu)]}^{\hat{E}^2} \overbrace{- \frac{1}{2g^2}\sum_{\Box} Tr[\hat{U}_\Box + \hat{U}_\Box^\dagger]}^{\hat{B}^2}$$





Hamiltonian Formalism of 2+1D Quantum Link Model

$$\hat{E}^2 \qquad\qquad \hat{B}^2$$

$$\hat{H} = \frac{g^2}{2}\sum_{x,\mu} Tr[\hat{E}_L^2(x,\mu) + \hat{E}_R^2(x,\mu)] - \frac{1}{2g^2}\sum_{\Box} Tr[\hat{U}_\Box + \hat{U}_\Box^\dagger]$$

For QED (U(1)): $E_L = -E_R = -i\partial_\theta, \quad U = \exp(i\theta)$

Algebra: $[E_L, U] = U, \quad [E_L, U^\dagger] = -U^\dagger$

Into spin algebra

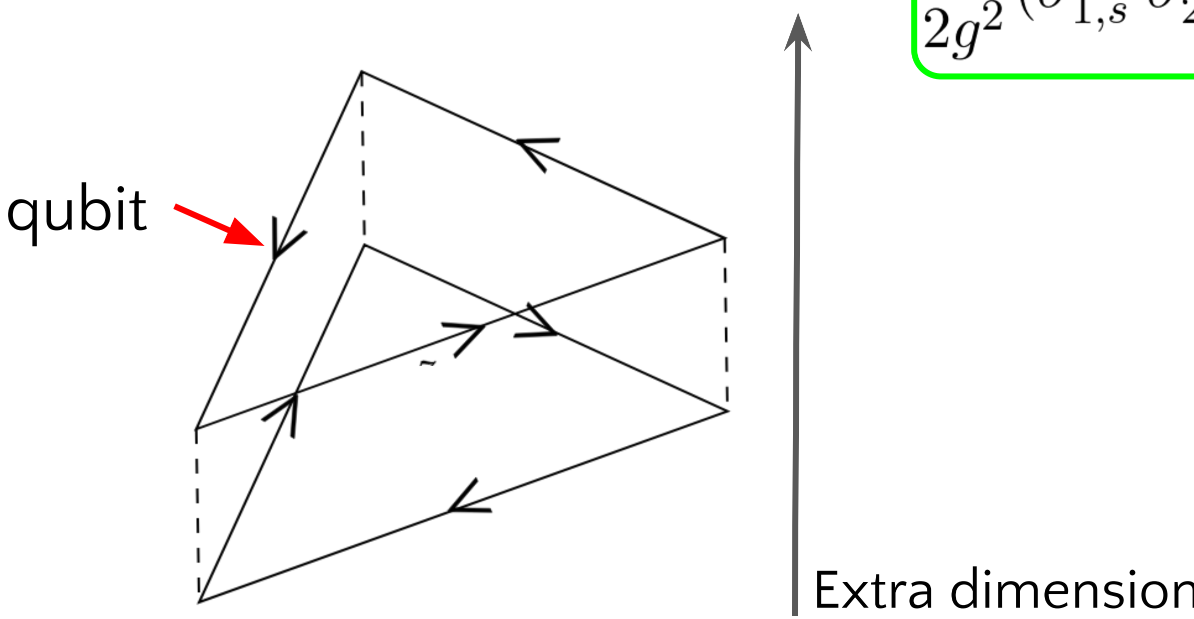
$$E_L = -E_R \to \sigma^z$$
$$U \to \sigma^+, \quad U^\dagger \to \sigma^-$$

$$[\sigma^z, \sigma^+] = \sigma^+, \quad [\sigma^z, \sigma^-] = -\sigma^-$$

Now we have the spin model

For a single triangle plaquette:

$$\hat{H} = \sum_s \Bigg[\underbrace{\frac{g^2}{2} \sum_{j=1}^{3} (\sigma^z_{j,s} + \sigma^z_{j,s+1})^2}_{\hat{H}_E} + \underbrace{\frac{\alpha}{2g^2} \sum_{j=1}^{3} (\sigma^+_{j,s}\,\sigma^-_{j,s+1} + \sigma^-_{j,s}\,\sigma^+_{j,s+1})}_{\hat{H}_{XY}} - \underbrace{\frac{1}{2g^2} (\sigma^+_{1,s}\,\sigma^+_{2,s}\,\sigma^+_{3,s} + \sigma^-_{1,s}\,\sigma^-_{2,s}\,\sigma^-_{3,s})}_{\hat{H}_B} \Bigg]$$



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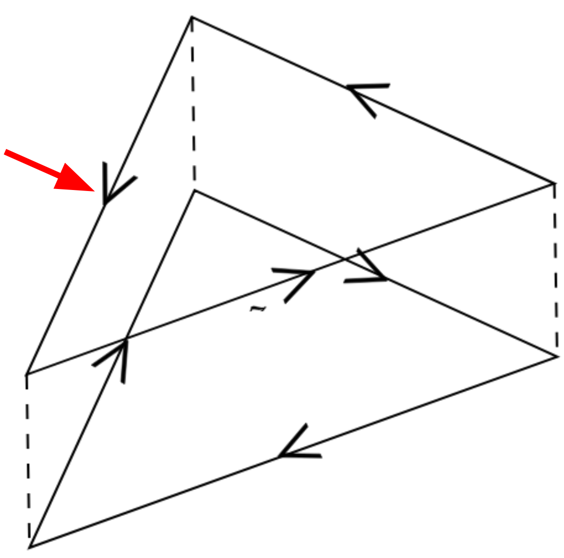
1d XXZ chain ($\hat{H}_E + \hat{H}_{XY}$)

qubit



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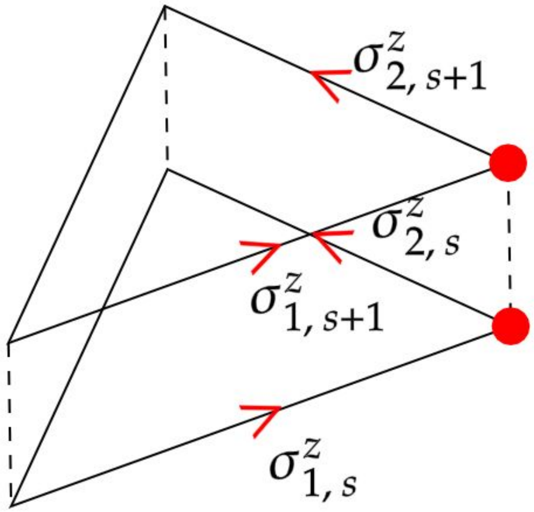


gauge generators:

$$G_{ij} = \sum_s (\sigma^z_{i,s} - \sigma^z_{j,s})$$

G_{ij}=0 corresponds to the Gauss law (physical) sector.

Now we have the spin model

For a single triangle plaquette:

$$\hat{H} = \sum_s \Bigg[\underbrace{\frac{g^2}{2} \sum_{j=1}^{3} (\sigma^z_{j,s} + \sigma^z_{j,s+1})^2}_{\hat{H}_E} + \underbrace{\frac{\alpha}{2g^2} \sum_{j=1}^{3} (\sigma^+_{j,s}\,\sigma^-_{j,s+1} + \sigma^-_{j,s}\,\sigma^+_{j,s+1})}_{\hat{H}_{XY}}$$

$$- \underbrace{\frac{1}{2g^2} (\sigma^+_{1,s}\,\sigma^+_{2,s}\,\sigma^+_{3,s} + \sigma^-_{1,s}\,\sigma^-_{2,s}\,\sigma^-_{3,s})}_{\hat{H}_B} \Bigg]$$

We are interested in time evolution (w/ Trotterization):

$$|\psi(t)\rangle = e^{-i\hat{H}t} |\psi(0)\rangle$$

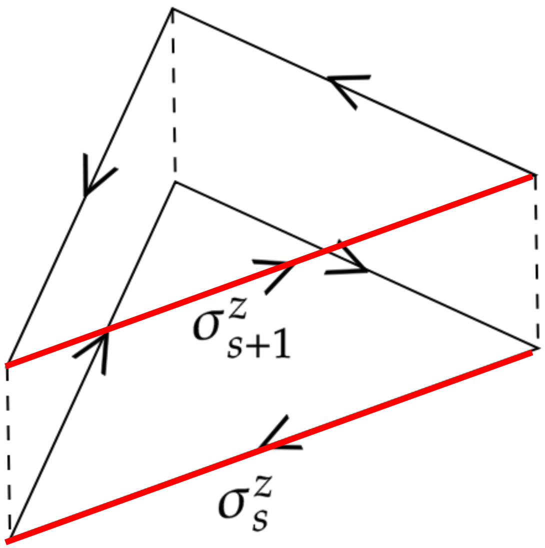
$$\approx \left(e^{-i\hat{H}_E \frac{t}{n}} e^{-i\hat{H}_{XY} \frac{t}{n}} e^{-i\hat{H}_B \frac{t}{n}} \right)^n |\psi(0)\rangle$$

with t/n small enough.

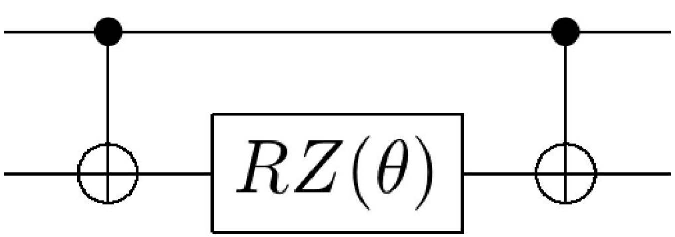
Electric term:

For a single triangle plaquette:

$$\hat{H} = \sum_{s} \Bigg[\frac{g^2}{2} \sum_{j=1}^{3} (\sigma^z_{j,s} + \sigma^z_{j,s+1})^2 + \frac{\alpha}{2g^2} \sum_{j=1}^{3} (\sigma^+_{j,s}\,\sigma^-_{j,s+1} + \sigma^-_{j,s}\,\sigma^+_{j,s+1}) - \frac{1}{2g^2} (\sigma^+_{1,s}\,\sigma^+_{2,s}\,\sigma^+_{3,s} + \sigma^-_{1,s}\,\sigma^-_{2,s}\,\sigma^-_{3,s}) \Bigg]$$



Circuit



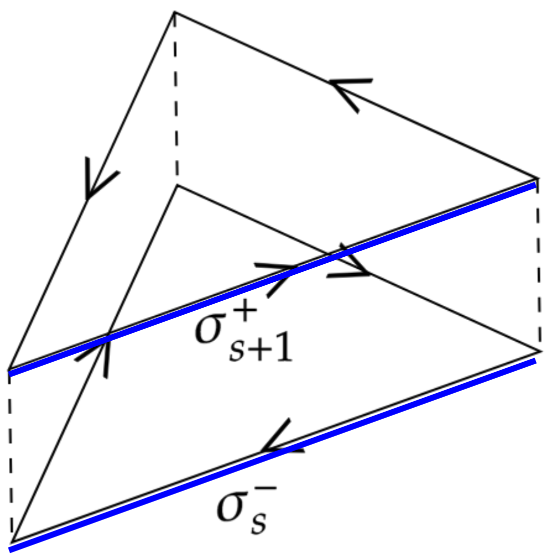
$\cong$

$$e^{-i\hat{H}_E\theta}$$

XY coupling term:

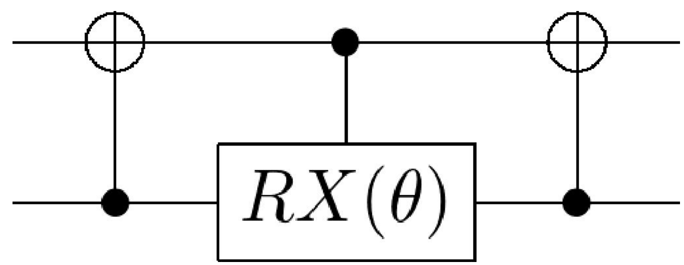
For a single triangle plaquette:

$$\hat{H} = \sum_s \Bigg[\frac{g^2}{2} \sum_{j=1}^{3} (\sigma^z_{j,s} + \sigma^z_{j,s+1})^2 + \boxed{\frac{\alpha}{2g^2} \sum_{j=1}^{3} (\sigma^+_{j,s}\, \sigma^-_{j,s+1} + \sigma^-_{j,s}\, \sigma^+_{j,s+1})}$$
$$- \frac{1}{2g^2} (\sigma^+_{1,s}\, \sigma^+_{2,s}\, \sigma^+_{3,s} + \sigma^-_{1,s}\, \sigma^-_{2,s}\, \sigma^-_{3,s}) \Bigg]$$



$+h.c.$

Circuit



$$\cong$$

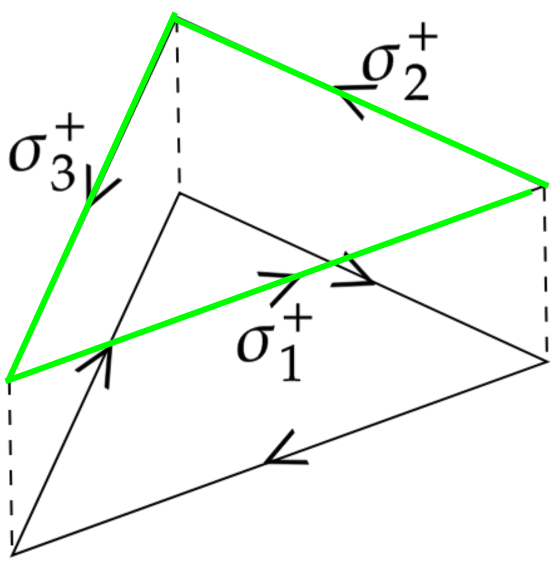
$$e^{-i\hat{H}_{XY}\theta}$$

Plaquette term:

For a single triangle plaquette:

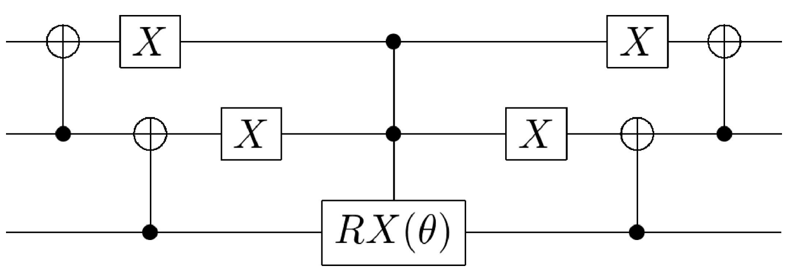
$$\hat{H} = \sum_s \Bigg[\quad \frac{g^2}{2}\sum_{j=1}^{3}(\sigma^z_{j,s} + \sigma^z_{j,s+1})^2 + \frac{\alpha}{2g^2}\sum_{j=1}^{3}(\sigma^+_{j,s}\,\sigma^-_{j,s+1} + \sigma^-_{j,s}\,\sigma^+_{j,s+1})$$

$$- \frac{1}{2g^2}(\sigma^+_{1,s}\,\sigma^+_{2,s}\,\sigma^+_{3,s} + \sigma^-_{1,s}\,\sigma^-_{2,s}\,\sigma^-_{3,s}) \Bigg]$$



$+h.c.$

Circuit



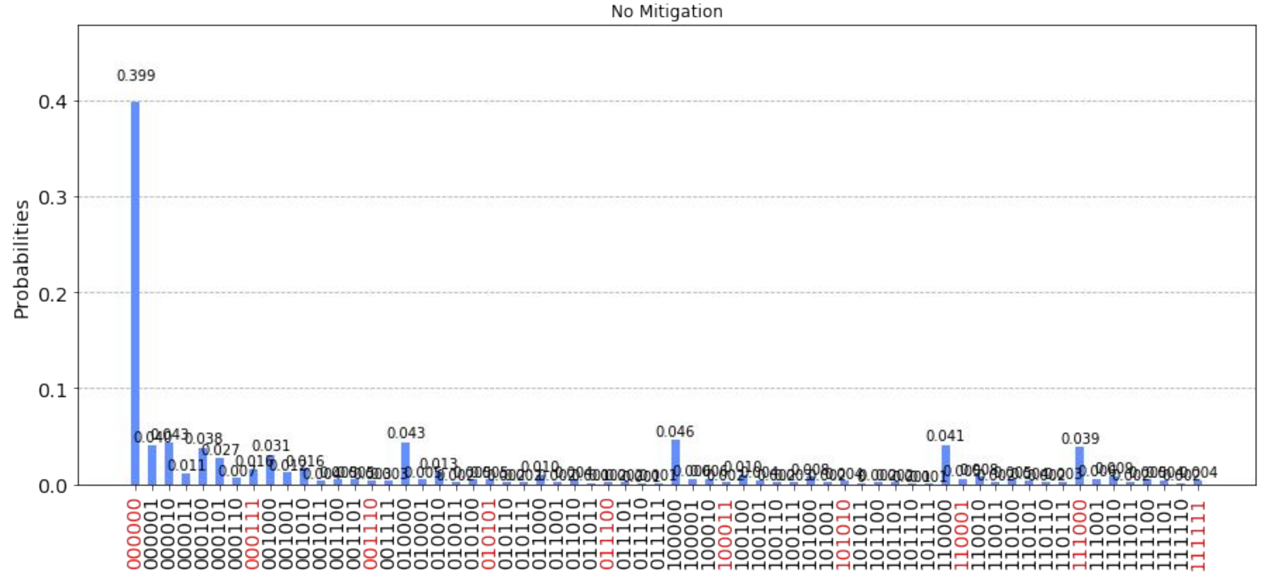
$$\cong$$

$$e^{-i\hat{H}_B\theta}$$

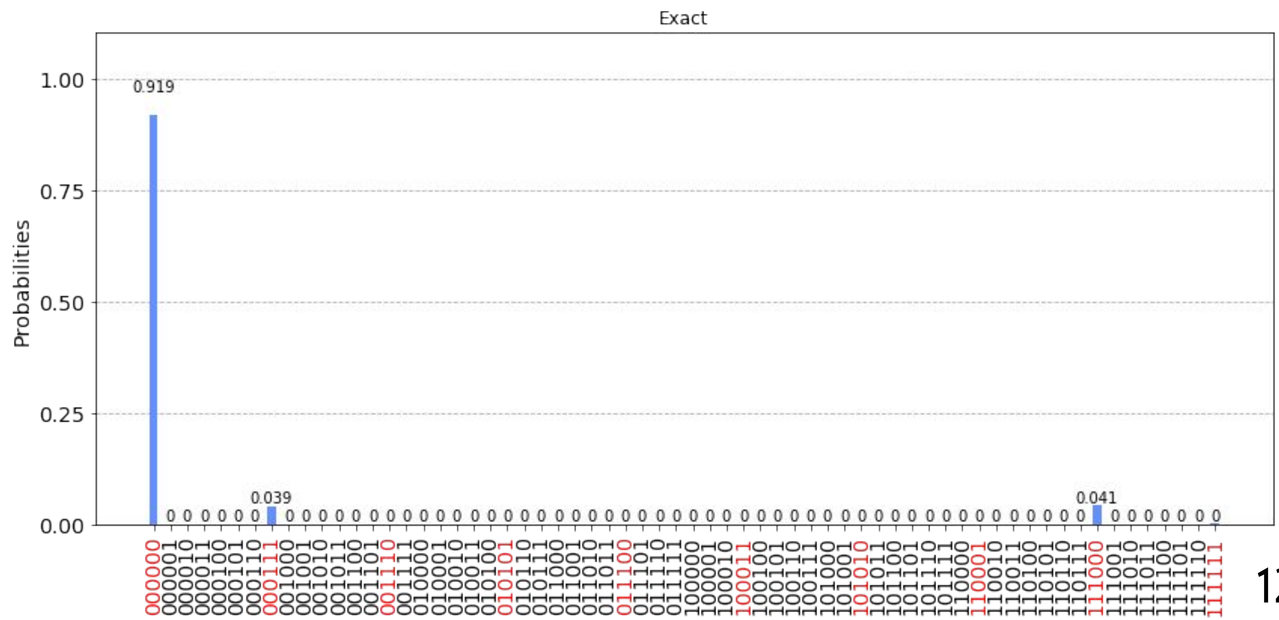
On IBM Q "Johannesburg"

PD of measuring the state after one trotter step to |0> with t=0.1

IBM Q



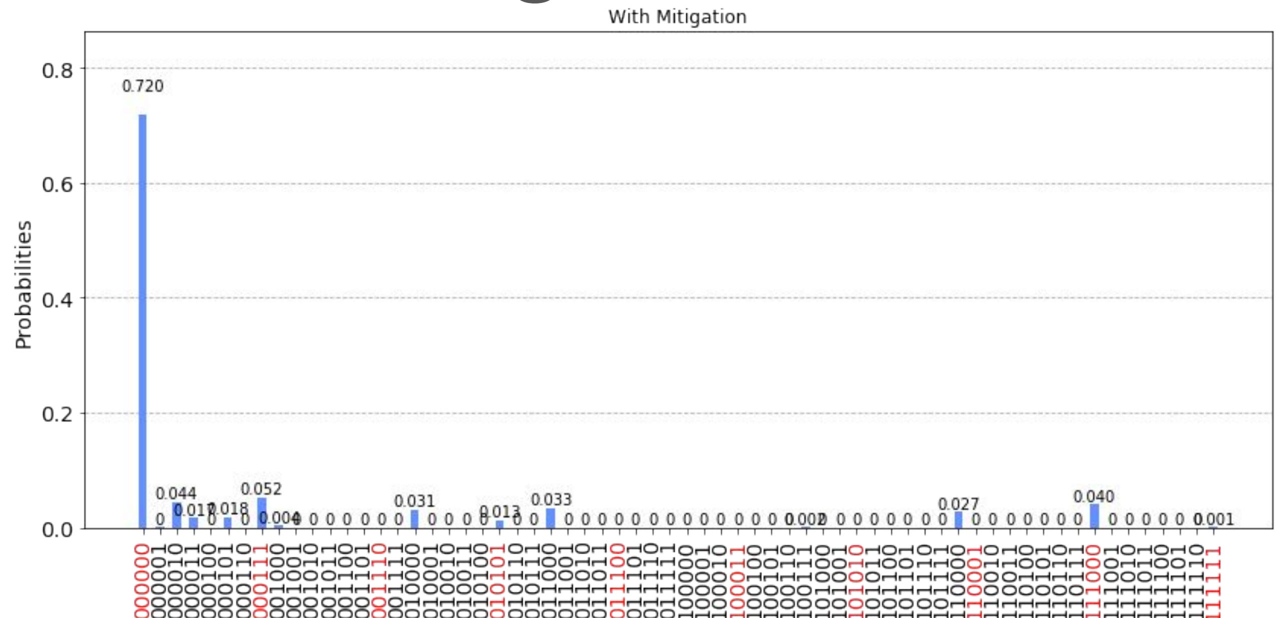
Ideal



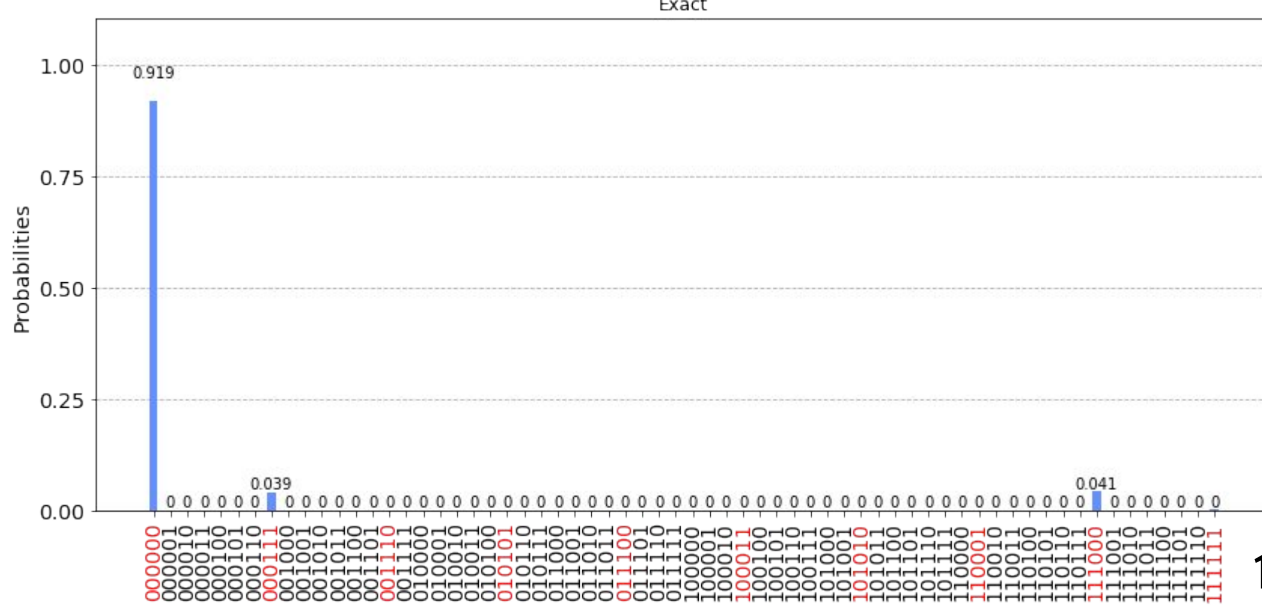
On IBM Q (w/ error mitigation)

PD of measuring the state after one trotter step to |0> with t=0.1

IBM Q



Ideal



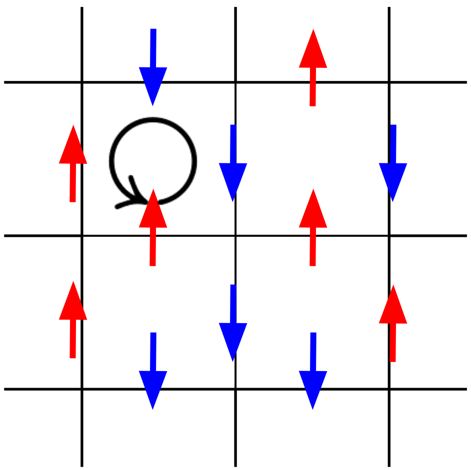
Need error correction!

- Mitigation is not enough; still leakage to invalid states (i.e. out of the Gauss's law sector)
- Need exact error correction/(at least) detection.
- Maybe we can get a clue from investigating the simplified LGT model: Z_2 gauge theory

Z2 gauge model: gauge transformation

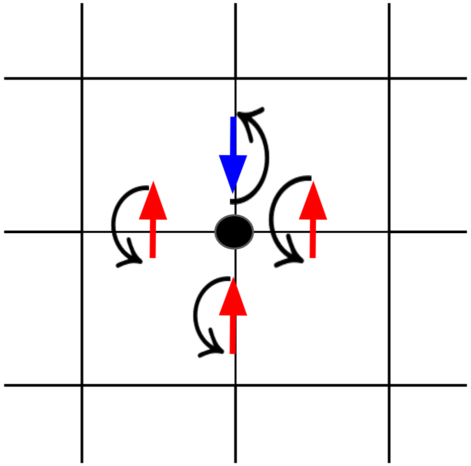
Z2 gauge model in 2D:

$$H_{\mathbb{Z}_2} = -J_e \sum_j \sigma_j^z - J_m \sum_{\Box} \prod_{j \in \Box} \sigma_j^x$$



Z2 local transformation flips the spins around the site:

$$G_{\mathbb{Z}_2,x} = \exp[i\frac{\pi}{2} \sum_{y \in \langle x,y \rangle} \sigma_{xy}^z] = \prod_{y \in \langle x,y \rangle} \sigma_{xy}^z$$



How about our U(1) case??

- It has the Z2 local symmetry as well (for the single layer):

$$[H, G_{\mathbb{Z}_2,x}] = 0 \quad \forall x$$

- The Gauss law (U(1)) sector ⊆ Z2 "+1" sector
- If we can detect the leakage from the Z2 sector (should be easier than U(1) cuz it can be done by measuring a single Pauli op), that means the violation of the U(1) sector as well.

→ Can we correct it??

How about our U(1) case??

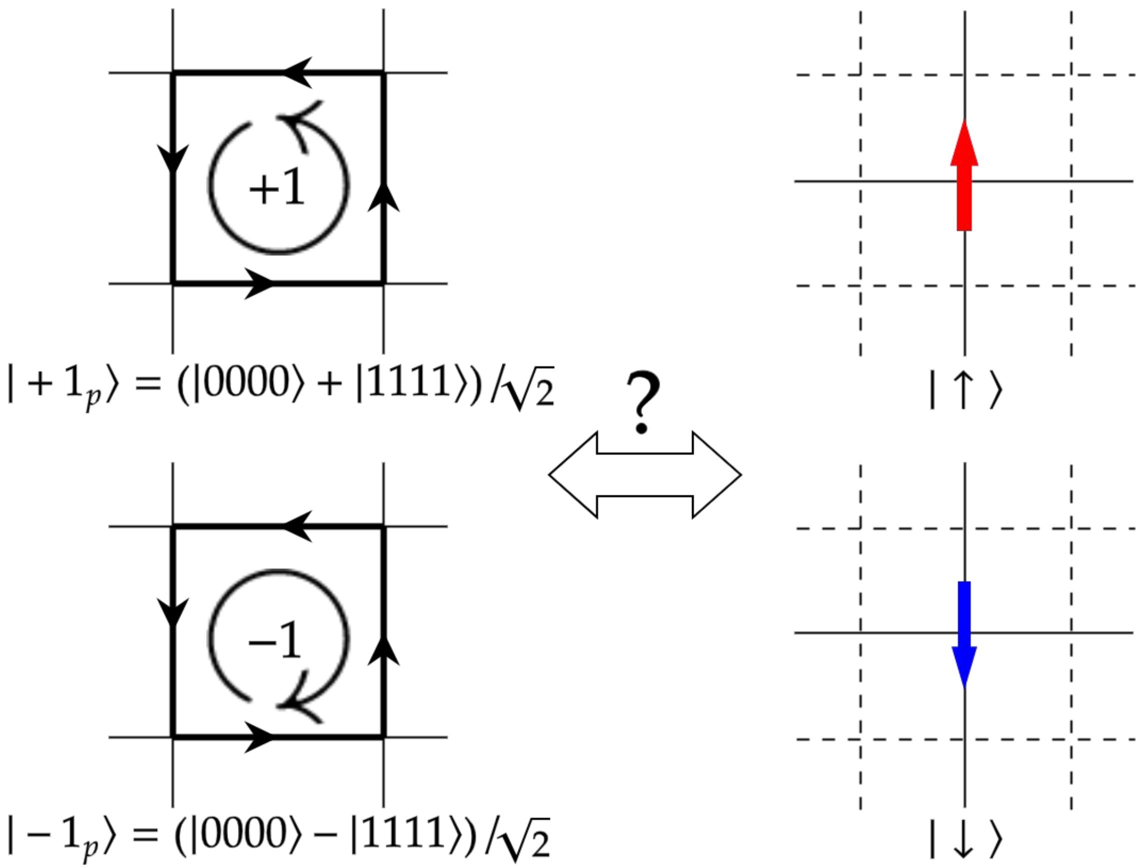
- Reduction to the Z2 model:
  - With a single layer, the E and XY terms become trivial, and the magnetic term can be decomposed as:

$$\sum_s \sigma^+_{1,s}\sigma^+_{2,s}\sigma^+_{3,s} + h.c. = \sum_s \sigma^x_{1,s}\sigma^x_{2,s}\sigma^x_{3,s} - \sum_s \sum_{i,j,k\in\mathrm{perm}(1,2,3)} \sigma^x_{i,s}\sigma^y_{j,s}\sigma^y_{k,s}$$

    - so we can turn off the additional term to see how the symmetry is broken U(1) → Z2.
  - If you add the gauge terms to the Z2 model (and omit the electric term), we can construct a model called *toric code*, which has topological order and an application to the quantum error correction.

How about our U(1) case??

- Can we construct a dual model as well as Z2 ↔ Ising?
  - E.g. single layer truncated case

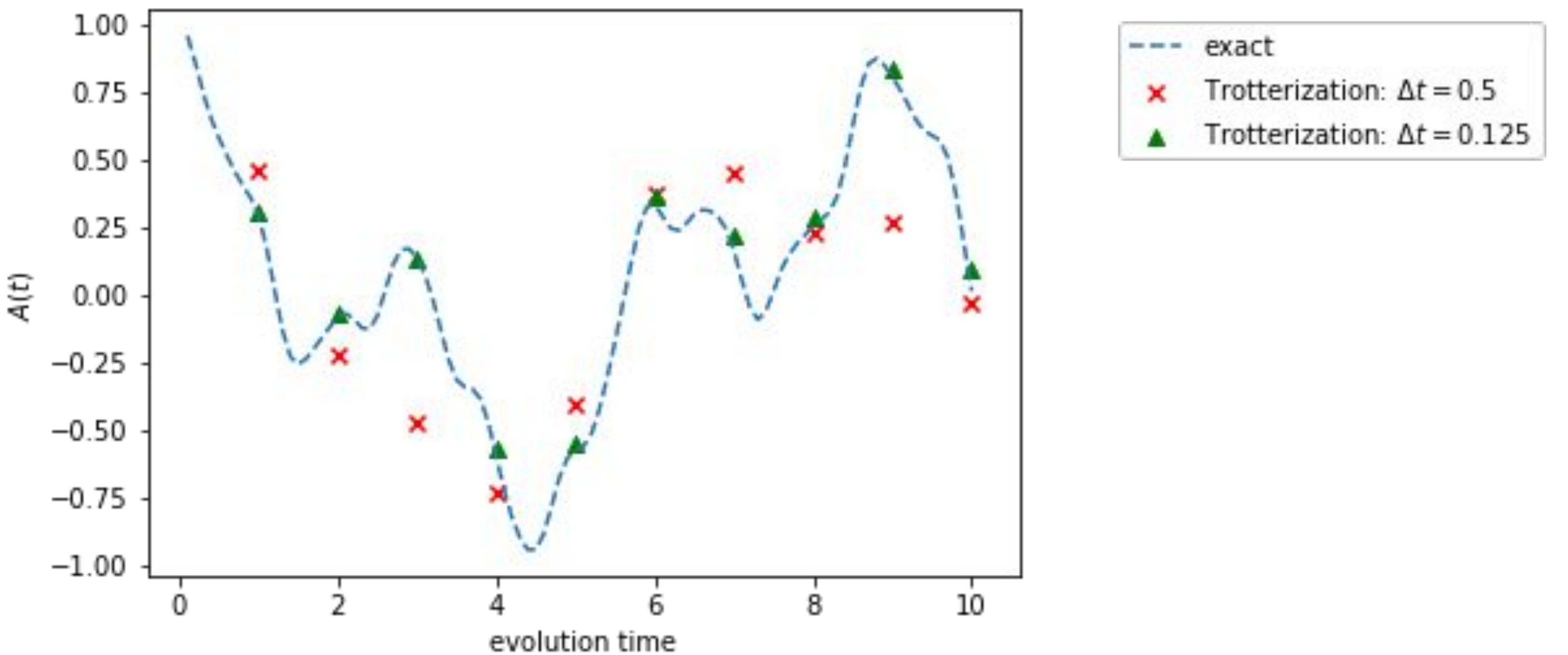


Summary and follow-up questions

- Constructed the 2+1D U(1) quantum link model embedded in SU(2) algebra.
    - and implemented its quantum circuit on IBM Q.
    - very noisy.
- Can we apply the idea of Z2 GT for error correction to U(1)?
- There at least exists a similarity between them.
    - can we find an interesting physics taking advantage of the similarity?

Trotterization

$$U(t) = \left(e^{-i\hat{H}_E \frac{t}{n}} e^{-i\hat{H}_{XY} \frac{t}{n}} e^{-i\hat{H}_B \frac{t}{n}} \right)^n$$



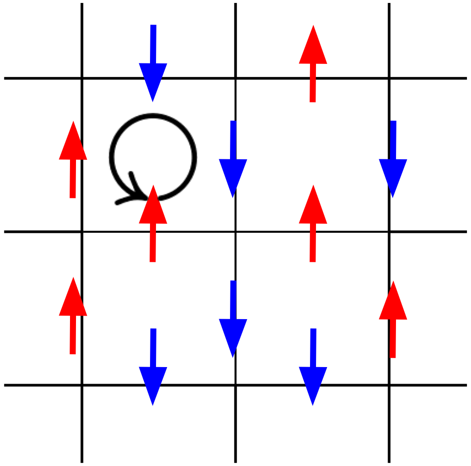
Computation of <0|U(t)|0> (exact).
Around t/n ≃ 0.1 for useful simulation.

Z2 gauge model: dual to Ising model

Z2 gauge model in 2D:

$$H_{\mathbb{Z}_2} = -J_e \sum_j \sigma_j^z - J_m \sum_{\square} \prod_{j \in \square} \sigma_j^x$$

spins on the links



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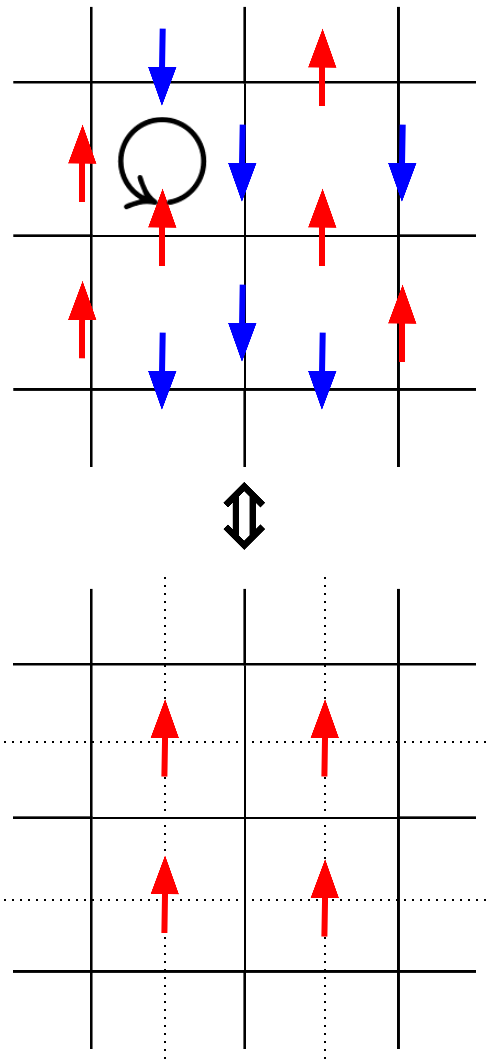

Ising model:

$$H_{TFIM} = -J \sum_{\langle i,j \rangle} \sigma_i^x \sigma_j^x - g \sum_j \sigma_j^z$$

Z_2 gauge model: toric code

One interesting exactly solvable model is called toric code:

$$H_{TC} = -J_g \sum_x G_x - J_m \sum_{\square} \prod_{j \in \square} \sigma_j^x$$

I.e. Z_2 model with J_e=0 and the additional gauge term

with the PBC

Why QEC --- embed logical states to physical GSs

→→ Local or "open" path errors brings the state out of the gauge sector → detectable!!

→→ Errors on a "closed" path is gauge redundant

→→ Degeneracy of GSs (4ˆg) gives enough DoF for qubit ops