Studies on meson-baryon interactions in the HAL QCD method with all-to-all propagators

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• 1st step: NK interactions

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Motivation

Backgrounds

• Various exotic hadrons have been found from experiments ($X, Y, Z, P_c$, etc.).

• Although there have been lots of theoretical and experimental approaches to explain such hadrons, they are still not understood well.

• On the other hand, QCD may describe all hadrons.

Our ultimate goal: reveal the properties of all hadrons including exotic hadrons from lattice QCD
Motivation

Recent studies on resonances in lattice QCD

meson-meson scatterings → mesonic resonances

Finite volume method
well investigated
- $\rho$ [M. Werner et al., 2019]
- $\sigma, f_0, f_2$ [R. Briceno et al., 2018]
- $\kappa, K^*$ [G. Rendon et al., 2020]

HAL QCD method
l=1 P-wave $\pi\pi \rightarrow \rho$
(cf. Y. Akahoshi’s talk)

meson-baryon scatterings → baryonic resonances

Finite volume method
l=3/2 P-wave $N\pi \rightarrow \Delta$
[S. Paul et al., 2018]
[C. W. Andersen et al., 2017]

HAL QCD method
none
But this method may be efficient for meson-baryon systems!
Motivation

All-to-all propagators in the HAL QCD method

- To investigate meson-baryon scatterings that have resonances, we need all-to-all propagators.

- One-end trick [M. Foster, C. Michael, 1999] : very efficient for the HAL QCD method with all-to-all propagators.

(cf. Y. Akahoshi’s talk)

As a first step …

- S-wave NK scatterings check the effectiveness of the one-end trick for meson-baryon systems

- l=3/2 P-wave Nπ scatterings extract Δ resonance
Motivation

Our plan

1. S-wave NK (LO)
   Examinations of the effectiveness of the one-end trick for meson-baryon systems

2. I=3/2 P-wave Nπ with \(\Delta\) source (LO)
   Simulation at a heavy pion mass to see \(\Delta\) as a bound state

3. I=3/2 P-wave Nπ with \(\Delta\) and Nπ sources (NLO)
   Simulation near the physical point using the one-end trick to see \(\Delta\) as a resonance

Other resonances or pentaquarks

\(\square\) : today’s talk

We are here
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HAL QCD method

Idea of HAL QCD method

[ N. Ishii, S. Aoki, T. Hatsuda, 2007]

NBS wave function

\[ \Psi^W(r) = \langle 0 \mid O_1(x + r, 0)O_2(x, 0) \mid 2H, W \rangle \]

\[ \Psi^{W,l}(r) \propto \frac{\sin(kr - \frac{l}{2} \pi + \delta^l(k))}{kr} e^{i\delta^l(k)} \]

\[ \left( \frac{k^2}{2\mu} - H_0 \right) \Psi^W(r) = \int d^3r' \frac{U(r, r')}{\Psi^W(r')} \]

We can obtain a potential for two-hadron states from NBS wave functions.
HAL QCD method

4-pt correlation function

\[ F(t, r) = \langle 0 \mid O_1(x + r, t + t_0)O_2(x, t + t_0) \bar{J}(t_0) \mid 0 \rangle \]

\[ 1 = \sum_n |2H, W_n\rangle\langle 2H, W_n| + \cdots \]

\[ = \sum_n \Psi^n_{W_n}(r) \langle 2H, W_n \mid \bar{J}(t_0) \mid 0 \rangle e^{-W_n t} + \cdots \]

\[ (W_n = \sqrt{k^2_n + m_1^2} + \sqrt{k^2_n + m_2^2}) \]

NBS wave function

It is hard to extract a ground state if the system contains baryons.
HAL QCD method

Time-dependent HAL QCD method

R-correlator

\[ R(t, \mathbf{r}) = \frac{F(t, \mathbf{r})}{e^{-m_1 t} e^{-m_2 t}} = \sum_n A_n \Psi_n^{W_n}(\mathbf{r}) e^{-\Delta W_n t} + \text{(inelastic)} \]

elastic term satisfies

\[ \sum_n \left( \frac{k_n^2}{2\mu} - H_0 \right) A_n \Psi_n^{W_n}(\mathbf{r}) e^{-\Delta W_n t} = \sum_n \int d^3 \mathbf{r}' \ U(\mathbf{r}, \mathbf{r}') \ A_n \Psi_n^{W_n}(\mathbf{r}) e^{-\Delta W_n t} \]

\[ = \frac{1 + 3\delta^2}{8\mu} \Delta W_n^2 + \Delta W_n + O(\Delta W_n^3) \]

\( \delta = \frac{m_1 - m_2}{m_1 + m_2} \)

\[ \left( \frac{1 + 3\delta^2}{8\mu} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right) R(\mathbf{r}, t) \sim \int d^3 \mathbf{r}' \ U(\mathbf{r}, \mathbf{r}') R(\mathbf{r}', t) + O(\Delta W_n^3) \]

Once the inelastic states are suppressed, we can derive the potential even when the excited states remain in \( R(t, \mathbf{r}) \).
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1st step: NK interactions

About NK systems

- $K^+p\ (I = 1),\ K^+n - K^0p\ (I = 0)$
- S-wave NK $\rightarrow\ J^P = 1/2^−$
- **no quark annihilation diagrams** in NK system
  - all-to-all propagators play a role in **increasing statistics**
- NK for $I(J^P) = 0(1/2^-), 1(1/2^-)$: candidates for the channels of $\Theta^+(1540)$ pentaquark [LEPS Collab., 2003]
1st step: NK interactions

Brief review of one-end trick

: stochastic method that can be applied to only meson operators

\[ 1 \approx \frac{1}{N} \sum_{r=1}^{N} \eta^{[r]}(t_0) \otimes \eta^{[r]}(t_0)^\dagger \]

\[ \approx \frac{1}{N} \sum_{r=1}^{N} (D^{-1} \eta^{[r]}(t_0))(x_1, t_1) \otimes (D^{-1} \gamma_5 \Gamma^\dagger \eta^{[r]}(t_0))^\dagger(x_2, t_2) \gamma_5 \]

\[ \rightarrow \text{Solving } D\psi^{[r](t_0)} = \eta^{[r](t_0)} \text{ and } D\xi^{[r](t_0)} = \gamma_5 \Gamma^\dagger \eta^{[r](t_0)}, \]

we can calculate 2 all-to-all propagators using 1 noise vector!
1st step: NK interactions

Strategy for calculating all-to-all propagators in NK

- One example of quark contractions for l=1

\[ p_{K^+} + p_p = 0 \]

\[ p_{K^-} = 0 \]

One-end trick \((x, y): \text{summed}\)

\(z_0: \text{fixed}\)

\[ p_{\bar{p}} = 0 \]

Automatically

All-mode averaging (AMA) using independence of \(z_0\)

[E. Shintani et al., 2015]
1st step: NK interactions

**Setup**

- PACS-CS, (2+1)-flavor configurations (gauge fixed, 400 conf.)
- \( a = 0.0907 \) fm on \( 32^3 \times 64 \) lattices at \( m_\pi \approx 570 \) MeV, \( m_K \approx 710 \) MeV and \( m_N \approx 1400 \) MeV
- smearing quarks at the source
- **leading order analysis** in the derivative expansion of the non-local potential

\[
V_{0}^{LO}(r) = \frac{1}{R(r, t)} \left( \frac{1 + 3\delta^2}{8\mu} \frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t} - H_0 \right) R(r, t) + O(\Delta W_n^3)
\]
1st step: NK interactions

Results: potentials

\[ I = 0 \]

\[ I = 1 \]

- independent of \( t \)
  - inelastic states are suppressed, LO analysis is good
- repulsive core for both \( I = 0 \) and \( I = 1 \)
- shallow attractive pocket at middle distances for \( I = 0 \)
- both potentials go to zero at long distances
1st step: NK interactions

Results: phase shifts

![Graph showing phase shifts with LQCD (S_01, t=12) and LQCD (S_11, t=12) compared to experimental data with red (I = 0) and blue (I = 1) lines.](image)

- consistent qualitatively with the experimental ones
  - one-end trick is good for meson-baryon systems
- no bound or resonant states
  - We could not find \(\Theta^+(1540)\) at \(m_\pi \simeq 570\) MeV.

Experiment: INS GW Data Analysis Center [SAID] ([http://gwdac.phys.gwu.edu/](http://gwdac.phys.gwu.edu/))
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2nd step: Nπ interactions with Δ sources

About Nπ systems

- P-wave Nπ \( \rightarrow J^P = 3/2^+ \)
- \( l=3/2, J^P = 3/2^+ \) Nπ \( \rightarrow \Delta(1232) \)

3-pt correlation function

\[
F_{\alpha jz}(\mathbf{r}, t) = \langle \pi^+(\mathbf{r} + \mathbf{x}, t)N_{\alpha}(\mathbf{x}, t)\tilde{\Delta}_{jz}^{++}(t_0)\rangle
\]

where

\[
\tilde{\Delta}_{+3/2}^{++}(t_0) = -\sum_y \epsilon_{abc}(\bar{u}_b(y, t_0)\Gamma_+\bar{u}_c^T(y, t_0))\bar{u}_{a,0}(y, t_0)
\]

\[
\tilde{\Delta}_{+1/2}^{++}(t_0) = -\frac{1}{\sqrt{3}} \sum_y \epsilon_{abc}[\sqrt{2}(\bar{u}_b(y, t_0)\Gamma_z\bar{u}_c^T(y, t_0))\bar{u}_{a,0}(y, t_0) + (\bar{u}_b(y, t_0)\Gamma_+\bar{u}_c^T(y, t_0))\bar{u}_{a,1}(y, t_0)]
\]

\[
\tilde{\Delta}_{-1/2}^{++}(t_0) = \frac{1}{\sqrt{3}} \sum_y \epsilon_{abc}[\sqrt{2}(\bar{u}_b(y, t_0)\Gamma_z\bar{u}_c^T(y, t_0))\bar{u}_{a,1}(y, t_0) + (\bar{u}_b(y, t_0)\Gamma_-\bar{u}_c^T(y, t_0))\bar{u}_{a,0}(y, t_0)]
\]

\[
\tilde{\Delta}_{-3/2}^{++}(t_0) = \sum_y \epsilon_{abc}(\bar{u}_b(y, t_0)\Gamma_-\bar{u}_c^T(y, t_0))\bar{u}_{a,1}(y, t_0)
\]

\[
\left(\Gamma_\pm = \frac{1}{2}C(\gamma_2 \pm i\gamma_1), \Gamma_z = -i\frac{C\gamma_3}{\sqrt{2}}\right)
\]
2nd step: $N\pi$ interactions with $\Delta$ sources

Quark contraction diagrams

AMA using independence of $x$

$\pi^+$

$\bar{u}$

$\bar{d}$

$u$

$x + r$

$p$

$\bar{\Delta}$

$z$

$t + t_0$

$t_0$

$\leftarrow$ : (conventional) stochastic estimation

$z$ : summed, $x$ : fixed, $r$ : spatial coord. of NBS w.f.
2nd step: $N\pi$ interactions with $\Delta$ sources

Quark-antiquark pair and sink smearing

- We usually use point quark sinks, but …

It is impossible to fit this potential!

What does this behavior come from?
2nd step: $N\pi$ interactions with $\Delta$ sources

Quark-antiquark pair and sink smearing

- quark-antiquark pair has a singular behavior in short distances according to OPE

$$F(r, t) \sim \langle q(r)\bar{q}(0) \rangle \propto \frac{1}{r^3} \quad \Rightarrow \quad V(r) \propto \frac{1}{r^2}$$

- we consider P-wave

$$F(r, t) \propto \frac{1}{r^3} Y_{1,m}(\Omega)$$

sharp structure produces the spreading behavior!

(The same thing happens in $l=1$ P-wave $\pi\pi$ system)

One of the solutions to this problem: sink smearing
2nd step: $N\pi$ interactions with $\Delta$ sources

**Setup**

- CP-PACS+JLQCD, (2+1)-flavor configurations (gauge fixed, **50 conf.**)
- $a = 0.12$ fm on $16^3 \times 32$ lattices at $m_{\pi} \simeq 870$ MeV
- smearing quarks at the source and the sink
- leading order analysis in the derivative expansion
- $m_N \simeq 1820$ MeV, $m_\Delta \simeq 2030$ MeV from 2pt functions
  - $E_b \simeq 660$ MeV (?)
2nd step: $N\pi$ interactions with $\Delta$ sources

Results: potentials

the spreading behavior disappears!

binding energy: $E_b = 729 \pm 11$ MeV

larger than 660 MeV due to the smallness of the volume
Conclusions

• We investigate meson-baryon interactions in the HAL QCD method with all-to-all propagators.

• We study S-wave NK interactions and see the effectiveness of the one-end trick for meson-baryon systems.

• We are analyzing $l=3/2$ P-wave $N\pi$ interactions with $\Delta$ sources at a heavy pion mass to see $\Delta$ as a bound state.
Future works

- $N\pi$ with $\Delta$ sources on a larger volume at a lighter pion mass
  - $\Delta$ as a bound state
- 3rd step: $N\pi$ with $\Delta$ and $N\pi$ sources (NLO) near the physical point using the one-end trick
  - $\Delta$ as a resonance
- 4th step: Other systems
  - other resonances, pentaquarks
Back up
Two main methods to analyze hadron scatterings

**Finite volume method**
[M. Lüscher, 1991]
- extract phase shifts using boundary condition in the finite volume
- good at meson-meson systems
- difficult for systems including baryons
  - hard to extract the energy in such systems

**HAL QCD method**
[N. Ishii, S. Aoki, T. Hatsuda, 2007]
- derive interaction potentials from the NBS wave functions $\Psi^W(r)$
- very efficient for systems including baryons
  - no need to extract ground states
- very large computational cost
  - need for the dependences of the relative position between 2 hadrons
Detailed numerical setups

S-wave NK

• take summation of 4 timeslices $t_0$ to increase statistics

• each component of 4-pt corr. is projected onto $A_1^+$ irreps

• smear quark sources using smearing function $f_{A,B}(x)$ at $(A, B) = (1.2, 0.19)$ for u quarks and $(A, B) = (1.2, 0.25)$ for s quarks

\[
f_{A,B}(x) = \begin{cases} 
Ae^{-B|x|} & (|x| < \frac{L-1}{2}) \\
1 & (|x| = 0) \\
0 & (|x| \geq \frac{L-1}{2})
\end{cases}
\]

P-wave $N\pi$

• take summation of 32 timeslices $t_0$ to increase statistics

• 4-pt corr. is projected onto $H_g$ irreps

• use smearing function $f_{A,B}(x)$ at $(A, B) = (1.0, 0.38)$ for source quarks and $(A, B) = (1.0, 1/0.7)$ for sink quarks
Detailed numerical setups

Others

• diluted indices in one-end trick: time, color, spinor, s2 (spatial)
• AMA: 8 spatial points (0,0,0),(0,0,L/2) ··· (L/2,L/2,L/2), $\epsilon = 10^{-4}$

About configurations

• CP-PACS/JLQCD, (2+1)-flavor confs.: [CP-PACS/JLQCD Collab., 2006]
  renormalization-group improved Iwasaki gauge action
  + nonperturbatively O(a) improved Wilson-clover quark action
• PACS-CS, (2+1)-flavor confs.: [PACS-CS Collab., 2009]
  renormalization-group improved Iwasaki gauge action
  + nonperturbatively O(a) improved Wilson-clover quark action

\[ \eta^{(s_{\text{dil}})}(\mathbf{x}) = \begin{cases} 
\eta(\mathbf{x}) & (x + y + z \equiv s_{\text{dil}} \pmod{2}) \\
0 & (x + y + z \equiv s_{\text{dil}} + 1 \pmod{2}),
\end{cases} \quad s_{\text{dil}} = 0,1, \]
All quark contraction diagrams in NK

$l=1$

$l=0$

$l=1$ diagrams with different coefficients
NK potentials and their breakups

- $t=12$

$I = 0$

$I = 1$

$V_{\text{eff}}(r)$

$-H_0$

$-\partial/\partial t$

$(1 + 3\delta^2)/8\mu(\partial^2/\partial t^2)$
Fitting results for NK interactions

- fitting function: 4 Gaussians

\[ \chi^2 / \text{dof} = 0.27(0.09) \]

\[ \chi^2 / \text{dof} = 0.65(0.19) \]
Sink smearing and the singular behavior

3pt function at \( z=0, t=6 \) and \( \alpha = 0 \) (real part)
(normalized s.t. maximum value=1.0)

w/ point quark sinks

w/ smeared quark sinks

- the sharp structure is smeared by the smeard quark sinks
Phase shift results for $N\pi$ interactions with $\Delta$ sources

- sharp rise followed by gradual fall
- $\delta(E_{th} \to \infty) \to -180^\circ \rightarrow 1$ bound state
stochastic estimation

\[ \eta(x)^\alpha_a \quad \cdots \text{noise vector that satisfies} \]
\[ \left\langle \left\langle \eta(x)^\alpha_a \eta^*(y)^\beta_b \right\rangle \right\rangle = \delta_{xy} \delta_{ab} \delta_{\alpha\beta} \]
\[ \eta(x)^\alpha_a \eta^*(x)^\alpha_a = 1 \quad \text{(for all } x, a, \alpha) \]

Propagator \( D^{-1} \) can be written as

\[ q(x)^\alpha_a \quad \leftrightarrow \quad \bar{q}(y)^\beta_b = D^{-1}(x, y)^{ab} = \sum_{c, \gamma, z} D^{-1}(x, z)^{ac} \delta_{\gamma} \delta_{cb} \delta_{\gamma\beta} \]
\[ = \sum_{c, \gamma, z} D^{-1}(x, z)^{ac} \langle \langle \eta(z)^\gamma_\gamma \eta^*(y)^\beta_b \rangle \rangle \]
\[ = \langle \langle (D^{-1}\eta(x)^\alpha_a \eta^*(y)^\beta_b) \rangle \rangle = \langle \langle (\psi(x)^\alpha_a \eta^*(y)^\beta_b) \rangle \rangle \]
\[ \equiv \psi \]
stochastic estimation

\[ D^{-1}(x, y)^{ab\alpha\beta} = \lim_{N_r \to \infty} \frac{1}{N_r} \sum_{r=1}^{N_r} \psi_{[r]}(x)^{\alpha\eta^*[r]}(y)^{\beta} \]

\[ (\psi\text{-solution}) \sum_{b, \beta, y} D(x, y)^{ab\alpha\beta} \psi(y)^{\beta}_b = \eta(x)^a \]

Therefore, \( D^{-1} \) can be estimated by

\[ D^{-1}(x, y)^{ab\alpha\beta} \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \psi_{[r]}(x)^{\alpha\eta^*[r]}(y)^{\beta}_b \]

noisy estimation: very noisy \( \eta(x)^a \) itself has \( O(1) \) error

this noise can be reduced by using “dilution”
stochastic estimation (+ dilution)

ex) time dilution

decompose the noise vector

\[ \eta(x) = \sum_{j=0}^{N_t-1} \eta^{(j)}(x) \]

where \( \eta^{(j)}(x) = \begin{cases} \eta(x) & (\text{for } j = t) \\ 0 & (\text{for } j \neq t) \end{cases} \)

\[
\begin{bmatrix}
\eta(t = 0) \\
\eta(t = 1) \\
\eta(t = 2) \\
\vdots
\end{bmatrix}
= \begin{bmatrix}
\eta^{(0)}(t) \\
\eta^{(1)}(t) \\
\eta^{(2)}(t) \\
\vdots
\end{bmatrix}
+ \begin{bmatrix}
0 \\
\eta(t = 1) \\
0 \\
\vdots
\end{bmatrix}
+ \begin{bmatrix}
0 \\
0 \\
\eta(t = 2) \\
\vdots
\end{bmatrix}
+ \ldots
\]
stochastic estimation (+ dilution)

ex) time dilution

\[ D^{-1}(x, y)_{ab}^{\alpha\beta} = \sum_{c,\gamma,z} D^{-1}(x, z)_{ac}^{\alpha\gamma} \langle \langle \eta(z)_{\alpha\gamma}^{c} \eta^{*}(y)_{\beta}^{b} \rangle \rangle \]

\[ = \sum_{c,\gamma,z} D^{-1}(x, z)_{ac}^{\alpha\gamma} \sum_{j,k=0}^{N_{t}-1} \langle \langle \eta^{(j)}(z)_{\alpha\gamma}^{c} \eta^{(k)*}(y)_{\beta}^{b} \rangle \rangle \]

\[ j \neq k \text{ terms are noisy parts, not signals} \]

\[ \rightarrow \sum_{c,\gamma,z} D^{-1}(x, z)_{ac}^{\alpha\gamma} \sum_{j=0}^{N_{t}-1} \langle \langle \eta^{(j)}(z)_{\alpha\gamma}^{c} \eta^{(j)*}(y)_{\beta}^{b} \rangle \rangle \]
stochastic estimation (+ dilution)

\[ D^{-1}(x, y)^{ab}_{\alpha\beta} = \sum_{j=0}^{N_t-1} \langle \langle \psi(j)^{(x)}(x)_{a}^{\alpha} \eta(j)^{*}_{b}^{(y)} \rangle \rangle \]

\[ \sum_{b, \beta, y} D(x, y)^{ab}_{\alpha\beta} \psi(i)(y)_{b}^{\beta} = \eta(i)(x)_{a}^{\alpha} \]

Therefore,

\[ D^{-1}(x, y)^{ab}_{\alpha\beta} \approx \frac{1}{N_{r}} \sum_{r=1}^{N_{r}} \sum_{j} \psi^{(j)}_{[r]}(x)_{a}^{\alpha} \eta^{(j)^{*}}_{[r]}(y)_{b}^{\beta} \]
\[
\bar{q}_\beta(x_1, t_1) q^\alpha(x_2, t_2) \\
= \sum_y D^{-1}(x_1, t_1; y, t_0) \Gamma D^{-1}(y, t_0; x_2, t_2)
\]

\[
= \sum_y \sum_{z, t_z, t_y} D^{-1}(x_1, t_1; z, t_z) (\delta_{t_z, t_0} \delta_{t_y, t_0} \delta_{z, y}) \Gamma D^{-1}(y, t_y; x_2, t_2)
\]

\[
= \left\langle \langle \eta(t_0) \otimes \eta(t_0)^\dagger \rangle \right\rangle \approx \frac{1}{N} \sum_{r=1}^{N} \eta^{[r](t_0)} \otimes \eta^{[r](t_0)^\dagger}
\]

\[
(\eta^{[r](t_0)}(x, t_x) = \delta_{t_x, t_0} \Xi^{[r]}_{a,\alpha}(x))
\]

\[
(\langle \Xi^{[r]}_{a,\alpha}(x) \Xi^{[r]^\dagger}_{b,\beta}(y) \rangle = \delta_{a, b} \delta_{\alpha, \beta} \delta_{x, y})
\]

\[
\approx \sum_y \sum_{z, t_z, t_y} D^{-1}(x_1, t_1; z, t_z) \left( \frac{1}{N} \sum_{r=1}^{N} \eta^{[r](t_0)}(z, t_z) \otimes \eta^{[r](t_0)^\dagger}(y, t_y) \right) \Gamma D^{-1}(y, t_y; x_2, t_2)
\]
one-end trick

[M. Foster, C. Michael, 1999]

Using $\gamma_5$ hermiticity,

$$
\frac{1}{N} \sum_{r=1}^{N} \left( \sum_{z, t_z} D^{-1}(x_1, t_1; z, t_z) \eta^{[r](t_0)}(z, t_z) \right) \otimes \left( \sum_{y, t_y} \eta^{[r](t_0)\dagger}(y, t_y) \Gamma \gamma_5 D^{-1\dagger}(y, t_y; x_2, t_2) \gamma_5 \right)
$$

$$
= \frac{1}{N} \sum_{r=1}^{N} \left( D^{-1} \eta^{[r](t_0)}(x_1, t_1) \right) \otimes \left( (D^{-1} \gamma_5 \Gamma^\dagger \eta^{[r](t_0)})^\dagger(x_2, t_2) \gamma_5 \right)
$$

Therefore, solving 2N linear equations

$$
\begin{cases}
D \psi^{[r](t_0)} = \eta^{[r](t_0)} \\
D \xi^{[r](t_0)} = \gamma_5 \Gamma^\dagger \eta^{[r](t_0)}
\end{cases}
$$

$$
\sum_{y} G(x_1, t_1; y, t_0) \Gamma G(y, t_0; x_2, t_2) \approx \frac{1}{N} \sum_{r=1}^{N} \psi^{[r](t_0)}(x_1, t_1) \otimes (\xi^{[r](t_0)\dagger}(x_2, t_2) \gamma_5)
$$
All-mode averaging (AMA)

general idea: Covariant approximation
averaging (CAA)

$O[U] \cdots$ observable that is covariant under symmetry $G$

$\Leftrightarrow O[U^g] = O^g[U] \ \text{for all} \ g \in G$

( ex) $G \cdots$ translation $x \rightarrow x + a$

We define

$$O_G[U] = \frac{1}{N_G} \sum_{g \in G} O[U^g] = \frac{1}{N_G} \sum_{g \in G} O^g[U]$$

($N_G \cdots$ number of the element of $G$)

This variable satisfies

$$\langle O[U] \rangle = \langle O_G[U] \rangle \quad (\because \langle O[U^g] \rangle = \langle O[U] \rangle)$$
All-mode averaging (AMA)

general idea: Covariant approximation averaging (CAA)

\[ O^{(appx)}[U] \cdots \text{approximation of } G \text{ which reduces computational cost} \]

and we introduce

\[ O^{(appx)}_G[U] = \frac{1}{N_G} \sum_{g \in G} O^{(appx)}[U^g] = \frac{1}{N_G} \sum_{g \in G} O^{(appx)}_{g}[U] \]
All-mode averaging (AMA)

general idea: Covariant approximation averaging (CAA)

Improved estimator is defined by


and this satisfies

\[ \langle O^{(imp)}[U] \rangle = \langle O[U] \rangle - \langle O^{(appx)}[U] \rangle + \langle O^{(appx)}_G[U] \rangle = \langle O^{(appx)}[U] \rangle \]
All-mode averaging (AMA)

All-mode averaging

\[
O^{(AMA)} = O[S^{(all)}[U]]
\]

\[
O^{(AMA)}_G = \frac{1}{N_G} \sum_{g \in G} O[S^{(all)g}[U]]
\]

where

\[
(S^{(all)b})_i = \sum_{k=1}^{N_\lambda} \frac{1}{\lambda_k} (\psi_k^\dagger b)(\psi_k)_i + (f_\epsilon(H)b)_i
\]

\[
f_\epsilon(H)b = \sum_{i=1}^{N_{CG}} (H^i)c_i
\]

spectral decomposition for low mode

relaxed stopping criterion in the CG method
All-mode averaging (AMA)

strategy for all-mode averaging w/o low modes

\[ C(U; z_0) \]: correlation function at gauge conf. \( U \) with the hadron source operator at \( z_0 \)

\[ C^{(appx)}(U; z_i) \]: approximated correlation function at gauge conf. \( U \) with the hadron source operator at \( z_i \)

by relaxing stopping condition

\[ \| D\psi - s \| / \| s \| < \epsilon \] in BiCG solver
All-mode averaging (AMA)

strategy for all-mode averaging w/o low modes

1. For each gauge conf., we calculate $C(U; z_0)$ and $C^{(appx)}(U; z_0)$ for some $z_0$. 
All-mode averaging (AMA)

strategy for all-mode averaging w/o low modes

2. Translate $z_0$ and calculate $C^{(appx)}(U; z_i)$ at each source point.
All-mode averaging (AMA)

strategy for all-mode averaging w/o low modes

3. The improve estimator is constructed from $C(U; z_0)$ and $\{C^{(appx)}(U; z_i)\}_{i=0,1\ldots N_s}$

$$C^{(imp)}(U) = C(U; z_0) - C^{(appx)}(U; z_0) + \frac{1}{N_s} \sum_{i=1}^{N_s} C^{(appx)}(U; z_i)$$

this satisfies

$$\langle C^{(imp)}(U) \rangle = \langle C(U; z_0) \rangle - \langle C^{(appx)}(U; z_0) \rangle + \frac{1}{N_s} \sum_{i=1}^{N_s} \langle C^{(appx)}(U; z_i) \rangle$$

$$= \langle C(U; z_0) \rangle$$

$$= \langle C^{(appx)}(U; z_0) \rangle$$
Quark contractions in l=3/2 Nπ with Nπ sources

point-to-all + stochastic+ one-end trick