Exploring the 't Hooft limit of meson observables

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QCD and the large $N_c$ limit
The Large $N_c$ (’t Hooft) limit of QCD

- Consider a $SU(N_c)$ gauge theory with $N_f$ fundamental fermions, and let:
  - $N_c \to \infty$
  - $\lambda = g^2 N_c \sim \alpha_s N_c = \text{constant}$ [’t Hooft coupling]
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  - Confinement & Spontaneous chiral symmetry breaking.
  - Low energy mesons spectrum (pseudo-Goldstone bosons)
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- Some definite nonperturbative predictions at Large $N_c$!
The elusive “$\Delta I = 1/2$ rule”

✓ Most of the Large $N_c$ predictions work reasonably well (e.g. decay constant, LECs, scattering amplitudes... )

✗ It fails for the ratio of isospin amplitudes in $K \rightarrow \pi\pi$:

$$\frac{\text{Re } A_0}{\text{Re } A_2} \bigg|_{\text{Large } N_c} = \sqrt{2} + O\left(\frac{1}{N_c}\right) \ll 22 \sim \frac{\text{Re } A_0}{\text{Re } A_2} \bigg|_{\text{experiment}}$$

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What is the origin of the anomalously large $1/N_c$ effects?

- Heavy charm: penguin diagram dominance?
- Final state interactions?
- Intrinsic low-energy QCD effects?

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Our project: QCD at Large $N_c$ from lattice simulations

- Configurations generated with HiRep [Del Debbio, et al.]
- Iwasaki gauge action with $N_f = 4$ clover-improved Wilson fermions
  - When useful, mixed-action setup with twisted mass.

Summary of configurations

- $a = 0.075$ fm $\rightarrow \left[ 4 \text{ values of } M_\pi \right] \times \left[ N_c = 3 - 6 \right] = 16$ ensembles
- $a = 0.065$ fm $\rightarrow \left[ 2 \text{ values of } M_\pi \right] \times \left[ N_c = 3 \right] = 2$ ensembles
Dissecting the $\Delta I = 1/2$ rule at large $N_c$

What is the origin of large $1/N_c$ effects in $K \to \pi\pi$?

\[
\begin{aligned}
\frac{\text{Re } A_0}{\text{Re } A_2} \bigg|_{\text{experiment}} & \approx 22 \gg \sqrt{2} \\
\text{Large } N_c
\end{aligned}
\]
Revisiting an old strategy at Large $N_c$ [Giusti, Hernández, Laine, Weisz, Wittig 2004]

**Goal:** Testing the “intrinsic low-energy QCD” hypothesis in a simplified setup
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**Step 1:** Integrating out the $W$: EFT with 4-fermion operators

$$\mathcal{H}^{SM}(W, q\ldots) \Rightarrow \mathcal{H}^{\Delta S=1}_{W}(u, d, c, s) = \sqrt{2} G_F V_{ud} V_{us}^* (k^+ Q^+ + k^- Q^-)$$

$$Q^\pm = (\bar{s} \gamma^\mu_L u)(\bar{u} \gamma^\nu_L d) \pm (\bar{s} \gamma^\mu_L d)(\bar{u} \gamma^\nu_L u) - [u \leftrightarrow c]$$

♦ Only two operators, no penguins, only CP-conserving.
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- Only two operators, no penguins, only CP-conserving.
- **GIM limit:** $m_u = m_c$

1. Disentangle heavy charm effects.
2. No closed quark loops.
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$$H^{\Delta S=1}_W(u, d, c, s) \Rightarrow H^{\Delta S=1}_{\text{ChPT}} \propto g^+ O^+ + g^- O^-$$

$$O^\pm = \frac{F^4}{4} (U \partial_\mu U)_{us}(U \partial_\mu U)_{du^\pm}(U \partial_\mu U)_{ds}(U \partial_\mu U)_{uu} - \text{[} u \leftrightarrow c \text{]}$$

$[U \equiv \text{Matrix of } \pi, K, D, \eta \text{ fields}]$
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- Compute $g^\pm$ in lattice QCD from $K \rightarrow \pi$ amplitudes

$$A^\pm = \langle K | \mathcal{O}^\pm | \pi \rangle \sim \frac{C_3^\pm(x, y, z)}{C_2(x, z)C_2(y, z)}, \quad \text{with } g^\pm = \lim_{M_\pi \to 0} A^\pm$$
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✧ Use $g^\pm$ to describe $\Delta S = 1$ processes such as $K \to \pi \pi$.

$$\frac{A_0}{A_2} = \frac{1}{2\sqrt{2}} \left( 1 + 3 g^- \right) \xrightarrow{\text{Large } N_c} \sqrt{2}$$
Large $N_c$ scaling of the $K \rightarrow \pi$ amplitudes

\[
C_3^{\pm} = \begin{cases} 
\text{Color-disconnected } O(N_c^2) & \text{if positive} \\
\text{Color-connected } O(N_c) & \text{if negative}
\end{cases}
\]
Large $N_c$ scaling of the $K \to \pi$ amplitudes

$$\mathcal{C}_3^{\pm} = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Color-disconnected } O(N_c^2)
\end{array}
\end{array}
\end{array} \mp \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\text{Color-connected } O(N_c)
\end{array}
\end{array}
\end{array}$$

$$\langle K | \mathcal{O}^{\pm} | \pi \rangle = A^{\pm} = 1 + a \frac{1}{N_c^2} + b \frac{N_f}{N_c^3} \pm c \frac{1}{N_c} \pm d \frac{N_f}{N_c^2}$$
Large $N_c$ scaling of the $K \to \pi$ amplitudes

$$\mathcal{C}_3^\pm =$$

\begin{align*}
\text{Color-disconnected } & O(N_c^2) \\
\text{Color-connected } & O(N_c)
\end{align*}

(a) $O(N_c)$ (c) $O(N_c)$

(b) $O\left(\frac{N_f}{N_c}\right)$ (d) $O(N_f)$

$$\langle K|\mathcal{O}^\pm|\pi\rangle = A^\pm = 1 + a \frac{1}{N_c^2} + b \frac{N_f}{N_c^3} \pm c \frac{1}{N_c} \pm d \frac{N_f}{N_c^2}$$

source of enhancement in $A^-/A^+$?
\[ \langle K|O^\pm|\pi \rangle = A^\pm = 1 + a \frac{1}{N_c^2} + b \frac{N_f}{N_c^3} \pm c \frac{1}{N_c} \pm d \frac{N_f}{N_c^2} \]

\[ \frac{A^+ - A^-}{2} = 1 + a \frac{1}{N_c^2} + b \frac{N_f}{N_c^3} \]

\[ \frac{A^+ - A^-}{2} = c \frac{1}{N_c} + d \frac{N_f}{N_c^2} \]

\[ A_f = 0, \ M_\pi \sim 570 \text{ MeV} \quad a = 2.1(1) \]

\[ A_f = 4, \ M_\pi \sim 560 \text{ MeV} \quad a = 1.2(3) \quad b = 2.2(3) \quad c = -1.55(2) \quad d = -1.44(13) \]

\[ A_f = 4, \ M_\pi \sim 360 \text{ MeV} \quad a = 2.4(4) \quad b = 1.6(4) \quad c = -1.49(15) \quad d = -1.32(18) \]
\[ \langle K|O^{\pm}|\pi \rangle = A^{\pm} = 1 + a \frac{1}{N_c^2} + b \frac{N_f}{N_c^3} \pm c \frac{1}{N_c} \pm d \frac{N_f}{N_c^2} \]

- \( N_f = 0, \) 570 MeV \( a = 2.1(1) \)
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▶ Coefficients \( a - d \) are \( O(1) \): natural size
$K \rightarrow \pi$ amplitudes at Large $N_C$

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- $N_f = 0$, $M_\pi \sim 570$ MeV
- $N_f = 4$, $M_\pi \sim 560$ MeV
- $N_f = 4$, $M_\pi \sim 360$ MeV

$A^+ + A^- = 1 + a \frac{1}{N_C^2} + b \frac{N_f}{N_C^3}$

$A^+ - A^- = c \frac{1}{N_C} + d \frac{N_f}{N_C^2}$

Coefficients $a - d$ are $O(1)$: natural size

- Large unquenching effect consistent with $N_f, N_C$ scaling.
$K \rightarrow \pi$ amplitudes at Large $N_c$

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\langle K|O^\pm|\pi\rangle = A^\pm = 1 + a \frac{1}{N_c^2} + b \frac{N_f}{N_c^3} \pm c \frac{1}{N_c} \pm d \frac{N_f}{N_c^2}
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- Coefficients $a - d$ are $O(1)$: natural size
- Large unquenching effect consistent with $N_f$, $N_c$ scaling.
- Strong enhancement in $A^-/A^+$ coming from $c, d$. 
Determination of the effective couplings, \( g^\pm = \lim_{M_\pi \to 0} A^\pm \)

**NLO Chiral Perturbation Theory prediction**

\[
A^+ = g^+ \left[ 1 - 3 \frac{M^2_\pi}{(4\pi F_\pi)^2} \log \frac{M^2_\pi}{\Lambda^2_+} \right], \quad A^+ A^- = (g^+ g^-) \left[ 1 + 3 \frac{M^2_\pi}{(4\pi F_\pi)^2} \log \frac{\Lambda^2_+}{\Lambda^2_-} \right]
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\[ g^+ = 0.187(21), \quad (g^+ g^-) = 0.91(4) \]
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$$g^+ = 0.187(21), \quad (g^+ g^-) = 0.91(4)$$

$$\frac{A_0}{A_2} \bigg|_{N_f=4} = 24(5)_{\text{stat}}(7)_{\text{sys}}$$
In the theory with a light charm, we observe an enhancement compatible with the observed $\Delta I = 1/2$ rule:

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This supports the “intrinsic low-energy QCD effects” hypothesis.
Dissecting the $\Delta I = 1/2$ rule at Large $N_c$

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👍 This supports the “intrinsic low-energy QCD effects” hypothesis.

😊 Recent RBC/UKQCD ’20 direct computation of $K \to \pi\pi$ at the physical point also reproduces the $\Delta I = 1/2$ rule.

\[
\frac{A_0}{A_2} \bigg|_{\text{QCD}} = 20(5)
\]
Conclusion
Further observables at Large $N_c$

1. Large $N_c$ scaling of meson masses and decay constants.

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2. $\pi \pi$ scattering

\[
\begin{align*}
N_c &= 5, \quad N_f = 4 \\
15 \otimes 15 &= 1 \oplus 15 \oplus 84 \oplus 15 \oplus 20 \oplus 45 \oplus 45 \\
\text{s-wave channels}
\end{align*}
\]
Further observables at Large $N_c$

1. Large $N_c$ scaling of meson masses and decay constants. [arXiv:1907.11511, EPJC (2019), P. Hernández, C. Pena and FRL]

2. $\pi\pi$ scattering

3. Properties of the $\eta'$ meson

$M_\pi \sim 560$ MeV
$M_{\eta'} = 760(70)$ MeV

Witten-Veneziano:
$M_{\eta'} \sim 850$ MeV
Summary

✓ We have found a hierarchy in the $K \to \pi\pi$ isospin amplitudes originating in intrinsic QCD dynamics, and compatible with natural-size $1/N_c$ and $N_f/N_c$ effects. [arXiv:2003.10293]

✍ The study of other observables is underway ($\pi\pi$ scattering, $\eta'$)
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Somewhere over the rainbow...
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The aim is getting some understanding of QCD ...

Thanks for your attention! 😊
Back-up slides
Large $N_c$ prediction for $K \rightarrow \pi \pi$

- Weak decay with two isospin amplitudes:
  - $A_0 \equiv \mathcal{T}(K \rightarrow (\pi \pi)_{I=0})$, and so $\Delta I = 1/2$
  - $A_2 \equiv \mathcal{T}(K \rightarrow (\pi \pi)_{I=2})$, and so $\Delta I = 3/2$

\[ \text{Re} A_0 - \text{Re} A_2 = \sqrt{2} + O(1/N_c^3) \]
Large $N_c$ prediction for $K \to \pi\pi$

Weak decay with two isospin amplitudes:

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- $A_2 \equiv T(K \to (\pi\pi)_{I=2})$, and so $\Delta I = 3/2$

\[ O(N_c^2) \quad O(N_c) \quad O(N_c^2 g^4) \]

[Diagram showing $K \to \pi\pi$ decay with weak isospin amplitudes]
**Large $N_c$ prediction for $K \to \pi \pi$**

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![Diagram](image)

For neutral particles, $\text{Diag.}(a)$ is not present:

$$A(K_0 \to \pi_0 \pi_0) = A_0 - \sqrt{2} A_2 = 0 \text{ at Large } N_c$$

The Large $N_c$ prediction for $K \to \pi \pi$ is:

$$\text{Re } A_0 \text{ Re } A_2 = \sqrt{2} + O(N_c^3)$$

---

[Fukugita et al 1977; Chivukula, Flynn, Georgi 1986]
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![Diagram](image)

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  A(K^0 \rightarrow \pi^0\pi^0) = A_0 - \sqrt{2}A_2 = 0 \text{ at Large } N_c
  \]

The Large $N_c$ prediction for $K \rightarrow \pi\pi$ is:

$$\frac{\text{Re } A_0}{\text{Re } A_2} = \sqrt{2} + \mathcal{O}\left(\frac{1}{N_c}\right)$$

[Fukugita et al 1977; Chivukula, Flynn, Georgi 1986]
Use observables from Gradient Flow [Lüscher]:

\[
\langle E(t) \rangle = \frac{1}{4} \langle G^a_{\mu \nu} G^a_{\mu \nu}(t) \rangle = \frac{3}{128\pi^2 t^2} \frac{N_c^2 - 1}{N_c} \lambda_{GF} \left( \mu = \frac{1}{\sqrt{8t}} \right)
\]

with \( \lambda_{GF} = N_c g_{GF}^2 \) ('t Hooft coupling).
Scale Setting at Large $N_c$

Use observables from Gradient Flow [Lüscher]:

$$\langle E(t) \rangle = \frac{1}{4} \langle G^a G^\mu_\nu (t) \rangle = \frac{3}{128 \pi^2 t^2} \frac{N_c^2 - 1}{N_c} \lambda_{GF} \left( \mu = \frac{1}{\sqrt{8t}} \right)$$

with $\lambda_{GF} = N_c g_{GF}^2$ ('t Hooft coupling).

For QCD, the scale $\sqrt{t_0}$ is defined through the implicit equation:

$$\langle t^2 E(t) \rangle \bigg|_{t=t_0} = 0.3.$$ 

Input: $t_0$ from other lattice simulations [ALPHA collaboration, $N_f = 2, 3$]

Scale setting for arbitrary $N_c$:

$$\langle t^2 E(t) \rangle \bigg|_{t=t_0} = 0.1125 \frac{N_c^2 - 1}{N_c}, \quad (M \sqrt{t_0}) \bigg|_{M=420 \text{ MeV}} = 0.3090(83)$$
Simultaneous chiral and $N_c$ fit for $F_\pi$

![Graph showing the fit of $F_\pi$ for different $N_c$ values](graph.png)

- Green triangles: $N_c = 3$
- Red inverted triangles: $N_c = 4$
- Blue squares: $N_c = 5$
- Magenta circles: $N_c = 6$

- Dotted lines: SU(4) NLO
- Solid lines: U(4) NNLO

**No sign of relevant discretization effects**

- $a = 0.065$ fm

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results on $B_K$

$$B_K \sim \frac{3}{4} A^+$$

![Graph showing results on $B_K$](image)

- BBG $N_f = 0$
- BBG $N_f = 4$
- this work $N_f = 0$
- this work $N_f = 4$
$K \rightarrow \pi\pi$ at NLO

\[
\text{Re} \left. \frac{A_0}{A_2} \right|_{M_\pi, M_D \rightarrow 0, M_{K}^{\text{phys}}} = \frac{1}{2\sqrt{2}} \left( 1 + 3 \frac{g^\pi}{g^+} \right) \\
+ \frac{17}{12\sqrt{2}} \left( 1 + \frac{1}{17} \frac{g^-}{g^+} \right) \frac{M_K^2}{(4\pi F_K)^2} \log \frac{\Lambda_{\text{eff}}^2}{M_K^2},
\]

Experiment
NLO ChPT

\begin{axis} [grid=both,]
\addplot[blue] coordinates {(0.1, 15.0) (10.0, 35.0)}; \label{plot1}
\addplot[red] coordinates {(0.1, 20.0) (10.0, 30.0)}; \label{plot2}
\end{axis}
Details on $1/N_c$ expansion of $A^\pm$

\[ C^\pm \equiv \langle P^{d\bar{u}}(y)\bar{Q}^\pm(x)P^{u\bar{s}}(z) \rangle \]
\[ \equiv Z_Q^\pm(\mu)\langle P^{ud}(y)[O^{sud}(x) \pm O^{sd\bar{u}}(x)]P^{u\bar{s}}(z) \rangle \]

\[ \langle P^{ab}J^{ba}_\mu \rangle = N_c \left( a + b \frac{N_f}{N_c} \right) + \ldots, \]

Color-disconnected \[ \langle P^{d\bar{u}}O^{sud}P^{u\bar{s}} \rangle = \langle P^{d\bar{u}}J^{ud}_\mu \rangle \langle P^{s\bar{u}}J^{u\bar{s}}_\mu \rangle + c + d \frac{N_f}{N_c} + \ldots, \]

Color-connected \[ \langle P^{d\bar{u}}O^{sd\bar{u}}P^{u\bar{s}} \rangle = N_c \left( e + f \frac{N_f}{N_c} \right) + \ldots, \quad (11) \]

+ explicit check of similar behaviour of $Z^\pm$
$l = 2 \, \pi \pi$ scattering length

- Use Lüscher threshold expansion:
  $$E_{\pi \pi} - 2 M_\pi = - \frac{4 \pi a_0}{M_\pi L^3} \left[ 1 + c_1 \frac{a_0}{L} + c_2 \left( \frac{a_0}{L} \right)^2 \right]$$
- At large $N_c$, mesons become noninteracting: $a_0^{l=2} \sim \frac{1}{N_c}$

\[ M_\pi a_0^{l=2} = - \frac{M_\pi^2}{16 \pi F_\pi^2} \left[ 1 - \frac{16 M_\pi^2}{F_\pi^2} L_{\pi \pi}(\mu) - \frac{M_\pi^2}{32 \pi^2 F_\pi^2} \left( \frac{13}{4} \log \frac{M_\pi^2}{\mu^2} - \frac{3}{4} \right) \right] \]

\[ \frac{L_{\pi \pi}}{N_c} \times 10^3 = -0.25(4) - \frac{0.9(2)}{N_c} \]

\[ \chi^2 = 19.06099 \]

d.o.f. $= 16 - 1$
Other scattering channels

- With $N_f = 4$, we have more channels (channels with $s$-wave):

$$15 \otimes 15 = 1 \oplus 15 \oplus 84 \oplus 15 \oplus 20 \oplus 45 \oplus 45$$

\[ C_1 = D + \frac{1}{N_f} C - \frac{2(N_f^2 - 1)}{N_f} R + \frac{N_f^2 - 1}{2} VV \]

\[ C_{84} = D - C \]

\[ C_{15} = D + \frac{2}{N_f} C - \frac{N_f^2 - 4}{N_f} R \]

\[ C_{20} = D + C \]

$N_c = 5$, $N_f = 4$, $M_\pi \sim 560$ MeV
Witten-Veneziano equation at large $N_c$

- $\eta_4$ correlator with subtraction of excited states [arXiv:1310.1207]

$$C_{\eta_4}(t) = C_\pi(t) - N_f D(t) D(0)$$

- Preliminary result: $M_{\eta_4} = 760(70)$ MeV
- Prediction from Witten-Veneziano with $\chi_{top}$ from [Cè et al.]:
  $$M_{\eta_4}^2 = M_\pi^2 + \frac{2N_f\chi_{top}}{F_\pi^2} \quad \rightarrow \quad M_{\eta_4} \sim 850 \text{ MeV}$$

$N_c = 5, \ N_f = 4, \ M_\pi \sim 560 \text{ MeV}$