Gamma model - bosonization and gauge theory interpretation

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Introductory remarks

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This talk: bosonization method introduced in [Wosiek '82],

with an emphasis on recent progress in its understanding.

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The even subalgebra is generated by parity and hopping operators:

$$(-1)^{N_f(x)} = 1 - 2\phi^*(x)\phi(x), \tag{2a}$$

$$\mathfrak{s}(l) = X(s(l))X(t(l)), \tag{2b}$$

where $s(l)$ and $t(l)$ are the endpoints of the link l and $X = \phi + \phi^*$.

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There is only one more independent relation, the *loop relation*:

$$\mathfrak{s}(l_1)\dots\mathfrak{s}(l_n) = 1 \tag{3}$$

whenever the links $l_1, \dots, l_n$ form a closed path.

Even fermionic operators

The Fock space $\mathcal{F}$ decomposes into the even and odd subspace

$$\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1. \tag{4}$$

$\mathcal{F}_\alpha$ are the only irreps of the algebra $\mathcal{A}_0$ of even operators.

Any other representation is a direct sum of these.

Even fermionic operators

In order to construct an exact bosonization map one has to

1. Construct a representation of $\mathcal{A}_0$,
2. Understand its decomposition into simple factors.

For the first step it suffices to build operators obeying all relations.

Construction of the Γ model

On each site x we put the Clifford algebra generated by $\{\Gamma(x, l)\}$, where the index l runs through all links incident to the site x.

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Bosonization map takes the form

$$(-1)^{N_f(x)} \mapsto \Gamma_*(x) = \text{phase} \cdot \prod_l \Gamma(x,l), \tag{5a}$$

$$\mathfrak{s}(l) \mapsto S(l) = -i\Gamma(s(l),l)\Gamma(t(l),l). \tag{5b}$$

We note that this maps local hamiltonians to local hamiltonians.

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Thus the number of links incident to any site has to be even.

Then all relations of $\mathcal{A}_0$ except of the loop relation are satisfied. We impose the loop relation as a *constraint* on physical states.

Construction of the Γ model

Theorem (Szczerba '85, Bochniak, Ruba '20)

Γ *model Hilbert space with loop constraints imposed is isomorphic as a representation of* $\mathcal{A}_0$ *to one half* $\mathcal{F}_\alpha$ *of the Fock space.*

α *depends on the lattice geometry and the way one resolves the sign ambiguity in the definition of* $\Gamma_*(x)$ *(independent for each* x*).*

Constraints and gauge fields

Consider coupling fermions to an external $\mathbb{Z}_2$ gauge field A.

Minimal coupling: replace $\mathfrak{s}(l) \mapsto \mathfrak{s}_A(l) = (-1)^{A(l)}\mathfrak{s}(l)$.

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For these generators, loop relations are modified:

$$\mathfrak{s}_A(l_1)\dots\mathfrak{s}_A(l_n) = \underbrace{(-1)^{A(l_1)+\dots A(l_n)}}_{\text{holonomy}}. \tag{6}$$

This provides an interpretation for subspaces of the Γ model Hilbert space defined by modified constraints.

Constraints and gauge fields

Theorem (Bochniak, Ruba '20)

Γ model Hilbert space $\mathcal{H}$ decomposes as $\bigoplus_{[A]} \mathcal{H}_{[A]}$, with the sum running over all gauge orbits of $\mathbb{Z}_2$ gauge fields.

Constraints and gauge fields

Theorem (Bochniak, Ruba '20)

Γ model Hilbert space $\mathcal{H}$ decomposes as $\bigoplus_{[A]} \mathcal{H}_{[A]}$, with the sum running over all gauge orbits of $\mathbb{Z}_2$ gauge fields.

$\mathcal{H}_{[A]}$ describes fermions in the field A: $\mathcal{H}_{[A]} \cong \mathcal{F}_{\alpha+(A,\zeta)}$, where

$$(A, \zeta) = \sum_{l} A(l) \quad mod\ 2. \tag{7}$$

In words, only states of parity $\alpha + (A, \zeta)$ are implemented.

Constraints and gauge fields

Example

The quadratic fermionic hamiltonian

$$H = \sum_l h_l \ \phi(s(l))\phi(t(l))^* + \sum_x \nu_x \ \phi(x)^*\phi(x)$$

is bosonized to the form

$$H_\Gamma = \sum_l h_l \ \frac{1+\Gamma_*(s(l))}{2} S(l) \frac{1+\Gamma_*(t(l))}{2} + \sum_x \nu_x \ \frac{1-\Gamma_*(x)}{2}.$$

Modifying constraints is equivalent to replacing $h_l \mapsto h_l \cdot (-1)^{A(l)}$.

Constraints and gauge fields

Since the Γ model Hilbert space incorporates all possible A fields, it is natural to ask whether the gauge field can be made dynamical.

For this one needs a momentum W conjugate to the A field.
In the standard gauge theory this is the electric field.

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For this one needs a momentum W conjugate to the A field. In the standard gauge theory this is the electric field.

Existence of such W would contradict the relation $N_f \equiv \alpha + (A, \zeta)$. Secondly, in the standard $\mathbb{Z}_2$ gauge theory one has

$$\text{Gauss' law} \implies N_f \equiv 0. \tag{8}$$

Constraints and gauge fields

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It turns out that there exists a mapping of the Γ model to a $\mathbb{Z}_2$ gauge theory with a modified Gauss' law.

Modified Gauss' law means that gauge transformations $A_i \mapsto A_i + \partial_i\theta$ are implemented on the quantum level by $|A_i\rangle \mapsto e^{iI(A,\theta)}|A_i + \partial_i\theta\rangle$ for some nontrivial functional I.
Such mechanism exists in models with Chern-Simons like terms.

Constraints and gauge fields

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The main properties of the claimed mapping are:

1. Γ model operators $\leftrightarrow$ gauge invariant operators.
2. Even fermionic operators and Wilson lines are represented locally. Electric fields are nonlocal on the Γ model side, while the Γ field is nonlocal in the gauge theory.
3. The Gauss' law constraint in gauge theory is an exact identity in the Γ model. It implies the relation $N_f \equiv \alpha + (A, \zeta)$ between the gauge field and the number of fermions.

Constraints and gauge fields

From the gauge theory point of view, our basic field Γ acts as a composite of a single fermion and a lump of electromagnetic field:

$$\Gamma = \text{fermion} \times \text{flux}. \tag{9}$$

This is related to the so called flux attachment mechanism.

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Braiding of charges and fluxes leads to Aharonov-Bohm phases, which allow a fermion to become a boson. Constraints of the Γ model define the subspace in which no fluxes are present.

Constraints and gauge fields

Recently another exact bosonization has been developed [Chen, Kapustin and Radicevic, '18 and '19]:

Fermions in spatial dimension d can be mapped to $(d-1)$-form $\mathbb{Z}_2$ gauge theory with a topological term in action which for flat fields α reduces to the integral of the Steenrod square $\mathrm{Sq}^2(\alpha)$.

Constraints and gauge fields

	Γ model	$(d-1)$-form gauge theory
degrees of freedom	on sites	on $(d-1)$-cells
local constraints	on plaquettes	on $(d-2)$-cells (Gauss' law)
fermionic excitations	on sites	on d-cells (fluxes)
topological action	**not yet known**	Steenrod square

This table suggests that any direct relation between the two formulations would have to involve the dual lattice construction.

Summary

1. Γ model $\rightarrow$ bosonization in any dimension.
2. A practical difficulty: one has to deal with constraints.
3. Omitting constraints introduces a $\mathbb{Z}_2$ gauge field.
4. The $\mathbb{Z}_2$ gauge field obeys a modified Gauss' law,
   which resembles Chern-Simons theories.
5. Bosonization may be understood as "flux attachment".
6. It would be interesting to find a path integral formulation.