The lattice Yang-Mills theory with a gauge-invariant gluon mass in view of the gauge-invariant BEH mechanism towards confinement

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Introduction

What is the mechanism of quark confinement?

- A promising scenario is the **dual superconductor picture** of the QCD vacuum. [Nambu,1974]['t Hooft,1975][Mandelstam,1976]
- One of the remarkable facts on this picture found in the preceding studies is *Infrared Abelian dominance* : The Abelian part (or diagonal component) of the gauge field becomes dominant for quark confinement in the low-energy or long-distance region [Ezawa & Iwazaki,1982].

  This hypothesis was confrmed by:
  Abelian dominance of the string tension: The string tension of the linear potential in the static quark-antiquark potential can be reproduced by the Abelian part alone [Suzuki & Yotsuyanagi,1990].
  Dynamical generation of the off-diagonal gluon mass: The off-diagonal gluon propagator exhibits the exponential fall-off in the distance [Amemiya & Suganuma,1999].
- However, these results were obtained only in the specific gauge called the maximal Abelian (MA) gauge based on the idea of **Abelian projection method** proposed by ['t Hooft,1981].
  The gauge invariance or independence was not clear in the Abelian projection method

Introduction (cont')

The decomposition method :

- We have succeeded to demonstrate the Abelian dominance of the string tension in the gauge-invariant way based on the novel reformulation of the Yang-Mills theory in terms of the new field variables obtained from the gauge covariant decomposition method and the non-Abelian Stokes theorem for the Wilson loop operator .

For more details, see the review: K.-I. Kondo, S. Kato, T. Shinohara and A. Shibata, Phys. Rept **579***, 1–226 (2015). arXiv:1409.1599 [hep-th]*

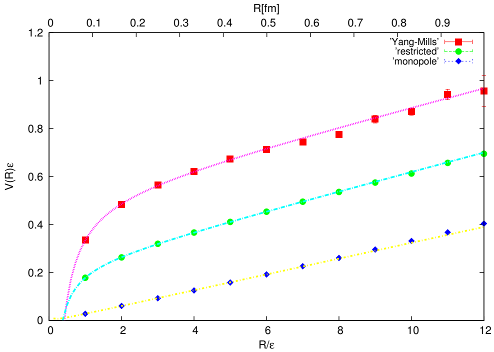
- In the decomposition method, the Lie algebra valued Yang-Mills filed is decomposed into two parats in the gauge-independent way: $\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x)$, where $\mathscr{V}_\mu$, called the restricted filed, corresponds to the "Abelian" or diagonal part in the Abalian projection method, and
  $\mathscr{X}_\mu$, called the remaining part, corresponds to the off-diagonal part.
- How about the Abelian dominance of the diagonal propagator?

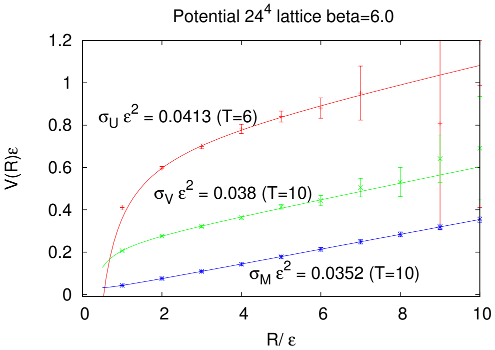
  The propagator can be obtained only after the gauge fixing. Therefore, Abelian dominance of the diagonal propagator cannot be extended in the gauge invariant way.
- Instead, however, we can give a gauge-invariant definition for the off-diagonal gluon mass.
- Therefore, we can study the mass generation of the off-diagonal gluon mass in the gauge-invariant way.

Lattice result for pure Yang-Mills theory

The followings are the results by the decomposition method.

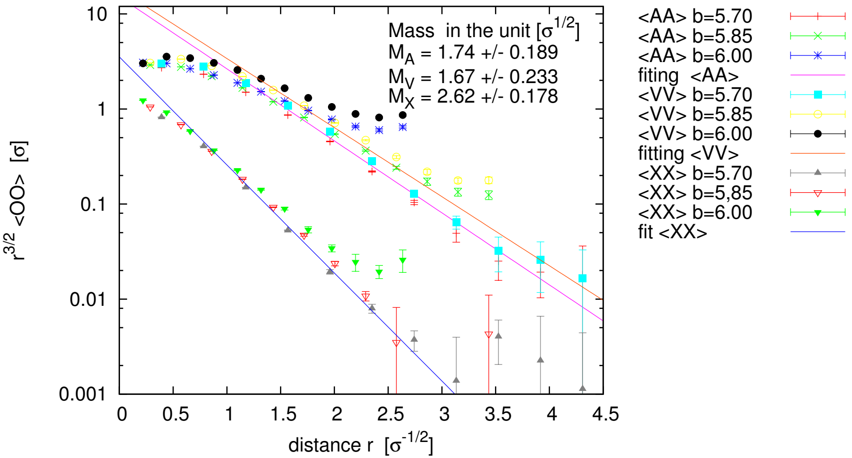
Abelian dominance of the string tension:





The static quark-antiquark potentials as functions of the quark-antiquark distance R:
(from above to below) full $V_{\rm full}(R)$, restricted part $V_{\rm rest}(R)$ and magnetic–monopole part $V_{\rm mono}(R)$.
(Left) $SU(2)$ at $\beta = 2.5$ on 24^4 lattice, [Kato, Kondo and Shibata, PRD**91**, 034506 (2015)]
(Right) $SU(3)$ at $\beta = 6.0$ on 24^4 lattice. [Kondo, Shibata, Shinohara & Kato, PRD**83**, 114016 (2011)]

Dynamical generation of the off-diagonal gluon mass:



The rescaled correlation functions $r^{3/2} \langle O(r)O(0) \rangle$ for $O = \boldsymbol{A, V, X}$ for 24^4 lattice with $\beta =$, 5.7, 5.85, 6.0. The physical scale is set in units of the string tension $\sigma_{\text{phys}}^{1/2}$.
[A. Shibata et al., PRD**87**, 054011 (2013)]

Introduction (cont')

- The decomposition method is based on the gauge-independent description of the Brout-Englert-Higgs (BEH) mechanism proposed recently by [Kondo, 2016, 2018], which needs

  neither the spontaneous breaking of gauge symmetry $G \to H$,

  nor the non-vanishing vacuum expectation value of the scalar field $\langle 0|\phi(x)|0\rangle := v \neq 0$.

- To explain it, we need to introduce a specific gauge-scalar model (complementary gauge-scalar model) which reduces to the Yang-Mills theory with a gauge-invariant gluon mass term (massive Yang-Mills theory) .
- The gauge-invariant gluon mass term
  simulates the dynamically generated mass to be extracted in the low-energy effective theory of the Yang-Mills theory
  and plays the role of a new probe to study confinement mechanism through the phase structure (Confinement phase, Higgs phase, deconfinement phase) in the gauge-invariant way.

Contents

In this talk, we discuss how the numerical method for the proposed massive Yang-Mills theory can be performed by taking into account the reduction condition in the complementary gauge-scalar model on a lattice.

Contents:

BEH mechanism for the gauge-scalar model

We consider $G = SU(2)$ gauge-scalar model with a single adjoint scalar field characterized by the gauge-invariant Lagrangian (no potential term):

$$\mathscr{L}_{\mathrm{GS}} = \mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{kin}} = -\frac{1}{2}\mathrm{tr}\{\mathscr{F}^{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)\} + \mathrm{tr}\{(\mathscr{D}^{\mu}[\mathscr{A}]\boldsymbol{\phi}(x))(\mathscr{D}_{\mu}[\mathscr{A}]\boldsymbol{\phi}(x))\},$$

where the Lie algebra valued Yang-Mills field $\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x) T_A$ $(A = 1, 2, 3)$ obey the gauge transformation:

$$\mathscr{A}_\mu(x) \to U(x)\mathscr{A}_\mu(x)U^{-1}(x) + ig^{-1}U(x)\partial_\mu U^{-1}(x), \quad U(x) \in G = SU(2)$$

and the Lie algebra valued scalar field $\boldsymbol{\phi}(x) = \phi^A(x) T_A$ $(A = 1, 2, 3)$ has the fixed radial length (modulus) $v > 0$:

$$\boldsymbol{\phi}(x) \cdot \boldsymbol{\phi}(x) \equiv 2\mathrm{tr}\{\boldsymbol{\phi}(x)\boldsymbol{\phi}(x)\} = \boldsymbol{\phi}^A(x)\boldsymbol{\phi}^A(x) = v^2.$$

and transforms according to the adjoint representation under the gauge transformation:

$$\boldsymbol{\phi}(x) \to U(x)\boldsymbol{\phi}(x)U^{-1}(x), \quad U(x) \in G = SU(2),$$

The covariant derivative $\mathscr{D}_\mu[\mathscr{A}] := \partial_\mu - ig[\mathscr{A}_\mu, \cdot]$ transforms according to the adjoint representation under the gauge transformation: $\mathscr{D}_\mu[\mathscr{A}] \to U(x)\mathscr{D}_\mu[\mathscr{A}]U^{-1}(x)$.

Conventional description for the BEH mechanism

- Suppose that the scalar field $\boldsymbol{\phi}(x)$ acquires a non-vanishing vacuum expectation value (VEV): $\langle \boldsymbol{\phi}(x) \rangle = \langle \boldsymbol{\phi} \rangle = \langle \boldsymbol{\phi}^A \rangle T_A$. Then the covariant derivative of the scalar field is

$$\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x) := \partial_\mu \boldsymbol{\phi}(x) - ig[\mathscr{A}_\mu(x), \boldsymbol{\phi}(x)] \to -ig[\mathscr{A}_\mu(x), \langle \boldsymbol{\phi} \rangle] + \dots .$$

Consequently, the kinetic term of the scalar field is modified into

$$\begin{aligned} \mathrm{tr}\{(\mathscr{D}^\mu[\mathscr{A}]\boldsymbol{\phi}(x))(\mathscr{D}_\mu[\mathscr{A}]\boldsymbol{\phi}(x))\} \to & -g^2 \mathrm{tr}_G\{[\mathscr{A}^\mu(x), \langle \boldsymbol{\phi} \rangle][\mathscr{A}_\mu(x), \langle \boldsymbol{\phi} \rangle]\} + \dots \\ & = -g^2 \mathrm{tr}_G\{[T_A, \langle \boldsymbol{\phi} \rangle][T_B, \langle \boldsymbol{\phi} \rangle]\} \mathscr{A}^{\mu A}(x) \mathscr{A}_\mu^B(x) + \dots \end{aligned}$$

If the non-vanishing VEV $\langle \boldsymbol{\phi} \rangle = \langle \boldsymbol{\phi}^A \rangle T_A$ of the scalar field $\boldsymbol{\phi}$ is chosen to a specific direction, e.g., $\langle \boldsymbol{\phi} \rangle_\infty = \langle \boldsymbol{\phi}^3 \rangle T_3$, [unitary gauge] uniformly over the spacetime, then the original local continuous gauge symmetry $G = SU(2)$ is spontaneously broken to a subgroup $H = U(1)$.

- Thus the kinetic term of the scalar field generates the mass term of the gauge field:

$$-g^2 \mathrm{tr}_G\{[T_A, vT_3][T_B, vT_3]\} \mathscr{A}^{\mu A} \mathscr{A}_\mu^B = \frac{1}{2}(gv)^2 (\mathscr{A}^{\mu 1}\mathscr{A}_\mu^1 + \mathscr{A}^{\mu 2}\mathscr{A}_\mu^2), \quad v := \langle \boldsymbol{\phi}^3 \rangle.$$

- The off-diagonal gluons $\mathscr{A}_\mu^1, \mathscr{A}_\mu^2$ acquire the same mass $M_W := gv = g\langle \boldsymbol{\phi} \rangle_\infty$,
- The diagonal gluon $\mathscr{A}_\mu^3$ remains massless.
  This description of the BEH mechanism depends on the specific gauge and is not gauge independent. Indeed, VEV $\langle \boldsymbol{\phi} \rangle_\infty$ is not a gauge invariant quantity.

Gauge-independent description for the BEH mechanism

We explain a gauge-independent description for the BEH mechanism, which does not rely on the SSB. [K.-I. Kondo, Phys. Lett. B**762**, 219–224 (2016). arXiv:1606.06194 [hep-th]].

- We construct a composite vector field $\mathscr{W}_\mu(x)$ which consists of $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}(x)$:

$$\mathscr{W}_\mu(x) := -ig^{-1}[\hat{\boldsymbol{\phi}}(x), \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)], \quad \hat{\boldsymbol{\phi}}(x) := \boldsymbol{\phi}(x)/v.$$

  We find that the kinetic term of the scalar field $\boldsymbol{\phi}$ is identical to the "mass term" of the vector field $\mathscr{W}_\mu$:

$$\mathscr{L}_{\rm kin} = \frac{1}{2}\mathscr{D}^\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x) \cdot \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x) = \frac{1}{2}M_W^2 \mathscr{W}^\mu(x) \cdot \mathscr{W}_\mu(x), \quad M_W := gv,$$

  as far as the constraint $(\hat{\boldsymbol{\phi}}(x) \cdot \hat{\boldsymbol{\phi}}(x) = 1)$ is satisfied.

- This "mass term" of $\mathscr{W}_\mu$ is gauge invariant, since $\mathscr{W}_\mu$ obeys the adjoint gauge transformation:

$$\mathscr{W}_\mu(x) \to U(x)\mathscr{W}_\mu(x)U^{-1}(x).$$

  The $\mathscr{W}_\mu$ gives a gauge-independent definition of the massive gluon mode in the operator level. The massive mode $\mathscr{W}_\mu$ can be described without breaking the original gauge symmetry. (We do not need to choose a specific vacuum from all possible degenerate ground states distinguished by the direction of $\boldsymbol{\phi}$.)

Complementary gauge-scalar model for the Yang-Mills theory

- In the gauge-scalar model, $\mathscr{A}_\mu(x)$ and $\boldsymbol{\phi}(x)$ are independent field variables.
- However, the Yang-Mills theory should be described by the Yang-Mills field $\mathscr{A}_\mu(x)$ alone and hence $\boldsymbol{\phi}$ must be supplied as a composite field made from the gauge field $\mathscr{A}_\mu(x)$ due to the strong interactions.
  [the scalar field $\boldsymbol{\phi}$ is to be given as a (complicated) functional of the gauge field $\mathscr{A}_\mu(x)$.]
- This is achieved by imposing the constraint which we call the **reduction condition**:

$$\boldsymbol{\chi}(x) := [\hat{\boldsymbol{\phi}}(x), \mathscr{D}_\mu[\mathscr{A}]\mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)] = \mathbf{0}$$

$$\Longleftrightarrow$$

$$\mathscr{D}_\mu[\mathscr{A}]\mathscr{W}_\mu(x) = \mathscr{D}_\mu[\mathscr{A}](-ig^{-1})[\hat{\boldsymbol{\phi}}(x), \mathscr{D}_\mu[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)] = \boldsymbol{\chi}(x) = \mathbf{0}$$

  This condition is gauge covariant, $\boldsymbol{\chi}(x) \to U(x)\boldsymbol{\chi}(x)U^{-1}(x)$.
- The reduction condition plays the role of eliminating the extra degrees of freedom introduced by the radially fixed adjoint scalar field into the Yang-Mills theory, since $\boldsymbol{\chi}$ represents two conditions due to

$$\boldsymbol{\chi}(x) \cdot \hat{\boldsymbol{\phi}}(x) = 0.$$

Complementary gauge-scalar model for the Yang-Mills theory (2)

- Therefore, the complementary gauge-scalar model (massive Yang-Mills theory) is characterized by the gauge-invariant Lagrangian (no potential term):

$$\begin{aligned}\mathscr{L}_{\mathrm{mYM}} &= \mathscr{L}_{\mathrm{YM}} + \mathscr{L}_{\mathrm{kin}} \\ &= -\frac{1}{2}\mathrm{tr}\{\mathscr{F}^{\mu\nu}(x)\mathscr{F}_{\mu\nu}(x)\} + \mathrm{tr}\{(\mathscr{D}^{\mu}[\mathscr{A}]\hat{\boldsymbol{\phi}}(x))(\mathscr{D}_{\mu}[\mathscr{A}]\hat{\boldsymbol{\phi}}(x))\}\end{aligned}$$

  where the scalar field $\hat{\boldsymbol{\phi}}$ is to be given by solving the reduction condition:

$$\chi(x) := [\hat{\boldsymbol{\phi}}(x), \mathscr{D}_{\mu}[\mathscr{A}]\mathscr{D}_{\mu}[\mathscr{A}]\hat{\boldsymbol{\phi}}(x)] = \mathbf{0}$$

- Thus, the "complementary" gauge-scalar model is defined by taking into account the Faddeev-Popov determinant $\widetilde{\Delta}^{\mathrm{red}}$ associated with the reduction condition $\chi = 0$ as

$$\tilde{Z}_{\mathrm{RF}} = \int \mathcal{D}\mathscr{A}\mathcal{D}\hat{\boldsymbol{\phi}}\ \delta(\chi)\,\Delta^{\mathrm{red}} e^{-S_{\mathrm{YM}}[\mathscr{A}]-S_{\mathrm{kin}}[\mathscr{A},\hat{\boldsymbol{\phi}}]}.$$

Contents

Massive Yang-Mills theory on the lattice

Then, we discuss the numerical simulation for the proposed massive Yang-Mills theory on the lattice.
By taking into account the reduction condition in the complementary gauge-scalar model, the gauge-invariant mass term is introduced:

$$Z_L = \int \mathcal{D}[U]\mathcal{D}[\boldsymbol{\phi}]\delta(\boldsymbol{\phi} - \hat{\boldsymbol{n}})e^{-\beta S_g - \gamma S_m}$$

$$S_g[U] := \sum_x \sum_{\mu>\nu} 2\,\mathrm{Re}\,\mathrm{tr}\left(\mathbf{1} - U_{x,\mu}U_{x+\mu,\nu}U^\dagger_{x+\nu,\mu}U^\dagger_{x,\nu}\right)$$

$$S_m[U,\boldsymbol{\phi}] := \sum_{x,\mu} \mathrm{tr}\left((D^\epsilon_\mu[U]\boldsymbol{\phi}_x)^\dagger(D^\epsilon_\mu[U]\boldsymbol{\phi}_x)\right), \quad D^\epsilon_\mu[U]\boldsymbol{\phi}_x := U_{x,\mu}\boldsymbol{\phi}_{x+\mu} - \boldsymbol{\phi}_x U_{x,\mu}$$

where $U_{x,\mu} \in SU(2)$ is the link variable, $\boldsymbol{\phi} = \boldsymbol{\phi}_A T^A \in su(2)$ is the color field (scalar field $\boldsymbol{\phi}$) with $\boldsymbol{\phi}\cdot\boldsymbol{\phi} = 1$, and $D^\epsilon_\mu[U]\boldsymbol{\phi}_x$ is the covariant derivative.
$\delta(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})$ represents the reduction condition in the complementary gauge-scalar model, and $\hat{\boldsymbol{\phi}}$ is the solution of the reduction condition for given gauge configuration, which is obtaine by minimizing the functional:

$$F_{\mathrm{red}}(\boldsymbol{\phi}; U) := \sum_{x,\mu} \mathrm{tr}\left((D^\epsilon_\mu[U]\boldsymbol{\phi}_x)^\dagger(D^\epsilon_\mu[U]\boldsymbol{\phi}_x)\right)$$

Nnmerical Simulation

Now, we perform the MC simuration to genarate the gauge configuration:

$$\rho[U, \boldsymbol{\phi}] := \frac{\delta(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})e^{-\beta S_g(U) - \gamma S_m(U,\boldsymbol{\phi})}}{Z_L}, \quad \hat{\boldsymbol{\phi}} : \text{ the solution minimizing } F_{\text{red}}(\boldsymbol{\phi}; U)$$

$$Z_L = \int \mathcal{D}[U]\mathcal{D}[\boldsymbol{\phi}]\delta(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})e^{-\beta S_g(U) - \gamma S_m(U,\boldsymbol{\phi})}$$

- Without the reduction condition (or $\delta(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}})$) , this model is reduced into the usual gauge-scalar model with a radially fixed scalar field.
- If $\gamma = 0$, the model is reduced into the usual Yang-Mills theory with the standard Wilson action.
- In the massive Yang-Mills theory, $U_{x,\mu}$ and $\boldsymbol{\phi}$ are no more independent field variables.
- Thus, the gauge configurations must be updated by solving the reduction condition simultaneously.

Reweighting method

- For the region $\gamma \sim 0$, we can use the rewaiting technique.
- Thus, we generate configurations for the standard Willson action

$$\rho[U] = \frac{e^{-\beta S_g(U)}}{Z_L}, \quad Z_L = \int \mathcal{D}[U] e^{-\beta S_g(U)}$$

- To obtain the color field (scalar field) configuration $\boldsymbol{\phi}$, we solve the reduction condition for each gauge configuration by minimizing the fanctional:

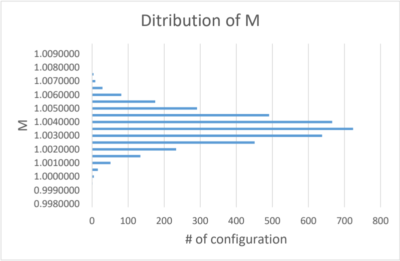
$$F_{\text{red}}(\boldsymbol{\phi}; U) := \sum_{x,\mu} \mathrm{tr}\left((D_\mu^\epsilon[U]\boldsymbol{\phi}_x)^\dagger (D_\mu^\epsilon[U]\boldsymbol{\phi}_x) \right).$$

  The color field $\hat{\boldsymbol{\phi}}$ is obtained as function of the gauge configuration. $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}[\boldsymbol{U}]$
- The observable $\mathcal{O}$ is measured by reweiting method.

$$\langle \mathcal{O} \rangle := \frac{\sum \mathcal{O}[U, \hat{\boldsymbol{\phi}}] e^{-\gamma S_m(U,\hat{\boldsymbol{\phi}})}}{\sum e^{-\gamma S_m(U,\hat{\boldsymbol{\phi}})}}$$

Results from the reweighting method

$$\mathcal{M} := \frac{1}{N_{\text{site}}} \sum_{x,\mu} \text{tr}((D^{\epsilon}_{\mu}[U]\boldsymbol{\phi}_x)^{\dagger}(D^{\epsilon}_{\mu}[U]\boldsymbol{\phi}_x)) \ :$$



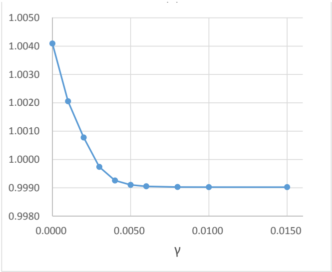


Figure: No error bars are ploted because they are very large for finite γ . (Left) The histgram of the mass term: $\mathcal{M}$. (Right) The mesurement of the $\langle \mathcal{M} \rangle$ for various γ

- In the last lattice conference, we give the first lattice calcuration of Yang-Mills theory with "a gauge-invariant gluon mass term" for small mass parater γ by using the reweithing technique.
- It has been found that the reweighting method can be applied only to a region where γ is very small.
- Therefore, the full simulation with gluon mass term to investigate the whole parameter space of the gauge coupling β and the mass term γ.

HMC algorithm

We first look at the gauge-scalar model.

$$\rho[U, \boldsymbol{\phi}] := \frac{e^{-\beta S_g(U) - \gamma S_m(U, \boldsymbol{\phi})}}{Z_L}, \qquad Z_L = \int \mathcal{D}[U]\mathcal{D}[\boldsymbol{\phi}] e^{-\beta S_g(U) - \gamma S_m(U, \boldsymbol{\phi})}$$

- When we introduce the cannonical momentums $\boldsymbol{\pi}_l = \pi_l^A T^A$ and $\boldsymbol{\rho}_l = \rho_l^A T^A$ which conjugate to $\boldsymbol{X}_l = T^A X_l^A$ with $U_l = \exp(i\boldsymbol{X}_l)$ and $\boldsymbol{\phi}_x = \phi_x^A T^A$, respectively, we obtain Hamitonian to evaluate the HMC method:

$$H = \frac{1}{2}\sum_l \boldsymbol{\pi}_l \cdot \boldsymbol{\pi}_l + \frac{1}{2}\sum_x \boldsymbol{\rho}_x \cdot \boldsymbol{\rho}_x + \beta S_g(U) + \gamma S_m(U, \boldsymbol{\phi}) + \sum_x \nu_x \left(\boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_x - 1\right)$$

  where the variables ν_x are Lagrange multipliers for the constraints.

- Usual HMC algorithm can be applicable.
- $U_{x,\mu}$ and $\boldsymbol{\phi}$ are independent, and the valiables can be updated separately.
- As for the constraint $\boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_x = 1$, it can be normalized for every step of solving the differentail equation.

HMC algorithm (2)

The massive Yang-Mills theory :

$$\rho[U, \boldsymbol{\phi}] := \frac{\delta(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}}) e^{-\beta S_g(U) - \gamma S_m(U, \boldsymbol{\phi})}}{Z_L}, \quad \hat{\boldsymbol{\phi}} : \text{ the solution minimizing } F_{\rm red}(\boldsymbol{\phi}; U)$$

$$Z_L = \int \mathcal{D}[U]\mathcal{D}[\boldsymbol{\phi}]\delta(\boldsymbol{\phi} - \hat{\boldsymbol{\phi}}) e^{-\beta S_g(U) - \gamma S_m(U, \boldsymbol{\phi})}$$

- We have further constraints called "reduction condition", i.e., $\hat{\boldsymbol{\phi}}$ is obtained by minimizing the function $F_{\rm red}[\boldsymbol{\phi}; U]$. However, this is not suitable for HMC, so we set other equavalent constraints, i.e., the stational condition for the finctional:

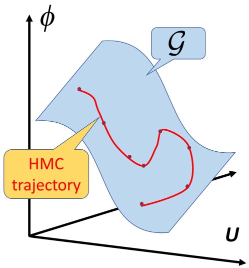
$$\frac{\partial}{\partial \phi_x^A} F_{\rm red}(\boldsymbol{\phi}, U) = 0, \quad \text{for all sites and A=1,2,3,}$$

Thus, we have the Hamiltonian with constraints for HMC:

$$H_T = \frac{1}{2}\sum_l \boldsymbol{\pi}_l \cdot \boldsymbol{\pi}_l + \frac{1}{2}\sum_x \boldsymbol{\rho}_x \cdot \boldsymbol{\rho}_x + \beta S_g(U) + \gamma S_m(U, \boldsymbol{\phi})$$
$$+ \sum_x \nu_x (\boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_x - 1) + \sum_{x,A} \lambda_x^A \frac{\partial}{\partial \phi_x^A} F_{\rm red}(\boldsymbol{\phi}, U),$$

where we have additional are Lagrange multipliers λ_x^A for the constraints.

HMC algorithm (3)



$$H_T = \frac{1}{2}\sum_l \boldsymbol{\pi}_l \cdot \boldsymbol{\pi}_l + \frac{1}{2}\sum_x \boldsymbol{\rho}_x \cdot \boldsymbol{\rho}_x$$
$$+\beta S_g(U) + \gamma S_m(U, \boldsymbol{\phi})$$
$$+\sum_x \nu_x (\boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_x - 1) + \sum_{x,A} \lambda_x^A \frac{\partial}{\partial \phi_x^A} F_{\mathrm{red}}(\boldsymbol{\phi}, U),$$

- $U_{x,\mu}$ and $\boldsymbol{\phi}$ are no more independent field variables.
- The gauge configurations must be updated by solving the reduction condition simultaneously.
- The intersection of the constraints defines the region that variable $(\boldsymbol{\phi}_x^A, U_{x,\mu})$ can teke:
  $\mathcal{G} := \{\boldsymbol{\phi}_x^A, U_{x,\mu} | \quad \frac{\partial}{\partial \phi_x^A} F_{\mathrm{red}}(\boldsymbol{\phi}, U) = 0; \quad \boldsymbol{\phi}_x \cdot \boldsymbol{\phi}_x = 1\}$
- Updating the configuration is the same as solving the motion of a particle on the constraining surface $\mathcal{G}$ under the potential $V = \beta S_g(U) + \gamma S_m(U, \boldsymbol{\phi})$.

Summary

- In order to clarify the mechanism of quark confinement in the Yang-Mills theory with the mass gap, we propose to investigate the massive Yang-Mills model, namely, Yang-Mills theory with "a gauge-invariant gluon mass term" to be deduced from a specific gauge-scalar model with a single radially-fixed scalar field under a suitable constraint called the reduction condition.
- We first explain why such a gauge-scalar model is constructed without breaking the gauge symmetry through the gauge-independent description of the Brout-Englert-Higgs mechanism which does not rely on the spontaneous breaking of gauge symmetry.
- This gives the massive Yang-Mills theory with gauge invariant mass term as the complementary gauge-scalar model for the Yang-Mills theory.
- Then we discuss how the numerical simulations for the proposed massive Yang-Mills theory can be performed by taking into account the reduction condition in the complementary gauge-scalar model on a lattice.

Outlook

- We will develop HMC algorithms with reduction condition constraints to investigate the whole parameter space of the gauge coupling β and the mass term γ.
- The gluon mass term simulates the dynamically generated mass to be extracted in the low-energy effective theory of the Yang-Mills theory and plays the role of a new probe to study the phase structure and confinement mechanism.
  - The phase diagram in the β–γ plain.
  - THe gluon mass dependence of string tension.
- We should take care of the fact that massive Yang-Mills models of distinct types are obtained depending on representations of the scalar field.
- For the fundamental representation,the massive Yang-Mills model is expected to have a single confining phase with continuously connecting confining and Higgs regions as suggested by the Fradkin-Shenker continuity.
- For the adjoint representation,the two regions will be separated by the phase transition and become two different phases showing confinement and deconfinement even at zero temperature.