Domain Wall and Overlap Fermions in 2+1D

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The Big Picture

- Lattice theories are needed to look at strongly coupled physics.
- Phase transitions and critical phenomena are important properties to investigate.
- Layered systems such as graphene, of immediate practical significance, may be described by 2+1D physics.
- Different lattice theories of the same continuum theory give different results for the location of critical phenomena.
- This presentation will focus on the locality of overlap and domain wall fermions in 2+1D and the recovery of U(2) symmetry in the continuum limit

Euclidean Continuum Formulation

$$S = S_F + S_G$$

QED3:

$$S_F[\psi_i, \bar{\psi}_i, A] = \int d^3x \; \bar{\psi}_i(\gamma_\mu(\partial_\mu + iA_\mu) + m)\psi_i$$

$$S_G[A] = \frac{1}{4g^2} \int d^3x \; F_{\mu\nu}F_{\mu\nu}$$

Thirring:

$$S[\psi_i, \bar{\psi}_i] = \int d^3x \; \bar{\psi}_i(\gamma_\mu\partial_\mu + m)\psi_i + \frac{g^2}{2N}(\bar{\psi}_i\gamma_\mu\psi_i)^2$$

$$S_G[A] = \frac{N}{2g^2} \int d^3x \; A_\mu^2$$

3+1D Continuum - symmetries

$$\Psi \to e^{i\alpha}\Psi \; ; \; \bar{\Psi} \to \bar{\Psi} e^{-i\alpha}$$

$$\Psi \to e^{i\alpha\gamma_5}\Psi \; ; \; \bar{\Psi} \to \bar{\Psi} e^{i\alpha\gamma_5}$$

$$U(1) \otimes U(1) \to U(1)$$

$$\gamma_5 D + D\gamma_5 = 0$$

2+1D Continuum – adds further symmetries

$$\Psi \to e^{i\alpha\gamma_3\gamma_5}\Psi \; ; \; \bar{\Psi} \to \bar{\Psi} e^{-i\alpha\gamma_3\gamma_5}$$

$$\Psi \to e^{i\alpha\gamma_3}\Psi \; ; \; \bar{\Psi} \to \bar{\Psi} e^{i\alpha\gamma_3}$$

$$U(2) \to U(1) \otimes U(1)$$

$$\gamma_3 D + D\gamma_3 = 0$$

… enabling the anti-hermitian masses

$$m \to -im\gamma_3 \qquad m \to -im\gamma_5$$

2+1D on the lattice – Ginsparg Wilson relations

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D$$

$$\gamma_3 D + D\gamma_3 = aD\gamma_3 D$$

$$\Psi \to e^{i\alpha\gamma_5(1-\frac{aD}{2})}\Psi \; ; \; \bar{\Psi} \to \bar{\Psi}e^{i\alpha\gamma_5(1-\frac{aD}{2})}$$

$$\Psi \to e^{i\alpha\gamma_3(1-\frac{aD}{2})}\Psi \; ; \; \bar{\Psi} \to \bar{\Psi}e^{i\alpha\gamma_3(1-\frac{aD}{2})}$$

Recovery of U(2) requires ...

$$aD \to 0 \text{ as } a \to 0$$

which requires D to be exponentially local ...

Overlap Fermions in 2+1D

$$D_{OL} = \frac{1+m}{2} + \frac{1-m}{2}V$$ **Standard**

$$D_{OL}^{3} = \frac{1-im\gamma_3}{2} + \frac{1+im\gamma_3}{2}V$$ **Alt. mass**

$$V = \mathrm{sgn}(H)$$ ***... must be approximated ...***

$$H_W = \gamma_{3/5} D_W$$ **Wilson kernel**

$$H_S = \gamma_{3/5}\frac{D_W}{2+D_W}$$ **Shamir kernel**

Truncated Overlap

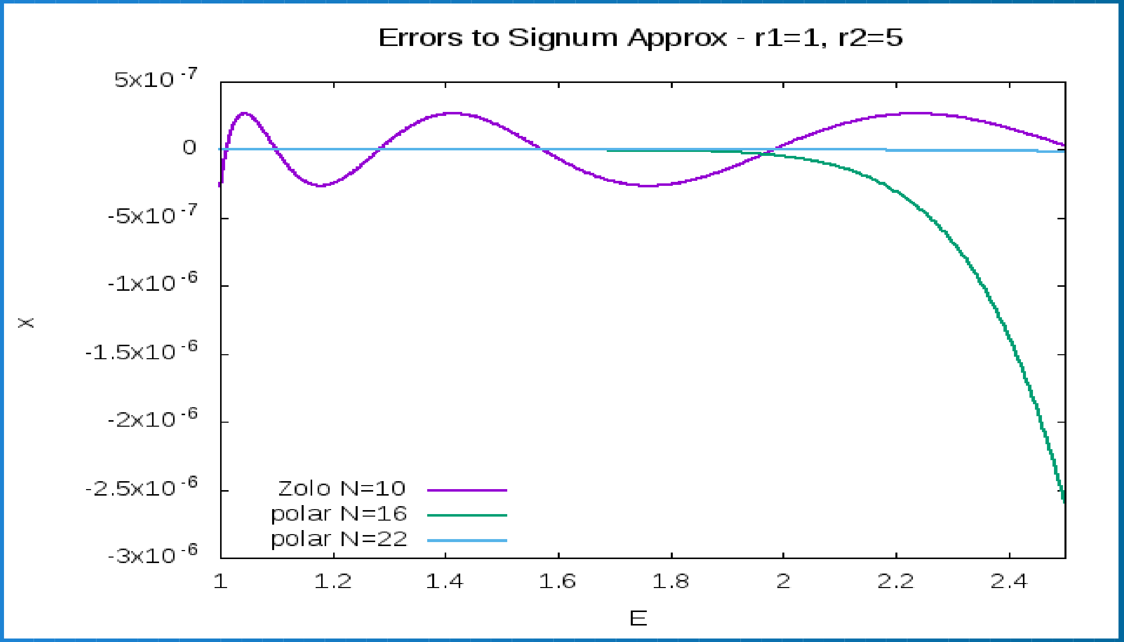
- Approximate sign function with rational function

$$\mathrm{sgn}(x) = cx\frac{\prod_i x^2 - n_i}{\prod_j x^2 - d_j}$$

$$\mathrm{sgn}(x) \approx \tanh(n\tanh^{-1}x) = xn\frac{\prod_{j=1}^{n/2-1}[x^2 + (\tan\frac{j\pi}{n})^2]}{\prod_{j=0}^{n/2-1}[x^2 + (\tan\frac{(j+1/2)\pi}{n})^2]}$$

Hyperbolic Tanh

Zolotarev



Kennedy, A, 2006 , arXiv:hep-lat/0607038

Domain Wall Fermions

With the massive Wilson Dirac operator and projectors ...

$$D_W \equiv D_W(-M) \qquad\qquad P_\pm = \frac{1 \pm \gamma_{3/5}}{2}$$

... the usual domain wall Dirac operator is ...

$$D_{DW}(m) = \begin{pmatrix} D_W + I & -P_- & 0 & mP_+ \\ -P_+ & D_W + I & -P_- & 0 \\ 0 & -P_+ & D_W + I & -P_- \\ mP_- & 0 & -P_+ & D_W + I \end{pmatrix}$$

... and the fermion is found on the walls via ...

$$q(x) = P_+ \Psi(x, N) + P_- \Psi(x, 1)$$
$$\bar{q}(x) = \bar{\Psi}(x, 1) P_+ + \bar{\Psi}(x, N) P_-$$

... allowing for fermionic measurements ... $C = \frac{1}{V} < \bar{q}(x) q(x) >$

Relation between (Truncated) Overlap and Domain Wall Operators

$$K_{DW} \equiv C^{\dagger} D_{DW}^{-1}(1) D_{DW}(m) C$$

$$C = \begin{pmatrix} P_- & P_+ & 0 & 0 \\ 0 & P_- & P_+ & 0 \\ 0 & 0 & P_- & P_+ \\ P_+ & 0 & 0 & P_- \end{pmatrix}$$

$$K_{DW} = \begin{pmatrix} D_{OL}(m) & 0 & 0 & 0 \\ -(1-m)\triangle_2^R & 1 & 0 & 0 \\ -(1-m)\triangle_3^R & 0 & 1 & 0 \\ -(1-m)\triangle_4^R & 0 & 0 & 1 \end{pmatrix}$$

Brower, R, Neff, H, Originos, K, Comput. Phys. Commun. 220 (2017) , arXiv:1206.5214

Alternative mass does not alter form of Pauli-Villars term:

$$K_{DW}^{M3} = C^{\dagger} D_{DW}^{-1}(1) D_{DW}^{M3}(m) C$$

Hands, S, Physics Letters B, Volume: 754, 2016, arXiv:1512.05885

Calculations

- Dynamic Thirring "Gauge" Fields
  - Nf=1 required for theory to have phase transition
  - RHMC (Rational Hybrid Monte Carlo) required for Nf=1
  - Ns=Nt=16

$$\frac{1}{g_c^2} \approx 0.3$$

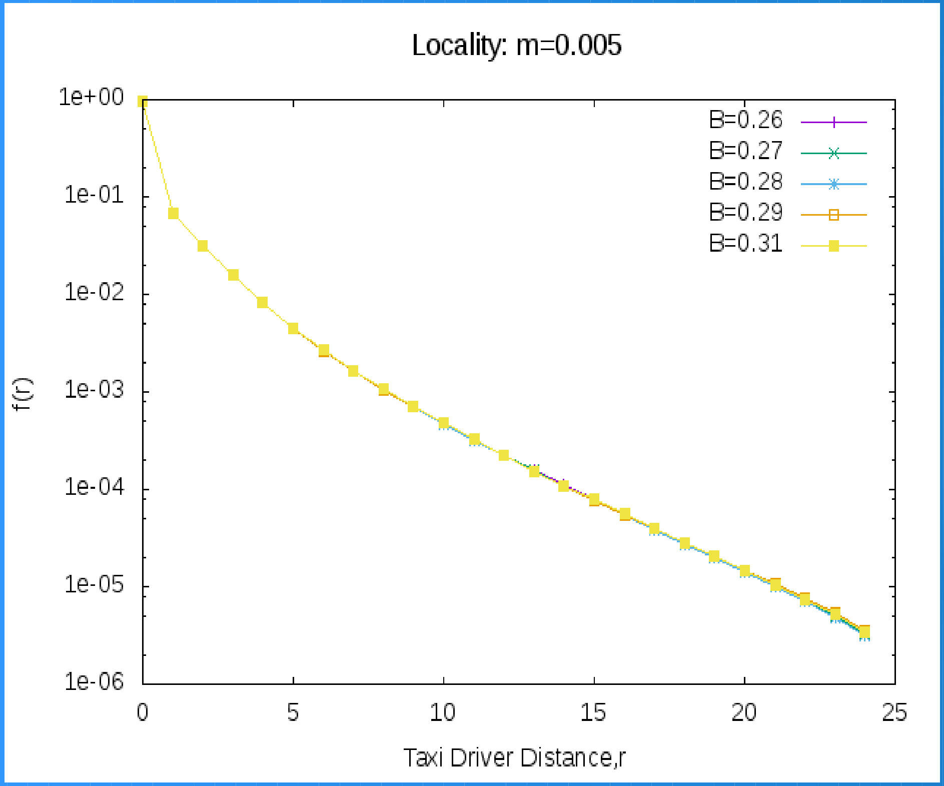
Truncated Overlap Operator
  - Direct Calculation with rational functions
  - Domain wall calculation with K_{DW}

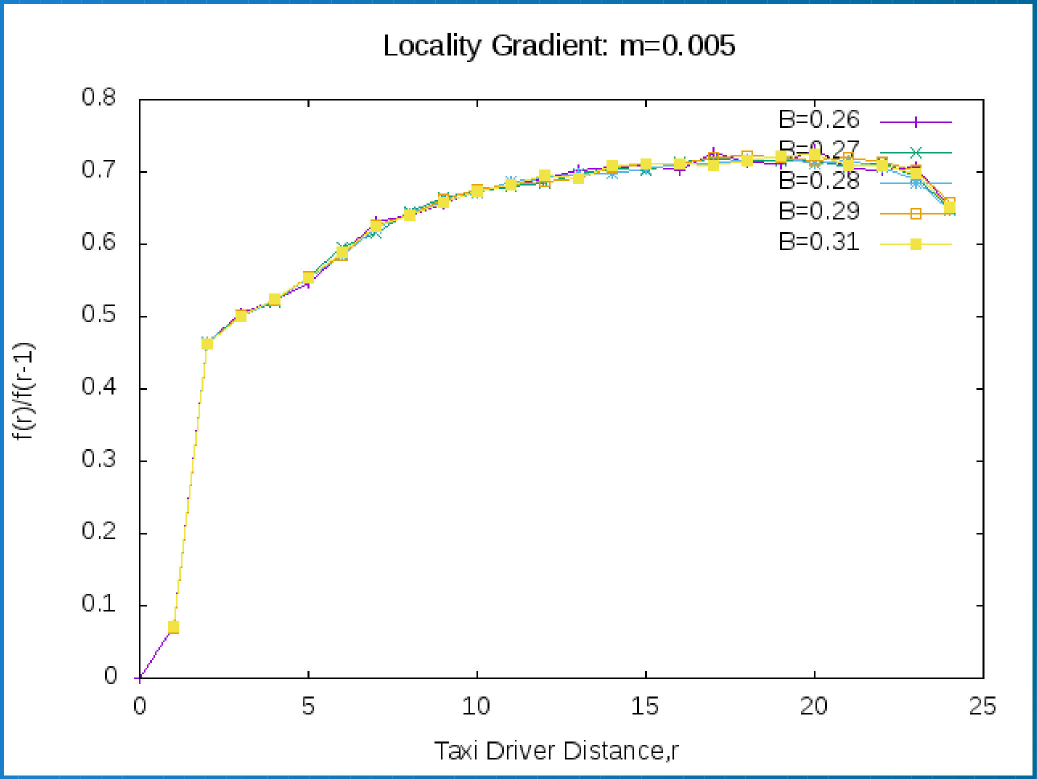
Hands, S, Phys. Rev. D 99, 2019, arXiv:1811.04818
Hands, S, Mesiti, M, Worthy, J, in preparation.

Locality

$$\psi(x) = D_{OL}\eta(x)$$

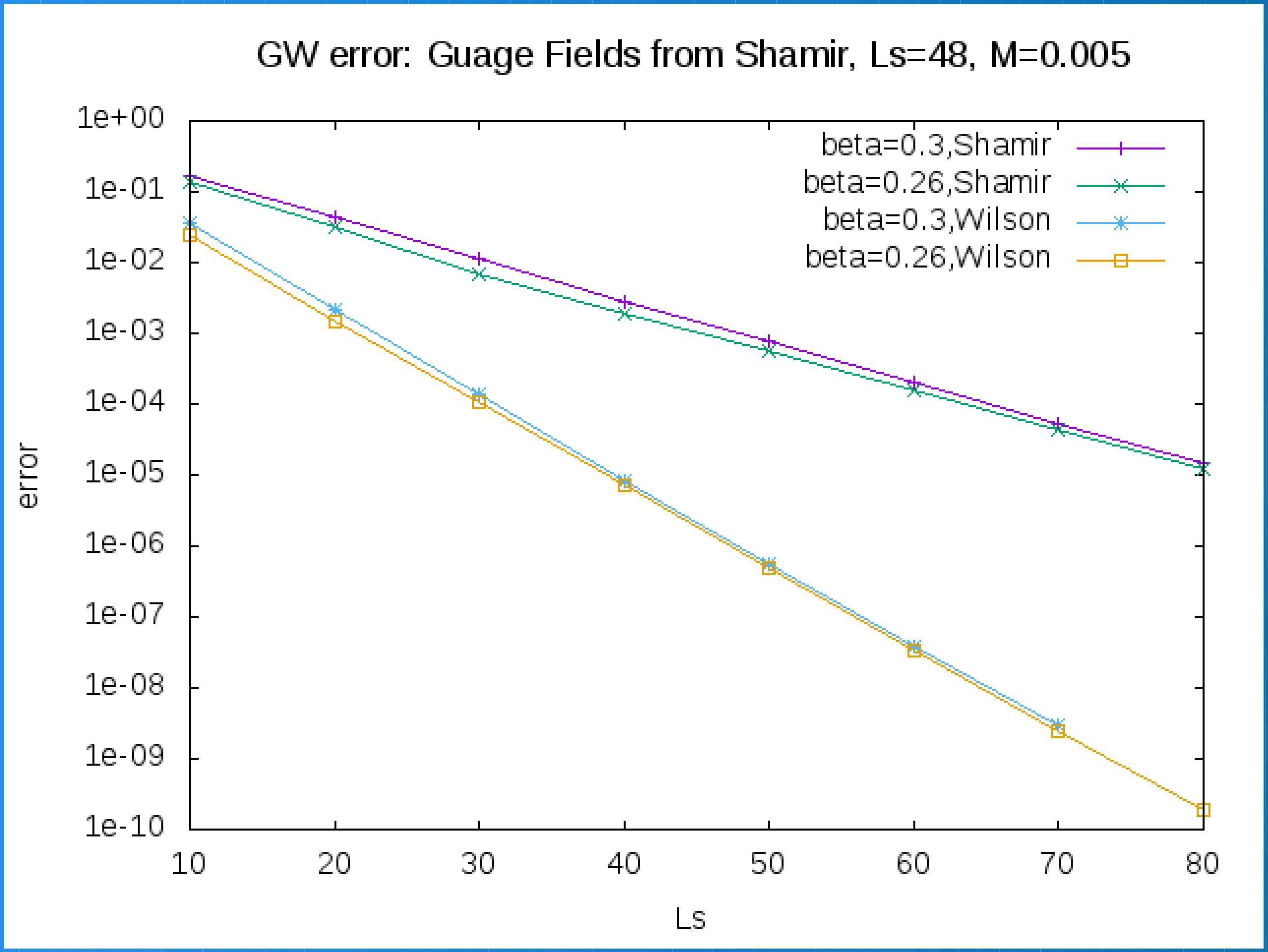
$$f(r) = \max\{||\psi(x)||_2 : ||x-y||_t = r\}$$





Ginsparg-Wilson error

$$\mathrm{err}_{GW} = ||(\gamma_3 D_{OL} + D_{OL}\gamma_3 - 2D_{OL}\gamma_3 D_{OL})\phi||_{\mathrm{inf}}$$



Summary

- Recovery of U(2) symmetry for GW Dirac operators in the continuum limit requires exponential locality
- We numerically demonstrated
  - locality of the (truncated) overlap operator, and hence of domain wall fermions near a critical region with a 2+1D Thirring model.
  - the recovery of the GW relations in the large Ls limit of the (truncated) overlap.

Formulation I - Partition Related Functions

$$Z \equiv \int \mathcal{D}[U] \exp(-S_G[U]) Z_F \quad Z_F \equiv \int \mathcal{D}[\Psi, \bar{\Psi}] \exp(-\bar{\Psi} D \Psi)$$

$$< O >_F \equiv \int \mathcal{D}[\Psi, \bar{\Psi}] O_F[\Psi, \bar{\Psi}] \exp(-\bar{\Psi} D \Psi)$$

$$< O >_G \equiv \int \mathcal{D}[U] O[U] \exp(-S_G[U])$$

$$< O > \equiv \frac{1}{Z} << O >_F >_G$$

$$Z = < Z_F >_G \qquad \frac{\partial Z_F}{\partial m} = -\mathrm{Tr}[D^m D^{-1}] \det(D)$$

$$\frac{\partial^2 Z_F}{\partial m^2} = (\mathrm{Tr}[D^m D^{-1}]^2 - \mathrm{Tr}[(D^m D^{-1})^2]) \det(D)$$

Formulation II - Condensate

$$D = D^0 + mD^m \quad D_{OL}^{M} = \frac{1}{2} - \frac{1}{2}V \quad D_{OL}^{M3} = \frac{-i\gamma_3}{2} + \frac{i\gamma_3}{2}V$$

$$C \equiv \frac{\partial \ln Z}{\partial m} =< \bar{\psi} D^m \psi >= \frac{1}{Z} < \mathrm{Tr}[D^m D^{-1}] Z_F >_G = \frac{1}{Z} < T Z_F >_G$$

$$T_{OL} = \mathrm{Tr}[\frac{1}{1-m}(D_{OL}^{-1} - 1)] \quad T_{OL}^3 = \mathrm{Tr}[\frac{1}{i\gamma_3 - m}((D_{OL}^3)^{-1} - 1)]$$

$$Z_F^{OL} = \det[D_{DW}(1)] Z_F^{DW}$$

$$C^{DW} = \frac{1}{Z^{DW}} < \frac{< \bar{\psi} D_{OL}^m \psi >_F}{\det[D_{DW}(1)]} >_G$$

Formulation III - Susceptibility

$$\chi \equiv \frac{\partial C}{\partial m} = \frac{\partial^2 \ln Z}{\partial m^2} = < \bar{\psi} D^m \psi \bar{\psi} D^m \psi > - < \bar{\psi} D^m \psi >^2$$

$$\chi = \frac{1}{Z} < \mathrm{Tr}[D^m D^{-1}]^2 Z_F^2 >_G - \frac{1}{Z} < \mathrm{Tr}[(D^m D^{-1})^2]) Z_F >_G$$

$$- (\frac{1}{Z} < \mathrm{Tr}[D^m D^{-1}] Z_F >_G)^2$$

$$\chi_{OL} = \frac{1}{Z} < T_{OL}^2 Z_F^2 >_G - \frac{1}{Z} < \mathrm{Tr}[\frac{1}{(1-m)^2}(D_{OL}^{-2} + 2D_{OL}^{-1} + 1))] Z_F >_G - C^2$$

$$\chi_{OL}^{M3} = \frac{1}{Z} < (T_{OL}^{M3})^2 Z_F^2 >_G - \frac{1}{Z} < \mathrm{Tr}[\frac{1}{(i\gamma_3 - m)^2}(D_{OL}^{-2} + 2D_{OL}^{-1} + 1))] Z_F >_G - C^2$$

$$\chi^{DW} = \frac{1}{Z^{DW}} < \mathrm{Tr}[R^{OL}]^2 (\frac{Z_F^{OL}}{\det[D_{DW}(1)] >)^2}_G$$

$$- \frac{1}{Z^{DW}} < \mathrm{Tr}[(R^{OL})^2]) \frac{Z_F^{OL}}{\det[D_{DW}(1)]} >_G$$

$$- \frac{1}{(Z^{DW})^2} < \mathrm{Tr}[R^{OL}] \frac{Z_F^{OL}}{\det[D_{DW}(1)]} >_G^2$$