Dirac eigenvalue spectrum and its relation to $U(1)_A$ symmetry breaking in high temperature $N_f=2+1$ QCD

Yu Zhang
Central China Normal University

in collaboration with
H.-T. Ding, S.-T. Li, S. Mukherjee, A. Tomiya and X.-D. Wang

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Outline

➢ Motivation
➢ Chiral & $U(1)_A$ symmetry and Dirac eigenvalue spectrum
➢ Lattice Setup
➢ Results
➢ Summary and Outlook
Motivation

At $T>T_c$, chiral symmetry is restored.

How about the fate of $U(1)_A$ symmetry?

Two possible scenarios:

Substantial $U(1)_A$ breaking

Negligible $U(1)_A$ breaking

This study
Pisarski, Wilczek (1984)

Philipsen, Pinke, PRD 93 (2016) 114507
Chiral & $U(1)_A$ symmetry and susceptibilities

\[ \chi_{\text{5,con}} \quad \pi : \bar{q} \gamma_5 \frac{T}{2} q \quad \text{SU}(2)_L \times \text{SU}(2)_R \quad \sigma : \bar{q} q \quad \chi_{\text{con}} + \chi_{\text{disc}} \]

\[ \chi_{\text{con}} \quad \delta : \bar{q} \frac{T}{2} q \quad \text{SU}(2)_L \times \text{SU}(2)_R \quad \eta : \bar{q} \gamma_5 q \quad \chi_{\text{5,con}} - \chi_{\text{5,disc}} \]

\[
\chi_\pi = \frac{N_f}{4} \frac{1}{V} \left\langle \text{Tr} \left( M^{-1} \gamma_5 M^{-1} \gamma_5 \right) \right\rangle \\
\chi_\delta = \frac{N_f}{4} \frac{1}{V} \left\langle \text{Tr} \left( M^{-1} M^{-1} \right) \right\rangle \\
\chi_{\text{disc}} = \left( \frac{N_f}{4} \right)^2 \frac{1}{V} \left[ \left\langle (\text{Tr} M^{-1})^2 \right\rangle - \left\langle \text{Tr} M^{-1} \right\rangle^2 \right]
\]

in the chirally symmetric phase, $\chi_{\text{disc}}$ can be used to probe the restoration of the $U(1)_A$ symmetry

[ See talk by Fukaya, Tues. 17:20 ]
signatures from Dirac eigenvalue spectrum

\[ \langle \bar{\psi} \psi \rangle = \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2} \xrightarrow{m \to 0} \pi \rho(0) \]

\[ \chi_\pi - \chi_\delta = \frac{N_f}{4} \int_0^\infty d\lambda \frac{4m^2 \rho(\lambda, m)}{(\lambda^2 + m^2)^2} \]

➢ the restoration of SU(2)_L × SU(2)_R symmetry
   • \( \rho(0) = 0 \)
   • Dilute Instanton gas approximation

➢ the restoration of U(1)_A symmetry
   • A gap in the near-zero modes
   • At high \( T \), \( \rho(\lambda) \) obey the Poisson distribution

we aim on determining the mass dependence of \( \rho(\lambda, m) \), so we propose to look at the derivatives of \( \rho(\lambda, m) \) with respect to the mass
Observables representation in terms of $\rho(\lambda)$

\[
\langle \bar{\psi} \psi \rangle = \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m\rho(\lambda, m)}{\lambda^2 + m^2}
\]

\[
\frac{d\langle \bar{\psi} \psi \rangle}{dm} = \frac{N_f}{4} \int_0^\infty d\lambda \rho(\lambda, m) \frac{\partial}{\partial m} \left( \frac{2m}{\lambda^2 + m^2} \right) + \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m \frac{\partial \rho(\lambda, m)}{\partial m}}{m^2 + \lambda^2}
\]

\[
= \chi_{\text{con}} + \chi_{\text{disc}}
\]

\[
\chi_{\text{disc}} = \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m \frac{\partial \rho(\lambda, m)}{\partial m}}{\lambda^2 + m^2}
\]

part of \[
\frac{d\chi_{\text{disc}}}{dm} = \frac{N_f}{4} \int_0^\infty d\lambda \frac{2m \frac{\partial^2 \rho(\lambda, m)}{\partial m^2}}{\lambda^2 + m^2}
\]

$\partial \rho(\lambda, m)/\partial m$ and $\partial^2 \rho(\lambda, m)/\partial m^2$ are related to $\chi_{\text{disc}}$ and $d\chi_{\text{disc}}/dm$
The derivatives of $\rho(\lambda)$ with respect to the quark mass

\[
\frac{\partial \rho(\lambda, m)}{\partial m} = \frac{N_f V}{4} \int_0^\infty d\lambda' \frac{2mC_2(\lambda, \lambda')}{\lambda'^2 + m^2}
\]

\[
\frac{\partial^2 \rho(\lambda, m)}{\partial m^2} = \left( \frac{N_f V}{4} \right)^2 \int_0^\infty d\lambda'' \int_0^\infty d\lambda' \frac{4m^2C_3(\lambda, \lambda', \lambda'')}{(\lambda''^2 + m^2)(\lambda'^2 + m^2)} + \frac{N_f V}{4} \int_0^\infty d\lambda' \frac{2(\lambda'^2 - m^2)C_2(\lambda, \lambda')}{(\lambda'^2 + m^2)^2}
\]

\[
C_2(\lambda, \lambda') = \langle \rho_u(\lambda)\rho_u(\lambda') \rangle - \langle \rho_u(\lambda') \rangle \langle \rho_u(\lambda) \rangle
\]

\[
C_3(\lambda, \lambda', \lambda'') : \text{three-point correlations of } \rho(\lambda) \text{ in } \lambda
\]

We establish an analytic relation between the mass derivatives of $\rho(\lambda)$

to the multi-point correlation functions of $\rho(\lambda)$ between different $\lambda$
Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues

- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

\[
\bar{n}[s, t] \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \sum_{j=0}^{p} g_j^p \gamma_j \langle \xi_r \dagger T_j(A) \xi_r \rangle
\]

- \( T_j \): Chebyshev polynomial
- \( \gamma_j \): coefficient
- \( p \): polynomial order

\[
\rho(\lambda, \delta) = \frac{1}{V} \frac{\bar{n}[\lambda - \delta/2, \lambda + \delta/2]}{\delta} \quad (\lambda \geq \delta/2)
\]

Giusti, Luscher, arXiv: 0812.3638
A. Patella, arXiv: 1204.4432
Di Napoli et al., arXiv: 1308.4275
Itou et al., arXiv: 1411.1155
Fodor et al., arXiv: 1605.08091
Cossu et al., arXiv: 1601.00744
Lattice Setup

➢ **Actions:**
- Tree level improved gauge action
- Highly improved staggered quark action

➢ **Lattice size:**
- \( N_{\tau} = 8, \ N_{\sigma} = 32, 40, 56 \)
- \( N_{\tau} = 12, \ N_{\sigma} = 48, 60, 72 \)
- \( N_{\tau} = 16, \ N_{\sigma} = 64 \)

➢ **Quark mass:**
- \( m_s \): set to its physical value
- \( m_l/m_s = 1/20, 1/27, 1/40, 1/80 \) (\( m_\pi = 160, 140, 110, 80 \) MeV)

➢ **Temperature:**
- \( T \approx 207 \) MeV
Chiral observables

\[ m_s < \bar{\psi}\psi > / T^4, \ T = 207\text{MeV} \]

\[ m_s^2 (\chi_\pi - \chi_\delta) / T^4, \ T = 207\text{MeV} \]

- \( \bar{\psi}\psi \) and \( \chi_\pi - \chi_\delta \) can be reproduced well from the eigenvalue spectrum.
- In the chiral limit, chiral symmetry is restored but \( U(1)_A \) symmetry is not at \( T \approx 207\text{MeV} \).
- \( \chi_\pi - \chi_\delta \) becomes larger towards continuum limit.
disconnected chiral susceptibilities

in the chiral limit, $\chi_{\text{disc}} \neq 0$

$U(1)_A$ symmetry is not restored

$\chi_{\text{disc}}$ can only be reproduced from the correlation function of eigenvalues but not from Poisson distribution

$\chi_{\text{disc}}/T^2$, $T = 207\text{MeV}$, $N_T = 12$

- stochastic method from $M^{-1}$
- constructed from $\frac{\partial \rho(\lambda, m)}{\partial m}$
- Poisson
Part of quark mass derivative of the disconnected chiral susceptibilities

The consistent result from two methods make us confident to extract information on \( \frac{\partial^2 \rho(\lambda, m)}{\partial m^2} \)
quark mass dependence of $\rho(\lambda)$

In the near-zero modes, quark mass dependence is significant
quark mass derivatives of $\rho(\lambda)$

$$\frac{\partial \rho(\lambda, m)}{\partial m} \bigg|_m \quad \text{and} \quad \frac{\partial^2 \rho(\lambda, m)}{\partial m^2} \bigg|_m$$

(a) $T = 207$ MeV, $N_\tau = 12$

$$m_\pi = 160 \text{ MeV}$$
$$m_\pi = 140 \text{ MeV}$$
$$m_\pi = 110 \text{ MeV}$$
$$m_\pi = 80 \text{ MeV}$$

(b) $T = 207$ MeV, $N_\tau = 12$

$$m_\pi = 160 \text{ MeV}$$
$$m_\pi = 140 \text{ MeV}$$
$$m_\pi = 110 \text{ MeV}$$
$$m_\pi = 80 \text{ MeV}$$

In the near-zero modes, $\rho(\lambda, m) \approx m^2 \delta(\lambda)$ have the same behavior as dilute instanton gas approximation (DIGA).
Summary

➢ We established the relation between the spectrum derivatives with respect to quark mass and the correlation function of the eigenvalues

➢ $U(1)_A$ symmetry is still broken at $T \approx 207\text{MeV}$ as the non-vanishing disconnected susceptibility

➢ The spectrum does not obey the Poisson distribution

➢ The spectrum behaves as $m^2\delta(\lambda)$ in the near-zero modes, as indicated from the dilute instanton gas approximation
Outlook

➢ Check the volume dependence
➢ Goes to the continuum limit
backup
possible behaviors for $\rho(\lambda,m)$

At high $T$, if $\rho(\lambda,m)$ obey the Poisson distribution

$$C_2(\lambda, \lambda') = \frac{1}{V} \langle \rho_u(\lambda) \rangle \delta(\lambda - \lambda') - \frac{1}{N} \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle$$

then

$$\chi_{\text{disc}} = \frac{N_f}{4} (\chi_\pi - \chi_\delta) - \frac{V}{N} \langle \bar{\psi} \psi \rangle^2$$

$(N$ is the number of nonzero Dirac eigenvalue pairs)
\[ C_2(\lambda, \lambda') = \langle \rho_u(\lambda)\rho_u(\lambda') \rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \]

\[ = \left( \frac{1}{V} \right)^2 \left\langle \sum_{k=1}^{N} \delta(\lambda - \lambda_k) \sum_{l=1}^{N} \delta(\lambda' - \lambda_l) \right\rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \]

\[ = \left( \frac{1}{V} \right)^2 \left\langle \sum_{k=1}^{N} \delta(\lambda - \lambda_k) \delta(\lambda' - \lambda_k) \right\rangle + \left( \frac{1}{V} \right)^2 \left\langle \sum_{k=1}^{N} \delta(\lambda - \lambda_k) \sum_{l \neq k} \delta(\lambda' - \lambda_l) \right\rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \]

\[ = \frac{1}{V} \langle \rho_u(\lambda) \rangle \delta(\lambda - \lambda') + \left( \frac{1}{V} \right)^2 \left\langle \sum_{k=1}^{N} \delta(\lambda - \lambda_k) \right\rangle \left\langle \sum_{l \neq k} \delta(\lambda' - \lambda_l) \right\rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \]

\[ = \frac{1}{V} \left\langle \sum_{l \neq k} \delta(\lambda' - \lambda_l) \right\rangle = \frac{N - 1}{N} \langle \rho_u(\lambda') \rangle \quad (N = V/2) \]

\[ C_2(\lambda, \lambda') = \frac{1}{V} \langle \rho_u(\lambda) \rangle \delta(\lambda - \lambda') - \frac{1}{N} \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \]
Three point correlations of $\rho(\lambda)$

$$C_3(\lambda, \lambda', \lambda'') = \langle \rho_u(\lambda)\rho_u(\lambda')\rho_u(\lambda'') \rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda')\rho_u(\lambda'') \rangle$$

$$- \langle \rho_u(\lambda') \rangle \langle \rho_u(\lambda)\rho_u(\lambda'') \rangle - \langle \rho_u(\lambda'') \rangle \langle \rho_u(\lambda)\rho_u(\lambda') \rangle \quad (1)$$

$$+ 2\langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \langle \rho_u(\lambda'') \rangle$$
Time history of the topological charge