Towards color superconductivity on the lattice
- perturbative predictions and the complex Langevin method -

Takeru Yokota

Institute for Solid State Physics, Univ. of Tokyo

Collaborators

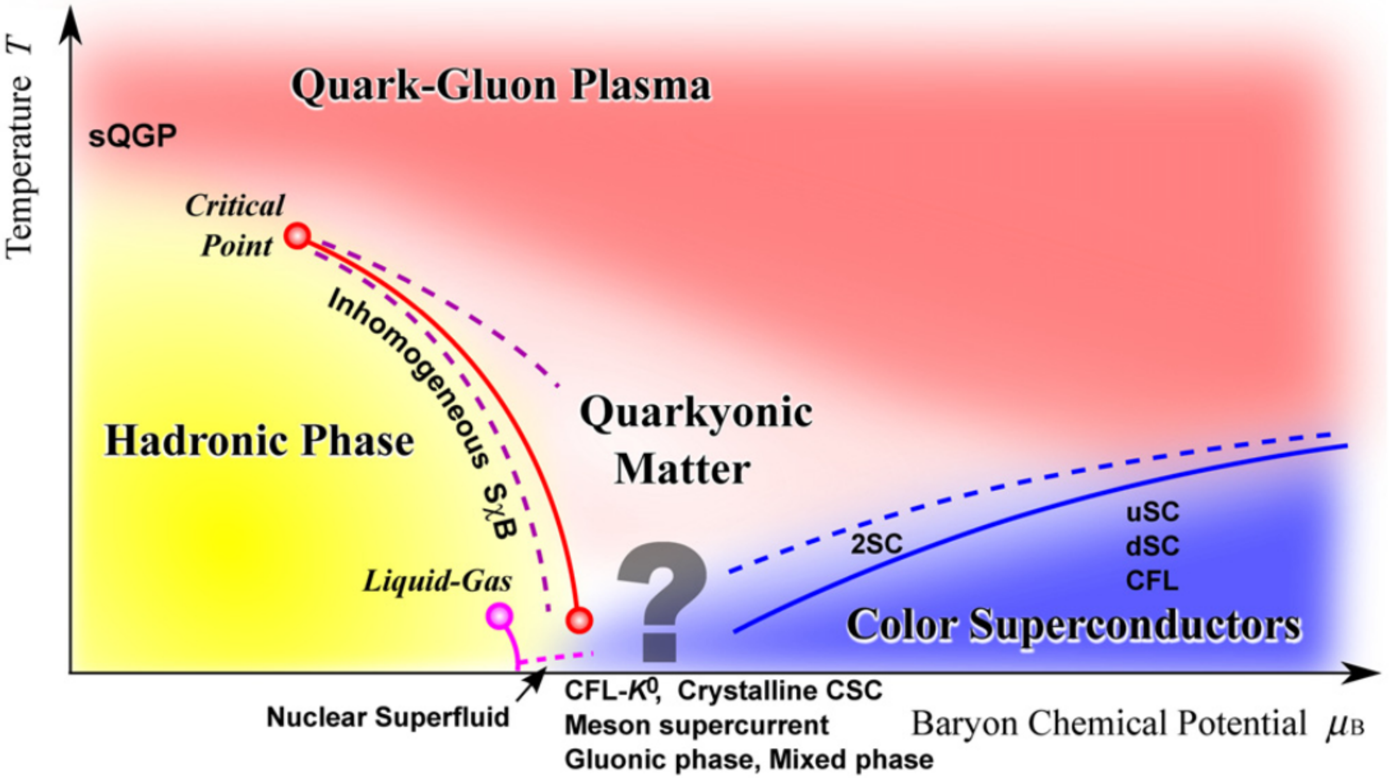
Yuhma Asano (KEK), Yuta Ito (Tokuyama National College of Technology), Hideo Matsufuru (KEK), Yusuke Namekawa (YITP), Jun Nishimura (KEK, SOKENDAI), Asato Tsuchiya (Shizuoka Univ.), Shoichiro Tsutsui (RIKEN)

Asia-Pacific Symposium for Lattice Field Theory (APLAT 2020), online, 2020/8/7

Contents

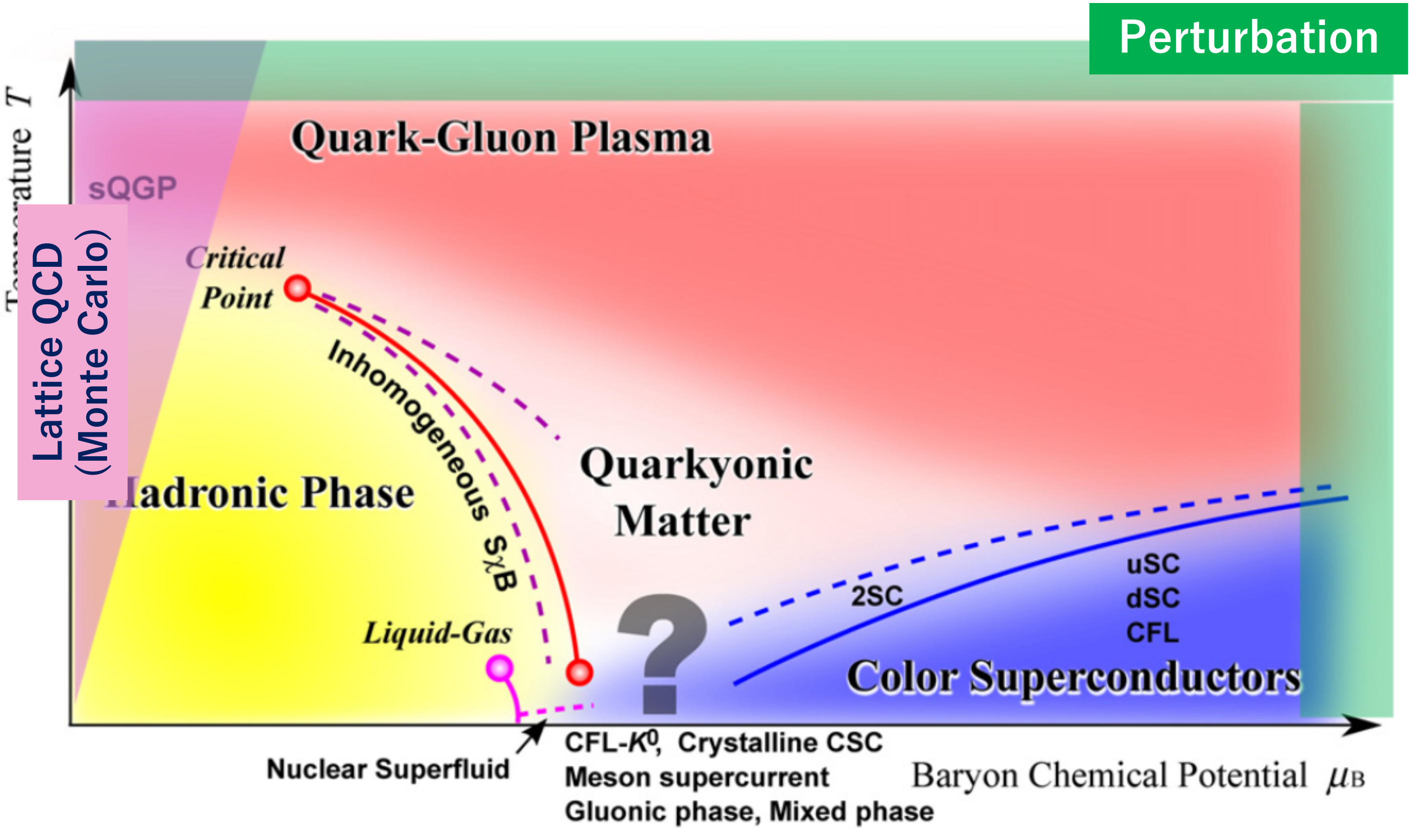
Contents

QCD phase diagram



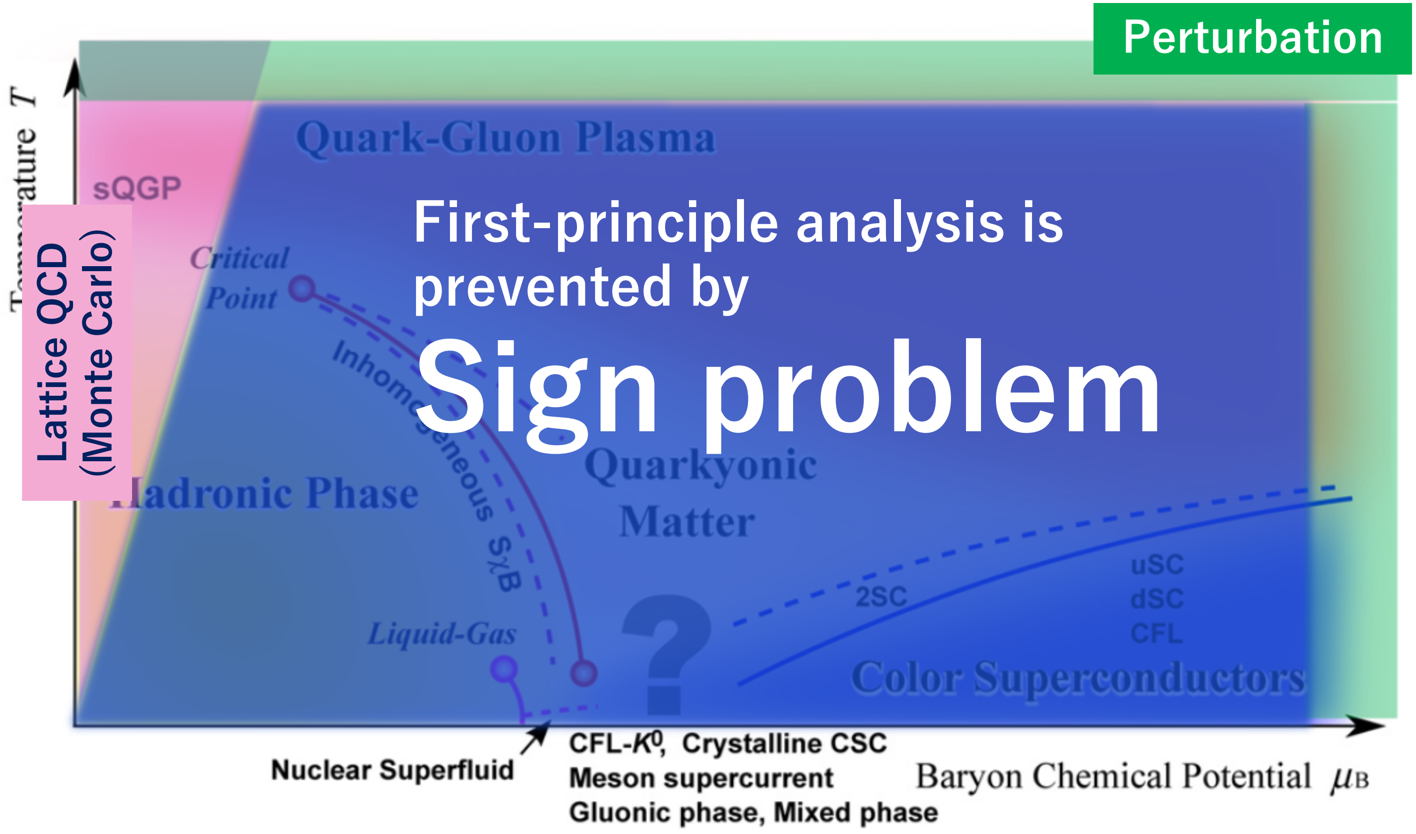
Fukushima, Hatsuda, RPP (2011)

QCD phase diagram



Fukushima, Hatsuda, RPP (2011)

QCD phase diagram



Perturbation

Lattice QCD (Monte Carlo)

First-principle analysis is prevented by

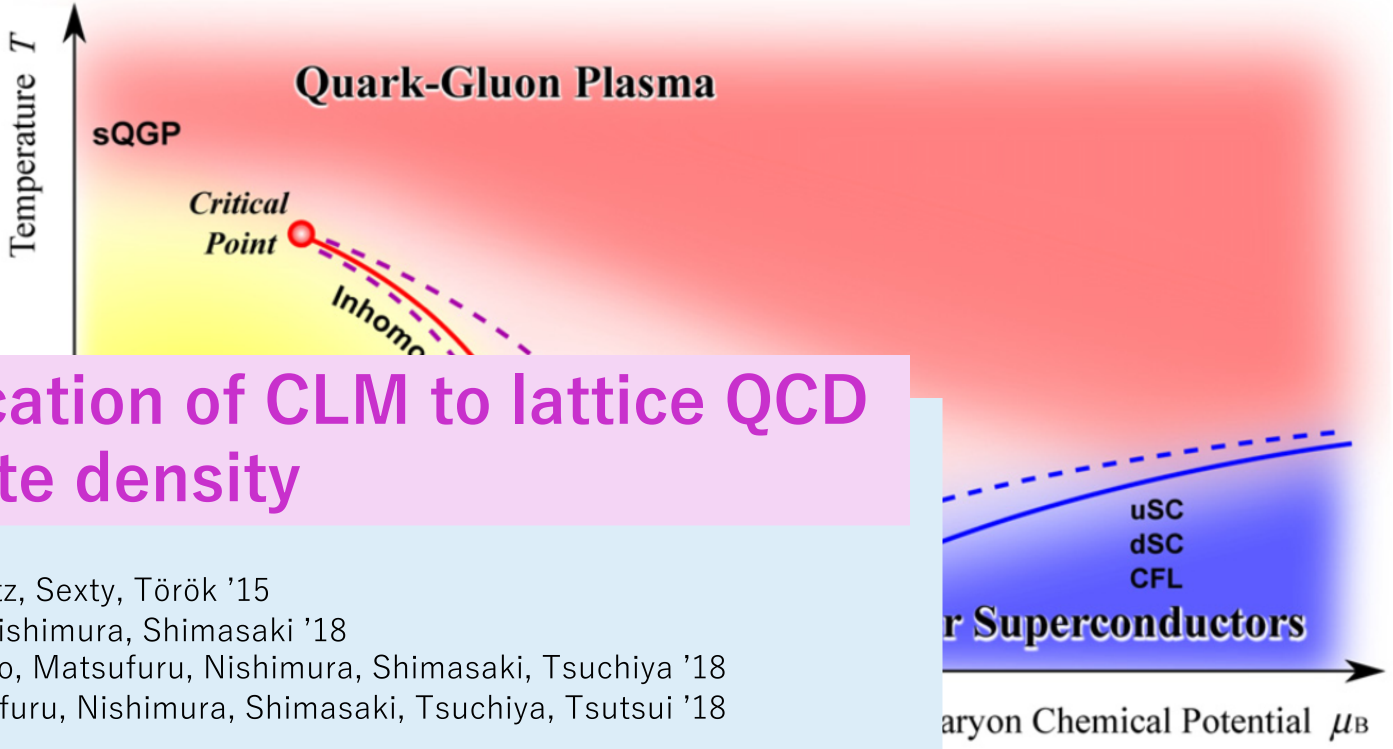
Sign problem

Fukushima, Hatsuda, RPP (2011)

QCD phase diagram

Methods to overcome sign problems

Lefschetz thimble, tensor renormaliztion group, complex Langevin method (CLM)…

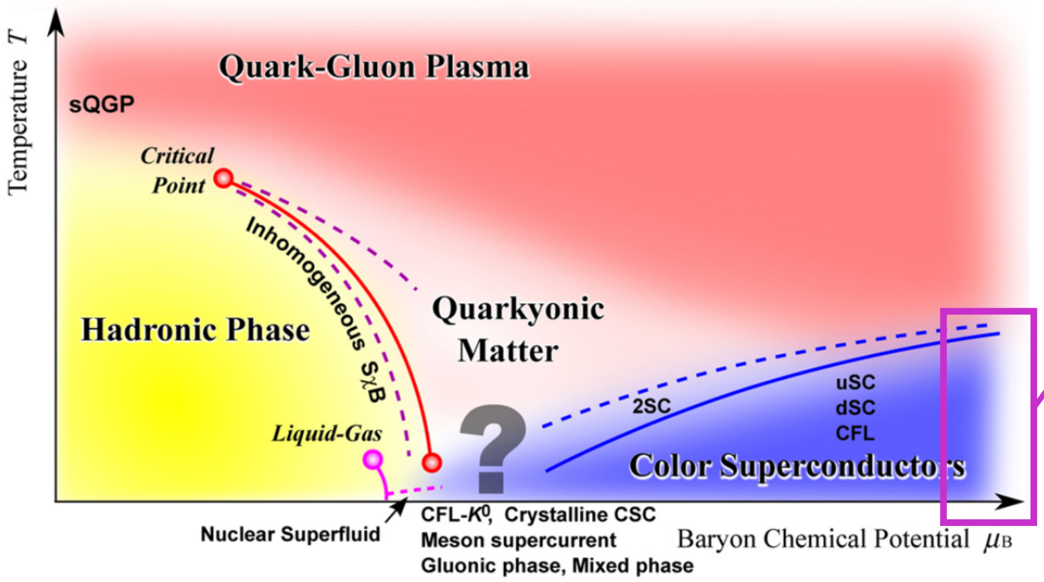


Application of CLM to lattice QCD at finite density

Sexty '14
Fodor, Katz, Sexty, Török '15
Nagata, Nishimura, Shimasaki '18
Tsutsui, Ito, Matsufuru, Nishimura, Shimasaki, Tsuchiya '18
Ito, Matsufuru, Nishimura, Shimasaki, Tsuchiya, Tsutsui '18
Sexty '19
Kogut, Sinclair '19
Sinclair, Kogut '19
Tsutsui, Ito, Matsufuru, Nishimura, Shimasaki, Tsuchiya '19
Scherzer, Sexty, Stamatescu '20
Ito, Matsufuru, Namekawa, Nishimura, Shimasaki, Tsuchiya, Tsutsui '20

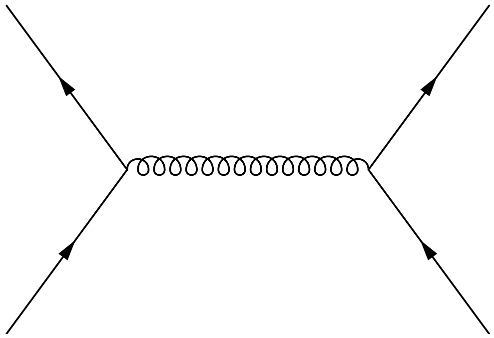
Fukushima, Hatsuda, RPP (2011)

Color superconductivity (CSC)



Barrois, NPB (1977); PhD thesis (1979)
Frautschi (1978), Bailin, Love, PR (1984)
Alford, Rajagopal, Wilczek, PLB (1998)

High μ_q (weak coupling), low T



Attractive in color-antitriplet channel

➡ Cooper instability

CSC with various symmetries of
Cooper pairs has been predicted
(2SC, CFL⋯)

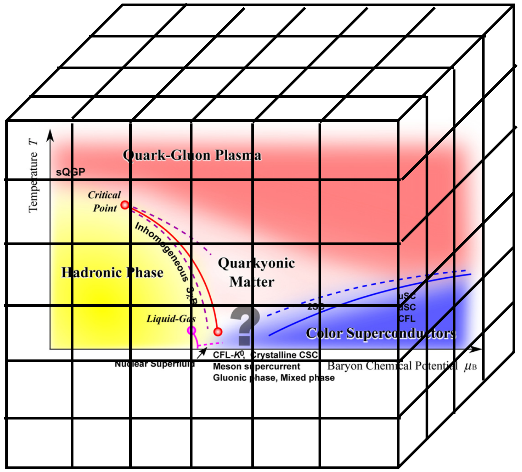
CSC on the lattice?

Analysis of CSC on the lattice by first-principle methods such as CLM?

Reasonable starting point:

high β region (small volume) $\beta = 2N_c/g_0^2$

Asymptotic freedom → weak coupling



But the region in which the CSC occurs has not been predicted so far based on lattice QCD.

c.f.) Calculation in effective models (e.g. NJL) in finite box

Hands, Walters, PLB (2002)

Amore, Birse, McGovern, Walet, PRD (2002)

Aim

We give the **perturbative prediction of the parameter region in Lattice QCD** for the first time.

- Derive the gap equation for Cooper pair with general shape and symmetry.
- Derive the condition to determine the critical β
    - The shape and symmetry of Cooper pairs also can be determined with our calculation method.
- Calculate critical β and give the prediction for the phase diagram on (μ_q, β)-plane.

Such a calculation in general setup is possible because **only finite number of coupled equations appear on the lattice**.
(↔functional equation in continuum case)

We also present some preliminary results of the CLM in the large β regime.

Contents

- Motivation
    - QCD phase diagram and sign problem
    - Possibility of investigating color superconductivity (CSC) on the lattice
- Perturbative predictions
    - Gap equation
    - Prediction of transition points
    - Phase diagram on (μ_q, β)-plane
- Current status of our analysis by the complex Langevin method
- Summary

Nambu-Gorkov formalism

Action for $N_{\mathrm{f}} = 4$ Staggered fermion

$$S = S_{\mathrm{fermi}} + S_{\mathrm{int}} + S_{\mathrm{gluon}}$$

$$S_{\mathrm{fermi}} = \sum_{N,N',\rho,\rho',a,a'} \overline{\chi}_{\rho}^{a}(N) D_{NN',\rho\rho'}^{aa'}(\mu_{\mathrm{q}}, m_{\mathrm{q}}) \chi_{\rho'}^{a'}(N')$$

Staggered fermion $\chi_{\rho}^{a}(N) = \chi^{a}(2N+\rho)$
$[\rho_\mu = 0$ or 1 $(\mu = 1\sim4)$, a: color index$]$

$$\Downarrow$$

Nambu spinor

$$\Psi_{\rho}^{a}(N) = \begin{pmatrix} \chi_{\rho}^{a}(N) \\ \overline{\chi}_{\rho}^{a}(N) \end{pmatrix}$$

$$S_{\mathrm{fermi}} = \frac{1}{2} \sum_{N,N',\rho,\rho',a,a'} \overline{\Psi}_{\rho}^{a}(N) \mathbf{D}_{NN',\rho\rho'}^{aa'} \Psi_{\rho'}^{a'}(N')$$

$$\mathbf{D}_{NN',\rho\rho'}^{aa'} = \begin{pmatrix} D_{NN',\rho\rho'}^{aa'}(\mu_{\mathrm{q}}, m_{\mathrm{q}}) & 0 \\ 0 & D_{NN',\rho\rho'}^{aa'}(-\mu_{\mathrm{q}}, -m_{\mathrm{q}}) \end{pmatrix}$$

Anomalous self-energy

propagator

$$\boldsymbol{\Sigma}_{NN',\rho\rho'}^{aa'} = \begin{pmatrix} \Sigma_{11,NN',\rho\rho'}^{aa'} & \Sigma_{12,NN',\rho\rho'}^{aa'} \\ \Sigma_{21,NN',\rho\rho'}^{aa'} & \Sigma_{22,NN',\rho\rho'}^{aa'} \end{pmatrix} \Rightarrow \begin{pmatrix} S_{11,qq',\rho\rho'}^{aa'} & S_{12,qq',\rho\rho'}^{aa'} \\ S_{21,qq',\rho\rho'}^{aa'} & S_{22,qq',\rho\rho'}^{aa'} \end{pmatrix} = [(\tilde{\mathbf{D}} + \tilde{\boldsymbol{\Sigma}})^{-1}]_{qq',\rho\rho'}^{aa'}$$

$$\begin{pmatrix} \langle \overline{\chi}\chi \rangle & \langle \chi\chi \rangle \\ \langle \overline{\chi}\,\overline{\chi} \rangle & \langle \chi\overline{\chi} \rangle \end{pmatrix}$$

If $\Sigma_{12(21)} \neq 0$, CSC emerges ($S_{12(21)} \neq 0$)

In the case of continuum QCD: Bailin, Love, PR (1984)⋯

Gap equation

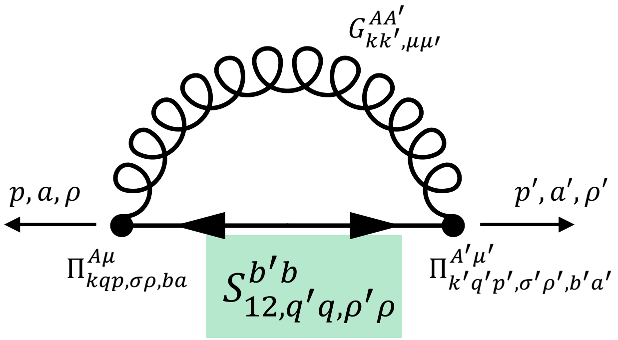
Gap equation: self-consistent equation to determine $\Sigma_{12\,(21)}$

We ignore $\Sigma_{11(22)}$ (higher-order contribution in the gap eq.)

$$\boldsymbol{\Sigma}^{aa'}_{NN',\rho\rho'} \approx \begin{pmatrix} 0 & \Sigma^{aa'}_{12,NN',\rho\rho'} \\ \Sigma^{aa'}_{21,NN',\rho\rho'} & 0 \end{pmatrix}$$

Gap equation (lowest-order, momentum space)

$$\tilde{\Sigma}^{a'a}_{12,p'p,\rho'\rho} =$$



$+O(g_0^4)$

determined by $\tilde{\Sigma}_{12(21)}$

$G^{AA'}_{kk',\mu\mu'}$: gluon propagator

$\Pi^{A\mu}_{kqp,\sigma\rho,ba}$: 2quark-gluon vertex

$S^{b'b}_{12,q'q,\rho'\rho}$: anomalous propagator

If this eq. has $\Sigma_{12(21)} \neq 0$ solution, CSC emerges.

Transition point

At the transition point, $S^{a'a}_{12,q'q,\rho'\rho} \approx -[\tilde{D}^{-1}_{11}\tilde{\Sigma}_{12}\tilde{D}^{-1}_{22}]^{a'a}_{q'q,\rho'\rho} + \mathcal{O}((\Sigma_{12(21)})^2)$ since $\Sigma_{12(21)} \to 0$

Gap eq. is reduced to linear equation

Gap eq. for color antitriplet channel (Σ^{-}_{12})

$$\sum_{q,\tau,\tau'} \left(\mathbf{1} - \frac{1}{\beta_c}\mathcal{M}'\right)_{(p\rho\rho'),(q\tau\tau')} \mathcal{V}_{(q\tau\tau')} = 0$$

Critical β

$\mathbf{1} - \frac{1}{\beta_c}\mathcal{M}'$ should have zero eigenvalue.

($\mathcal{V} \neq \mathbf{0}$ solution)

$\beta_c = \lambda'_{\max}$ (maximum eigenvalue of $\mathcal{M}'$)

Corresponding eigenvector $\mathcal{V}$ determines Σ^{-}_{12}

(i.e., shape and symmetry of Cooper pairs)

cf. another derivation: Condition for T-matrix being singular (argument like Thouless criterion) Thouless, AP (1960)

We use power iteration to calculate $\lambda'_{\max}$ & $\mathcal{V}$.

$$\mathcal{M}'_{(p\rho\rho'),(q\tau\tau')} = \frac{N_c+1}{V} \sum_{k\in\mathrm{BZ}} \frac{\tilde{\delta}_{2(q-p-k)}}{4\sum_\nu \sin^2\frac{k_\nu}{2}} \sum_\mu \left|\cos\left(\frac{\overline{p}_\mu+\overline{q}_\mu}{2}\right)\right|^2$$

$$\times \frac{\sum_\alpha[\Gamma^\alpha\Gamma^\mu]_{\rho'\tau'}\sin\overline{p}_\alpha + im_{\mathrm{q}}\Gamma^\mu_{\rho'\tau'}}{\sum_\nu \sin^2\overline{p}_\nu + m_{\mathrm{q}}^2} \frac{\sum_\beta[\Gamma^\mu\Gamma^\beta]_{\rho\tau}\sin\overline{q}^*_\beta - im_{\mathrm{q}}\Gamma^\mu_{\rho\tau}}{\sum_\nu \sin^2\overline{q}^*_\nu + m_{\mathrm{q}}^2}$$

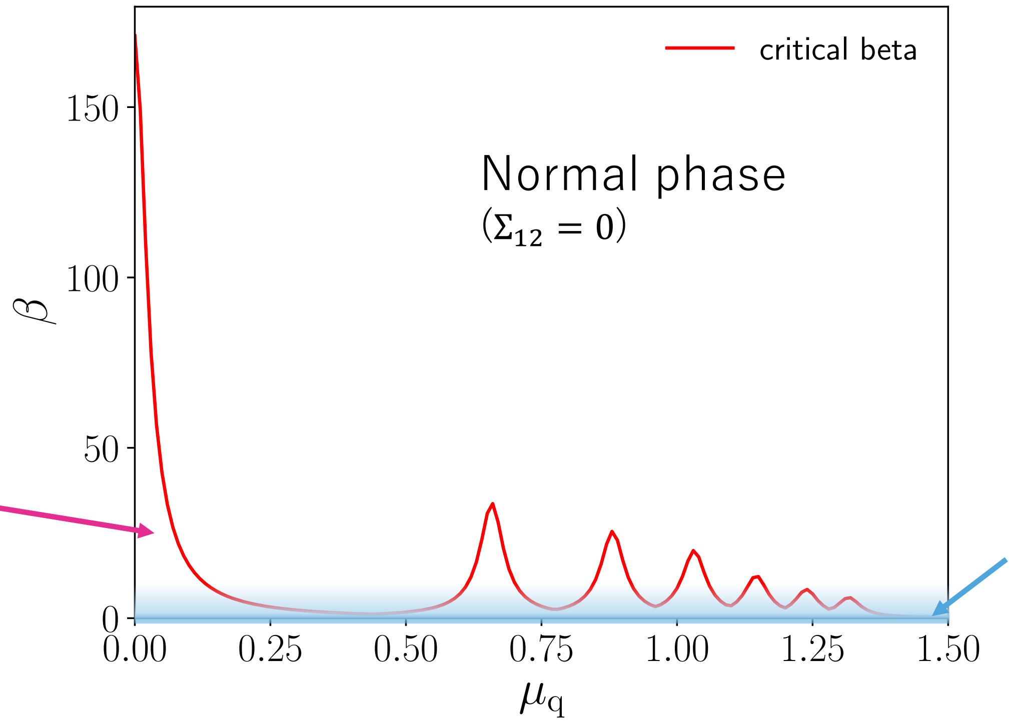
$*$ We omit $\boldsymbol{k} = \mathbf{0}$ mode by hand to avoid divergence caused by lattice artifact

$$\mathcal{V}_{(q\tau\tau')} = [L^{-1}\tilde{\Sigma}^{-}_{12}]_{q\tau'\tau}$$

Prediction of phase diagram in $(\boldsymbol{\mu}_{\mathbf{q}}, \boldsymbol{\beta})$-plane

$8^3 \times 128$ lattice

$m_q = 0$



CSC phase
$(\Sigma_{12} \neq 0)$

Non-perturbative region (small β)

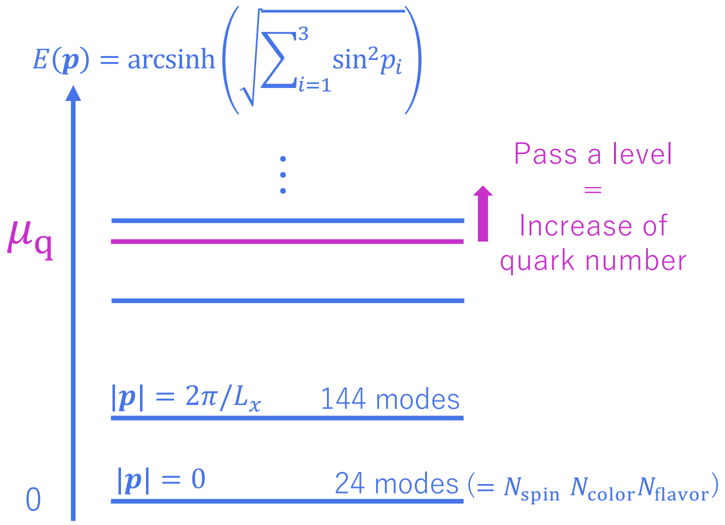
- Peak structure: CSC phase spreads in perturbative region (large β) at specific $\mu_{\mathbf{q}}$.

Origin of peak structure

The origin of peak is elucidated by comparing with the results of quark number

Energy levels of quarks

(discretized because $\boldsymbol{p}$ is discretized in finite volume)



$$E(\boldsymbol{p}) = \mathrm{arcsinh}\left(\sqrt{\sum_{i=1}^{3} \sin^2 p_i}\right)$$

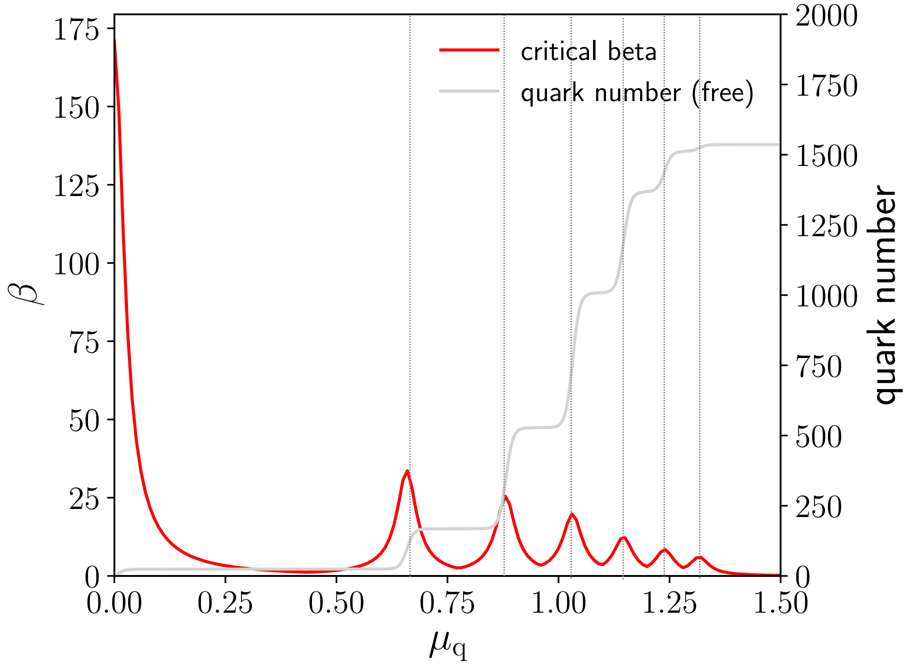
μ_q

Pass a level = Increase of quark number

$|\boldsymbol{p}| = 2\pi/L_x$ 144 modes

$|\boldsymbol{p}| = 0$ 24 modes ($= N_\mathrm{spin}\, N_\mathrm{color} N_\mathrm{flavor}$)

0

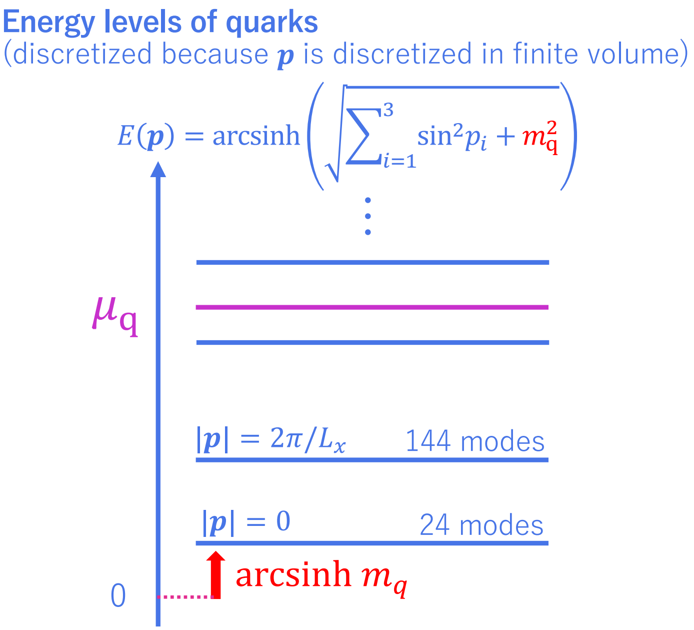


- Peak positions correspond to μ_q at which quark number changes.
  - At such μ_q, modes of fermions exist on the Fermi surface.

    ➡ These modes form Cooper pairs.
  - Consistent with results in NJL model in small box.
    - Open of the energy gap at discrete μ_q was observed.

Amore, Birse, McGovern, Walet, PRD (2002)

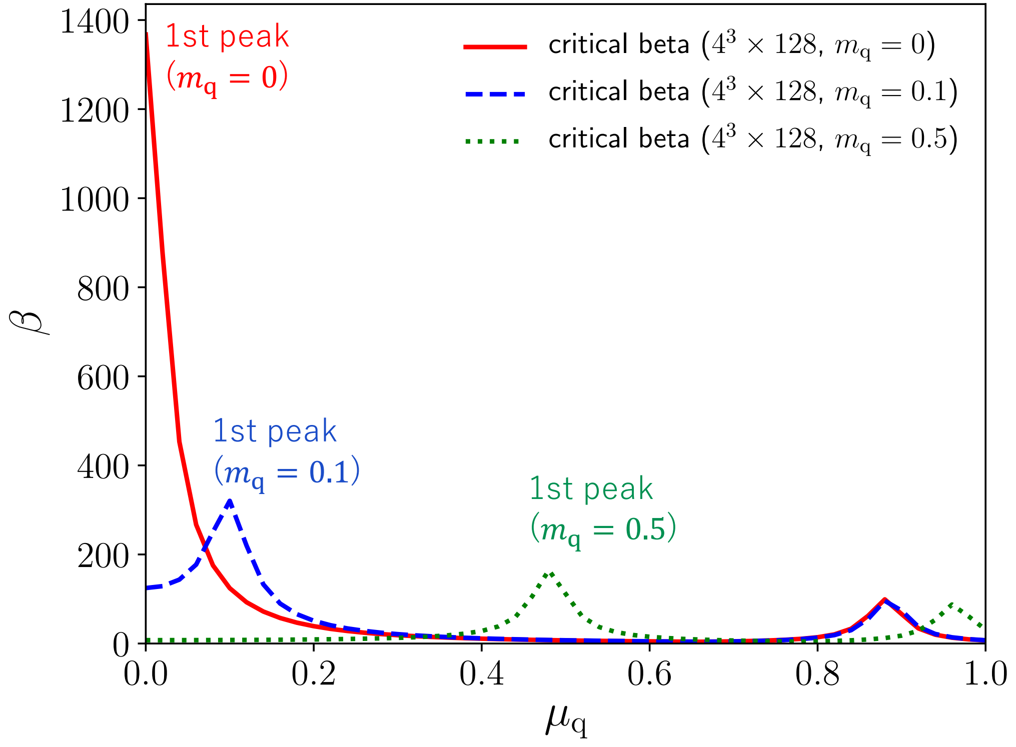
Quark-mass dependence



Energy levels of quarks

(discretized because $\boldsymbol{p}$ is discretized in finite volume)

$$E(\boldsymbol{p}) = \mathrm{arcsinh}\left(\sqrt{\sum_{i=1}^{3}\sin^2 p_i + m_\mathrm{q}^2}\right)$$

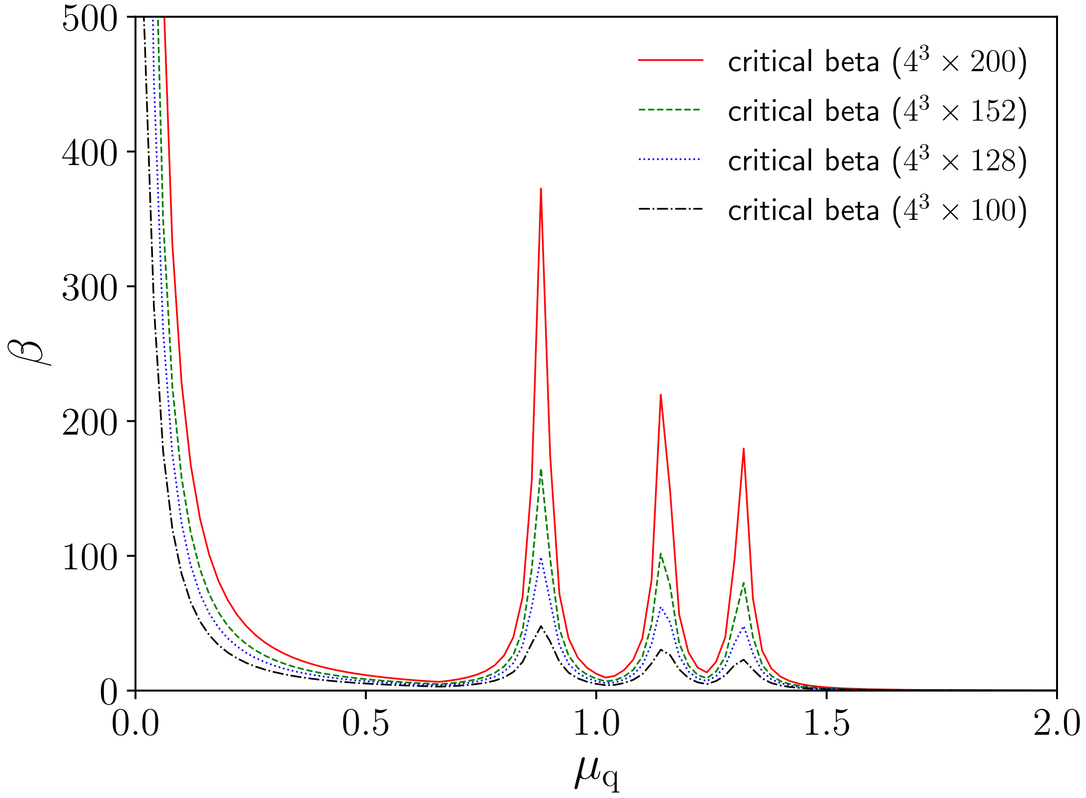


The peaks are shifted when m_q is changed.

➡ Consistent with the interpretation that the peak positions are determined by the energy spectrum of quark.

Temperature dependence

$m_q = 0$



- β_c is sensitive to the temperature.

➡ Low temperature is preferable to observe CSC in the perturbative region.

Contents

Complex Langevin method (CLM)

Parisi ’83; Klauder ’84;
Aarts, Seiler, Stamatescu ’09;
Aarts, James,Seiler, Stamatescu ’11;
Seiler, Sexty, Stamatescu ‘13; Sexty ’14;
Fodor, Katz, Sexty, Torok ’15;
Nishimura, Shimasaki ’15;
Nagata, Nishimura, Shimasaki ’15;
Sinclair, Kogut ’16

Complexification of gauge field

$$U_{x,\mu} \in \mathrm{SU}(3) \to \mathcal{U}_{x,\mu} \in \mathrm{SL}(3,\mathbb{C})$$

Complex Langevin equation

$$\mathcal{U}_{x,\mu}(\tau+\epsilon) = \exp(i(-\underbrace{\epsilon D_{x,\mu} S[\mathcal{U},\mu_{\mathrm{q}}]}_{\text{Drift term}} + \underbrace{\sqrt{\epsilon}\eta_{x,\mu}(\tau)}_{\text{Noise term}}))\mathcal{U}_{x,\mu}(\tau)$$

Condition for justifying $\frac{1}{Z}\int dU \mathcal{O}[U] e^{-S[U,\mu]} = \lim_{\tau\to\infty} \langle \mathcal{O}[\mathcal{U}(\tau)] \rangle_\eta$

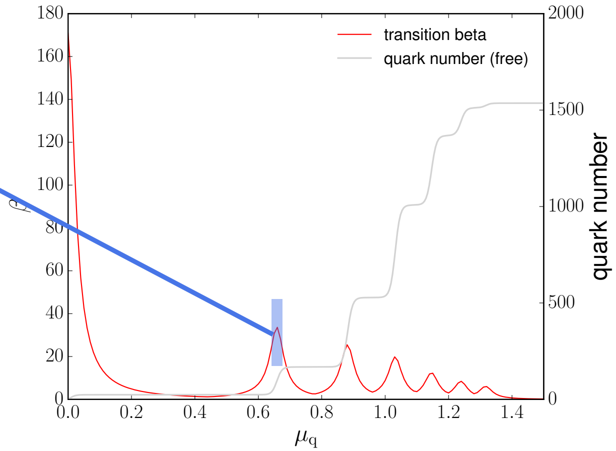
The probability of the drift term should be suppressed exponentially at large magnitude.

Nagata, Nishimura, Shimasaki, PRD94 (2016)

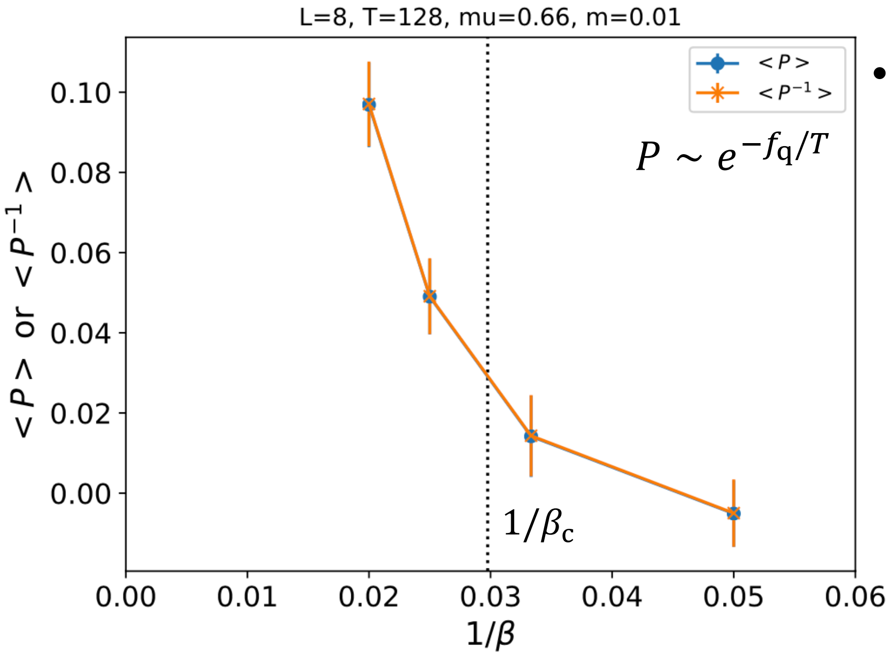
CLM on the lattice with a small aspect ratio & β

Calculation on $8^3 \times 128$, $\mu_\mathrm{q} = 0.66$, $\beta = 20, 30, 40{,}50$

- We confirm CLM is justified at these parameters.



Signal of phase transition in the Polyakov line?

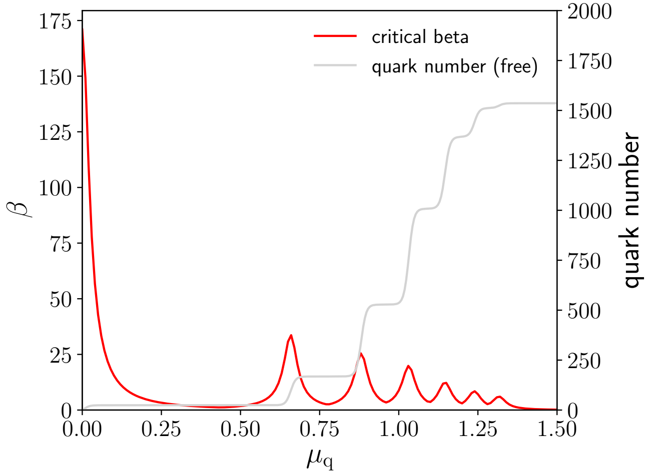


- Polyakov line rapidly changes near $1/\beta_\mathrm{c}$
  - Normal phase in $\beta > \beta_\mathrm{c}$?
    - Polyakov line may be enhanced due to gapless excitation from Fermi surface.

      cf) perturbative calculations on small S^3
      Hands, Hollowood, Myers, JHEP (2010)
  - CSC in $\beta < \beta_\mathrm{c}$?

    → Calculation of CSC order parameter is ongoing.

Summary



Based on lattice perturbation for QCD,
we explicitly give the prediction for the parameter region in which CSC appears.

- Gap equation is only finite number of coupled equations on the lattice with general setup.
- Phase diagram on (μ_q, β)-plane without assumption on the form and symmetry of Cooper pairs.
- Peak structure is found
  - Peak appears when modes on quark exists on the fermi surface
- Small aspect ratio (e.g. $8^3 \times 128$) is preferable to observe CSC.

Outlook

- **Investigation of CSC with complex Langevin method (CLM) is ongoing.**
  - CLM works on lattice with a small aspect ratio and β.
  - Signal of phase transition in Polyakov line?
- Determination of the form & symmetry of the Cooper pair from eigenvector.
  - Find better operators for cleaner signals of CSC.