Symmetry protected topological phases and generalized (co)homology theory

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@Zoom
Brief review to symmetry protected topological (SPT) phases
- Classification of SPT phases
- Bulk-boundary correspondence

- SPT phases as a generalized (co)homology theory

- The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems.
“Topological” equivalence: If there exists a pass connecting two phases A and B without a phase transition, A and B are considered to be in a same phase.

- Ice $\neq$ water
- Water = water vapor
- SSB of translation symmetry between \{ice\} and \{water, water vapor\}
There may exist a phase distinction without SSB in a certain class of phases of matter.

In the topological phase, we consider the following rule of the game:

✓ Zero temperature

✓ Gapped (there exists a finite energy gap between the ground and the first excited state.)

✓ With symmetry (Z2 Ising, U(1) particle conservation, time-reversal, crystalline, …)
• A schematic picture of a phase diagram of the topological phase
• A topological phase := an equivalence class under the equivalence relation by the existence of path connecting two points without SSB or gapless phases.
Symmetry Protected Topological phases

- In general, there exists a ground state degeneracy that depends on the global topology of the closed space manifold.

- An SPT phase := a gapped phase that has a unique ground state for any closed space manifold (+\(\alpha\)).
  Exs: Haldane chain, topological insulators/superconductors

- A long-range entangled topological phase = a gapped phase that has a ground state degeneracy for a closed space manifold.
  Exs: Toric code, fluctuational quantum Hall effect
How to classify SPT phases?

Recall:

In SPT phases, all information is encoded only in the ground state.
- No excited states, no scale
  - Topological field theory (TFT)

Hilbert space is one-dimensional, therefore, we conclude:

An SPT phase $\sim$ a set of $U(1)$-valued partition functions
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$$Z(M) = \int \prod_{\text{mat. fields}} D\phi_{\text{mat}} e^{-S_M^{\text{mat}}}[\phi_{\text{mat}}]$$

$$= |Z(M)| \times e^{iS_M}$$

Euclidian spacetime manifold

Excited states

Characterizes SPT phases ??
An SPT phase \(\sim\) a set of U(1)-valued partition functions

- The partition function over a closed space manifold \(X\) and time circle \(S^1\) is always unity:
  \[
  Z(X \times S^1) = \text{Tr}(1) = \langle GS|GS\rangle = 1.
  \]

- Therefore, to distinguish different SPT phases, we should employ generic closed spacetime manifolds.
  
  - Orientation-reversing symmetry \(\rightarrow\) unoriented manifolds
    
    Exs: Klein bottle, real projective spaces, …
  
  - Global G symmetry \(\rightarrow\) background gauge fields \(A\)
An SPT phase $\sim$ a set of $U(1)$-valued partition functions

- In sum,

\[
Z(M, A) = \int \prod_{\text{mat. fields}} D\phi e^{-S_M^{\text{mat}}}(\phi_{\text{mat.}, A})
= |Z(M, A)| \times e^{iS_M(A)}
\]

- Comment: For matter degrees of freedom with fermions, spacetime manifold must be equipped with a (variants of) spin structure.
Deformation invariance

- In SPT phases, we are interested in theories which are deformation invariant.

Ex: the (3+1)D theta term of the background U(1)-field

$$\text{Exp} \left( \frac{2\pi i \theta}{4\pi^2} \int_M F^2 \right)$$

has a continuous parameter $\theta \in [0,2\pi]$. This is not a partition function of an SPT phase (but a partition function of a gapped phase).

However, if $\theta$ is quantized by some symmetry (time-reversal symmetry, for example) this (partially) describes an SPT phase.
Ex: the (2+1)D Chern-Simons form of the background U(1)-field

\[ \text{Exp} \left( \frac{2\pi ik}{4\pi^2} \int_M A dA \right), \quad k \in \mathbb{Z} \]

has a quantized parameter \( k \in \mathbb{Z} \). This is a partition function of an SPT phase.

An SPT phase = a set of U(1)-valued deformation invariant partition functions
Classification of SPT phases
[Chen-Gu-Liu-Wen, Levin-Gu, Gu-Wen, Kapustin, Freed-Hopkins, Yonekura, …]

- It was shown that SPT phases are classified by the “Anderson dual” of the “cobordism group”,

\[ \Omega^d_{str}(BG) \cong \text{Free} \ \Omega^{str}_{d+1}(BG) \times \text{Tor} \ \Omega^{str}_d(BG). \]

- Free part (\( Z \)) : theta term in \((d+1)D \to CS\) term in \(dD\).

- Torsion part (\( Z_q \)) : no continuous parameter
Outline

• Brief review to symmetry protected topological (SPT) phases
  ◆ Classification of SPT phases
  ◆ Bulk-boundary correspondence

• SPT phases as a generalized (co)homology theory

• The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems.
Bulk-boundary correspondence

- For an SPT phase, bulk has no signatures because it is a gapped theory.
- All physical signatures come from the boundary of space manifold.
- A bulk U(1)-valued partition function corresponds to a quantum anomaly of the boundary.
Ex: Cluster model [Chen-Lu-Vishwanath]

- (1+1)D bosonic SPT phase with $Z_2 \times Z_2$ symmetry

- Matter degrees: spin 1/2 at integer and half integer sites. $\sigma_j^\mu, \tau_j^{\mu+1}, j \in Z$.

- Hamiltonian (cluster model [Briegel-Raussendorf])
  \[
  H = - \sum_j A_{j+\frac{1}{2}} - \sum_j B_j = - \sum_{j \in Z} \sigma_j^z \tau_j^x \sigma_{j+1}^z - \sum_{j \in Z} \tau_{j-\frac{1}{2}} \sigma_j^x \tau_{j+\frac{1}{2}}^z
  \]

- $Z_2 \times Z_2$ symmetry operators:
  \[
  U_\sigma = \prod_j \sigma_j^x, \quad U_\tau = \prod_j \tau_j^{x+\frac{1}{2}}
  \]

- All terms are commuted with each other (frustration-free). The ground state of $H$ is given by $A_{j+\frac{1}{2}} = B_j = 1$. An excited state has at least a finite energy $E = 2 \rightarrow$ gapped.
- Hamiltonian on the open chain

\[
H = - \sum_{j=1}^{N-1} \sigma_j^z \tau_{j+\frac{1}{2}}^x \sigma_{j+1}^z - \sum_{j=1}^{N} \tau_j^z \sigma_{j-\frac{1}{2}}^x \tau_{j+\frac{1}{2}}^z
\]

\[\tau_{\frac{1}{2}} \quad \sigma_1 \quad \tau_{\frac{3}{2}} \quad \cdots \quad \tau_{N-\frac{1}{2}} \quad \sigma_N \quad \tau_{N+\frac{1}{2}}\]

- \# (dof) = 2N+1, \#(A_{j+\frac{1}{2}}, B_j) = 2N-1

-> The ground state is 4-fold degenerate.

\[
|\Psi(a,b)\rangle = \sum_{DDWs} |(\tau_{\frac{1}{2}}^x = a) - bulk - (\tau_{N+\frac{1}{2}}^x = b)\rangle, \quad a, b \in \{+, -\}.
\]

- This 4-fold degeneracy is not an accident, but is protected by the $Z_2 \times Z_2$ symmetry.
To see this, let’s consider how \( Z_2 \times Z_2 \) symmetry operators act on the ground states manifold.

\[
U_\sigma |\psi\rangle = \prod_j \sigma_j^x |\psi\rangle = \prod_j (\tau_j^z \tau_{j+1}^z) = \tau_1^z \otimes \tau_{N+\frac{1}{2}}^z =: U_\sigma^L \otimes U_\sigma^R,
\]

\[
U_\tau |\psi\rangle = \prod_j \tau_j^x |\psi\rangle = \tau_1^x \left( \prod_j (\sigma_j^z \sigma_{j+1}^z) \right) \tau_{N+\frac{1}{2}}^x = \tau_1^x \sigma_1^z \otimes \sigma_N^z \tau_{N+\frac{1}{2}}^x =: U_\tau^L \otimes U_\tau^R.
\]

\( Z_2 \times Z_2 \) symmetry operations split into ones for the left and the right edge spins.

\( Z_2 \times Z_2 \) acts on the right spin as a nontrivial projective representation, which can be seen in the algebra

\[
U_\sigma^R U_\tau^R = -U_\tau^R U_\sigma^R.
\]

This is a sort of quantum anomaly.
Why quantum anomaly?

- Let’s consider a (0+1)D system with $Z_2 \times Z_2$ symmetry without quantum anomaly.

- An anomaly-free system is invariant under the gauge transformation of background field, implying the group low of the $G$-actions is preserved.

\[ U_\sigma U_\tau = U_{\sigma \tau} = U_\tau U_\sigma \]

- Therefore, the breaking of group structure $U_\sigma^R U_\tau^R = -U_\tau^R U_\sigma^R$ signals a quantum anomaly.
Edge perspective: projective representations

- Let $G$ be a finite group. A set of matrices $\{D_g\}_{g \in G}$ is called a projective representation when it is a group representation up to a $U(1)$ phase

$$D_g D_h = \omega_{g,h} D_{gh}, \quad \omega_{g,h} \in U(1).$$

- The associativity $(D_g D_h) D_k = D_g (D_h D_k)$ yields the 2-cocyle condition

$$\omega_{g,h} \omega_{gh,k} = \omega_{g,hk} \omega_{h,k}.$$

- A redefinition $D_g \mapsto \alpha_g D_g, \alpha_g \in U(1)$ yields the equivalence relation (2nd coboundary)

$$\omega_{g,h} \sim \omega_{g,h} \alpha_h \alpha_{gh}^{-1} \alpha_g$$

- The factor system $\omega_{g,h}$ is classified by the 2nd group cohomology

$$H^2(G, U(1)) = Z^2(G, U(1))/B^2(G, U(1)).$$
Let’s consider the relationship between edge anomaly and the bulk $U(1)$-valued partition function.

A $G$-field $A$ over a spacetime manifold $M$ is a symmetry defect network over $M$.

When a matter field passes a defect line labeled by $g \in G$, the matter field is charged by $g$. 

![Diagram of a network with labeled edges $g$, $h$, and $gh$.]
The ansatz for U(1)-valued topological action:

\[ e^{i S_M(A)} = \prod_{\text{junctions}} \omega_{g,h}, \quad \omega_{g,h} \in U(1). \]

Gauge invariance requires the 2-cocycle condition on \( \omega_{g,h} \).

We get the (1+1)D Dijkgraaf-Witten topological action labeled by a group cocycle \( \omega_{g,h} \).
Anomaly cancellation

- The total system composed of the bulk and the boundary is anomaly free, namely, invariant under gauge transformations.

\[ U^R_g U^R_h = \omega_{g,h} U^R_{gh} \]
For an SPT phase, bulk has no signatures because it is a gapped theory.

A physical signature comes from the boundary of space manifold.

A bulk $U(1)$-valued partition function corresponds a quantum anomaly of the boundary.

**Bulk:**
Gapped
Nontrivial SPT phase

**Boundary:** low-energy excitation with a quantum anomaly
SPT phases are classified by U(1)-valued partition function of which coefficients are quantized.

A characteristic of SPT phase is the bulk-boundary correspondence: There is one-to-one correspondence between an SPT phase of bulk and a quantum anomaly of boundary.
Let us write the ground state with the bases of $\sigma_j^z = \{\uparrow, \downarrow\}$, $\tau_{j+\frac{1}{2}}^x = \{+. -.\}$.

1st terms -> $\sigma_j^z \sigma_{j+1}^z = \tau_{j+\frac{1}{2}}^x$ -> decorated domain walls (DDWs)

2nd terms $\sum_{j \in \mathbb{Z}} \tau_{j-\frac{1}{2}}^z \sigma_j^x \tau_{j+\frac{1}{2}}^z$ fluctuate the decorated domain walls

-> The ground state is the equal-weight superposition of the decorated domain walls.

$$|\Psi\rangle = \sum_{DDWs} |DDW\rangle$$
(2+1)d example: the integer quantum Hall state

- Matter: Dirac fermion with U(1) symmetry.

- Bulk U(1)-valued partition function: Chern-Simons form \( \exp \frac{i}{4\pi} \int_M A dA \).

- Boundary: chiral Dirac fermion

\[
\hat{H}_{\text{bdy}} = \sum_{k \in \mathbb{Z} + \frac{\theta}{2\pi}} vk \psi_k^\dagger \psi_k
\]

- The partition function is not invariant under the large gauge transformation \( \theta \mapsto \theta + 2\pi \).

- The anomaly on the boundary is cancelled by the bulk CS action.
Outline

Part 2  SPT phases as a generalized (co)homology theory.

Part 3  The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems. (if I have much time)

The following talk is based on KS-Xiong-Gomi, arXiv:1810.00801.

Related works:
Teo-Kane, Kitaev, Xiang, Gaiotto-Johnson-Freyd, Song-Huang-Fu-Hermele, Po-Watanabe-Jian-Zaletel, Thorngren-Else, Shenghan-Meng-Yang, Song-Fang-Yang, Okuma-Sato-KS, Freed-Hopkins, ...
Kitaev proposed that SPT phases are described by a generalized (co)homology theory. [11]

Gaiotto and Johnson-Freyd studied this proposal in the perspective of field theory [18].

In studying crystalline SPT phase in condensed matter physics, two similar strategies appeared.

- “Dimensional reduction” to classify SPT phases with crystalline symmetry [Song-Huang-Fu-Hermele 16],
- “Lattice homotopy” to classify LSM-type theorems [Po-Watanabe-Jian-Zaletel 17].

These two procedures would be summarized as “trivializing something nontrivial living in low-dimensional spaces by using something nontrivial living in higher-dimensional spaces. (The image from [Huang-Song-Huang-Hermele, 17])

This reminds us the Atiyah-Hirzebruch spectral sequence (AHSS) of the generalized homology.

We reconstruct these studies in terms of the AHSS of the generalized homology theory. [KS-Xiong-Gomi]
Why generalized (co)homology?

Two important “physical” observations (not mathematically rigorous):

1. The space of short-range entangled (SRE) states (\(\equiv\) unique gapped ground states) forms an \(\Omega\)-spectrum of a generalized (co)homology theory. [Kitaev 11,13,15]

2. The bulk-boundary correspondence is regarded as the boundary map of a generalized homology theory.
SRE states and an $\Omega$-spectrum [Kitaev]

- Let $F_n$ be the “space of SRE states in $n$-spatial dimensions”.
- $F_n$ is a based topological space. The trivial tensor product state can be regarded as a base point $* = |0\rangle \in F_n$.
- Kitaev proposed that $\{F_n\}_{n \in \mathbb{Z}}$ forms an $\Omega$-spectrum, i.e., $F_n$ is homotopically equivalent to the loop space $\Omega F_{n+1}$,

$$F_n \sim \Omega F_{n+1},$$

where $\Omega X = \{\ell : [0, 1] \to X|\ell(0) = \ell(1) = *\}$ is the based loop space of $X$.
- As a matter of mathematical fact, given an $\Omega$-spectrum $\{F_n\}_{n \in \mathbb{Z}}$, one can construct generalized cohomology and homology theories.

$$h^n(X, Y) = [X/Y, F_n],$$

$$h_n(X, Y) = \operatorname{colim}_{k \to \infty} [S^{n+k}, (X/Y) \wedge F_k].$$

- cf. $F(Y) =$"interacting Hamiltonians over $Y". F_n := F(D^n, \partial D^n)$. [Kitaev 11]
Discussion of the homotopy equivalence $F_n \to \Omega F_{n+1}$ [Kitaev]

- A characteristic of SRE state is the existence of its inverse

$$|\chi\rangle \otimes |\bar{\chi}\rangle \sim |1\rangle \otimes |1\rangle,$$

where $|\chi\rangle \in F_n$ and $|1\rangle = * \in F_n$ is a trivial tensor product state. ("invertible state")

- For a given $n$-dim. SRE state $|\chi\rangle \in F_n$, one can canonically construct an adiabatic pumping process that pumps the SRE state $|\chi\rangle$ from the right to the left boundaries.

$$F_n \to \Omega F_{n+1} = \{|\psi(\lambda)\rangle : [0, 1] \to F_n|\psi(0)\rangle = |\psi(1)\rangle = |\text{triv}\rangle\}$$
Some useful mathematical facts and physical interpretation

- By design, the classification of $n$-dim. SPT phases is given by the disconnected parts of $F_n$,

\[ \pi_0(F_n) = [pt, F_n] = h^n(pt). \]

- From the Poincaré duality and the suspension isomorphism,

\[ h^n(pt) = h_{-n}(pt) = h_0(D^n, \partial D^n). \]

- $h_0(D^n, \partial D^n)$ can be identified with the classification of SPT phases over $D^n$ relative to its boundary $\partial D^n$. 
Why generalized (co)homology?

Two important “physical” observations (not mathematically rigorous):

(1) The space of short-range entangled (SRE) states (= invertible states) forms an \(\Omega\)-spectrum of a generalized (co)homology theory. [Kitaev 11,13,15]

(2) The bulk-boundary correspondence is regarded as the boundary map of a generalized homology theory.
The bulk-boundary correspondence

A short-range entangled state

A boundary excitation (Hilbert space) with a quantum anomaly

Ex: Haldane chain protected by either TRS or $\mathbb{Z}_2 \times \mathbb{Z}_2$ onsite symm.

Ex: $2d$ and $3d$ topological insulator protected by TRS.

The bulk-boundary correspondence reminds us the boundary map of homology $\partial : X \rightarrow \partial X$. 
The axioms of generalized homology theories

- A generalized homology theory $h_n(X, Y)$, $Y \subset X$, $n \in \mathbb{Z}$, is a covariant functor from topological spaces to abelian groups.
- For a given map $f : (X, Y) \rightarrow (X', Y')$, we have a homomorphism $f_* : h_n(X, Y) \rightarrow h_n(X', Y')$ with the same direction.
Equipped with the boundary map

\[ \partial : h_n(X, Y) \to h_{n-1}(Y). \]
Axioms of generalized homology theory:

- (homotopy)
  If $f, f' : X \rightarrow X'$ are homotopic, then $f_* = f'_*$.  
- (excision)
  For $A, B \subset X$, the inclusion $A \rightarrow A \cup B$ induces an isomorphism $h_n(A, A \cap B) \rightarrow h_n(A \cup B, B)$.  
- (additivity)
  $h_n(\bigsqcup_{\lambda} X_{\lambda}, \bigsqcup_{\lambda} Y_{\lambda}) = \bigsqcup_{\lambda} h_n(X_{\lambda}, Y_{\lambda})$.
- (exactness)
  For $Y \subset X$, there is a long exact sequence
  \[ \cdots \rightarrow h_n(Y) \rightarrow h_n(X) \rightarrow h_n(X, Y) \xrightarrow{\partial} h_{n-1}(Y) \rightarrow \cdots \]

What is homology group $h_n(X, Y)$ for SPT phases?
From SPT phases to a generalized homology theory

- $h_0(X, Y) :=$ the abelian group of SPT phases over a real space $X$ which may have an quantum anomaly over a real space $Y \subset X$.

- We define the boundary map $\partial : h_0(X, Y) \to h_{-1}(Y)$ as the bulk-boundary correspondence.

- This implies $h_{-1}(Y)$ should be regarded as the abelian group of quantum anomaly over a real space $Y$.

**Ex:** Superconductors over $X = S^2$ that may have an anomalous edge state over the equator $Y = S^1$. We have

\[
h_0(S^2, S^1) = \mathbb{Z} \times \mathbb{Z}, \quad h_{-1}(S^1) = \mathbb{Z},
\]

$\partial : h_0(S^2, S^1) \to h_{-1}(S^1)$, $(n, m) \mapsto n - m$. 

![Diagram showing the correspondence between $S^2$, $S^1$, and the boundary map $\partial$]
The ordinary bulk-boundary correspondence is the special case of the boundary map $\partial$ where $X = D^n$ and $Y = \partial D^n$. 

\[
p(x + ip_y) \quad \partial \rightarrow S^1
\]
For generic $n \in \mathbb{Z}$

- The degree $n \in \mathbb{Z}$ of the generalized homology theory $h_n(X, Y)$ can be understood as a kind of a “degree of SPT phenomena”.
- The proper meaning of the $(n - 1)$-th homology is obtained by considering what the physical phenomenon living on the boundary of the $n$-th homology is.

\[ \partial \rightarrow n = 1 \quad \partial \rightarrow n = 0 \quad \partial \rightarrow n = -1 \quad \partial \rightarrow \]

Adiabatic pump \quad SPT phase \quad Anomaly

\[ \text{Ch=1} \quad \text{Chern ins.} \]
\[ \text{Ch=1} \quad \text{Chern ins.} \]
\[ \text{Ch=-1} \quad \text{Chern ins.} \]

chiral edge state
\[ h_1(X, Y) := \text{the abelian group of adiabatic pumps over a real space } X \text{ which may create a SRE state on } Y \subset X. \]

\[ h_0(X, Y) := \text{the abelian group of SPT phases over a real space } X \text{ which may have anomalous excitation on } Y \subset X. \]

\[ h_{-1}(X, Y) := \text{the abelian group of anomalous theories over a real space } X \text{ which may have a “source or sink” of an anomalous excitation on } Y \subset X. \]
“Physical definition” of $h_n(X, Y)$ v.s. the axioms

Let’s consider if the above identification of the group $h_n(X, Y)$ with a physical phenomenon related to SPT phases satisfies the axioms.

✓ A covariant functor (Because of the real-space picture)

✓ (homotopy)
  If $f, f' : X \to X'$ are homotopic, then $f_* = f'_*$. 

✓ (excision)
  For $A, B \subset X$, the inclusion $A \to A \cup B$ induces an isomorphism $h_n(A, A \cap B) \to h_n(A \cup B, B)$.

✓ (additivity)
  $h_n(\bigsqcup_{\lambda} X_{\lambda}, \bigsqcup_{\lambda} Y_{\lambda}) = \bigsqcup_{\lambda} h_n(X_{\lambda}, Y_{\lambda})$.

✓ (exactness)
  For $Y \subset X$, there is a long exact sequence
  $$\cdots \to h_n(Y) \to h_n(X) \to h_n(X, Y) \overset{\partial}{\to} h_{n-1}(Y) \to \cdots$$

... It looks OK.
Exactness

\[ \cdots \xrightarrow{\partial^1} h_0(Y) \xrightarrow{f_0^*} h_0(X) \xrightarrow{g_0^*} h_0(X, Y) \xrightarrow{\partial^0} h_{-1}(Y) \xrightarrow{f_{-1}^*} \cdots \]

- $f_*$ and $g_*$ are induced homomorphisms of inclusions $f : X \to Y$ and $g : (X, \emptyset) \to (X, Y)$, respectively.
- $f_0^*$ is regarded as embedding an SPT phase over $Y$ in $X$. 

![Diagram](Image)
\( g_*^0 \) is regarded as cutting out \( Y \) from \( X \), which leads to anomalous states over \( Y \) from an SPT phase over \( X \).

\( \partial^0 \) is the bulk-boundary correspondence.

From these physical interpretations, we can see the long exact sequence is compatible with properties of the SPT phases.
How useful is it?

- So far, I have discussed the abstract mathematical structure of SPT phases.
- There are several practical merits to study SPT phases.
- The real space $X$ can be an arbitrary real space, which can not be a manifold (a manifold is locally Euclidean). For example, we can ask what is the classification of SPT phases, adiabatic pumps, anomalies over a trijunction, the Klein bottle, ....

Since the generalized homology description is based on the real space on which SPT phases defined, it is straightforward to implement spatial symmetry (point group and space group).

$\Rightarrow$ Equivariant homology $h^n_G(X, Y)$.

In particular, the Atiyah-Hirzebruch Spectral Sequence (AHSS), a spectral sequence of generalized (co)homology theories, gives us a systematic way to thinking the interplay of crystalline symmetry and SPT phases.
Summary for part 2

- Invertibility of SRE states
  ⇒ an Ω-spectrum
  ⇒ a generalized (co)homology theory.

- The mathematical structure behind SPT phenomena such as SPT phases, anomalous excitations, adiabatic pumps, can be understood in the framework of the generalized homology theory.

- Open questions:
  - What is the generalized (co)homology description of the bulk-defect correspondence?
    → Probably, the $KK$-theory, which is like a combination of the homology and the cohomology, does work.
    Real space: homological
    Parameter space: cohomological
  - Can we find good physical interpretation for $h_{-2}(X, Y), h_{-3}(X, Y), \ldots$?

- Applying the Atiyah-Hirzebruch spectral sequence, which is a well-developed machinery in generalized (co)homology theories, to crystalline SPT phases gives us the comprehensive understanding of higher-order SPT phases and LSM theorems. (Part 3)
Outline

Part 2  SPT phases as a generalized homology theory.

Part 3  The Atiyah-Hirzebruch spectral sequence for crystalline SPT phases and LSM theorems. (if I have much time)
Atiyah-Hirzebruch Spectral Sequence (AHSS)

- The AHSS [Atiyah-Hirzebruch '61] is a spectral sequence to compute a generalized (co)homology theory $h_*$. 
- This is the mathematical structure behind the “dimensional reduction” [Song-Huang-Fu-Hermele 16] and the “lattice homotopy” [Po-Watanabe-Jian-Zaletel 17], but the AHSS goes beyond and complete their strategy. [KS-Xiong-Gomi, Song-Fang-Qi, Jiang-Cheng-Qi, Else-Thorngren]

- In general, a spectral sequence starts from the $E^1$-page, which is a something computable.
- We compute the $n$th differential ($n = 1, 2, \ldots$)
  \[ d^n_{p,q} : E^n_{p,q} \rightarrow E^n_{p-n,q+n-1}, \quad d^n \circ d^n = 0. \]
- The next page is defined as the homology of $d^n$,
  \[ E^{n+1}_{p,q} = \text{Ker} \ d^n_{p,q} / \text{Im} \ d^n_{p+n,q-n+1}. \]
- Assume that this iteration converges at some $E^r$-page.
  \[ E^1 \Rightarrow E^2 \Rightarrow \cdots E^r = E^{r+1} = \cdots =: E^\infty. \]
- The $E^\infty$-page approximates the homology theory $h^*(X, Y)$. (see below)
The starting point of the AHSS is to give a filtration of the space $X$, 

$$X_0 \subset X_1 \subset \cdots X.$$ 

A useful filtration is a cell-decomposition

$$X = \{0\text{-cells}\} \sqcup \{1\text{-cells}\} \sqcup \{2\text{-cells}\} \sqcup \cdots$$

with the following property: On each $p$-cell $D^p$ the crystalline symmetry $G$ behaves as an onsite symmetry of the little group $G_{D^p}$ over the $p$-cell $D^p$, which we call a uniform cell decomposition (only for this slide).

The $p$-skeleton $X_p$ is defined by

$$X_0 = \{0\text{-cells}\}, \quad X_p = X_{p-1} \cup \{p\text{-cells}\}.$$
A uniform cell decomposition

Ex: 2d real space with $C_4$ rotation $\rtimes \mathbb{Z}^2$ translation symmetry.

\begin{align*}
\{0\text{-cells}\} & \quad \{1\text{-cells}\} & \quad \{2\text{-cells}\} & \quad \text{[a]} \\
\{0\text{-cells}\} & \quad \text{[b]} \\
\{0\text{-cells}\} & \quad \{1\text{-cells}\} & \quad \{2\text{-cells}\}
\end{align*}
Topological Crystalline Liquid [Thorngren-Else 16]

- Please keep in your mind the following picture:
  The spatial scale $a$ of crystalline symmetry is much larger than the scale $\xi$ of microscopic degrees of freedom.

- Of course, this is not the case in cond-mat problems, however, it can be used to classify SPT phases because the effective theory would be “topological”. (Under debate.)

![Diagram of spatial scales $a$ and $\xi$]

FIG. 1. (a) In a smooth state, the lattice spacing and the correlation length $\xi$ are much less than the unit cell size $a$ and the radius of spatial variation. (b) The topological response of a crystalline topological liquid is captured by a spatially-dependent TQFT that captures the spatial dependence within each unit cell but “forgets” about the lattice.
The $E^1$-page is defined as

$$E^1_{p,-q} := h^G_{p-q}(X_p, X_{p-1}),$$

the $(p - q)$-th homology over $X_p$ relative to $X_{p-1}$.

For instance, $E^1_{p,-p} = h^G_0(X_p, X_{p-1})$ is the abelian group of SPT phases over $X_p$ which can be anomalous over $X_{p-1}$.

For a uniform cell decomposition, we have equalities

$$E^1_{p,-q} \cong \prod_{j \in p\text{-cells}} h^{G^p_{D_j}}_{p-q}(D^p_j, \partial D^p_j) \cong \prod_{j \in p\text{-cells}} h^{G^p_{D_j}}_{p-q}(D^p_j / \partial D^p_j (= S^p))$$

$$\cong \prod_{j \in p\text{-cells}} h^{G^p_{D_j}}_{-q}(pt) \quad \text{(suspension iso.)}$$

$$\cong \prod_{j \in p\text{-cells}} h^0_{p-j}(D^q_j, \partial D^q_j)$$

Therefore, $E^1_{p,-q}$ is the abelian group of $q$-dim. SPT phases (we denote them by SPT$^q$) with the onsite $G^p_{D_j}$ symmetry.
The first differential $d^1$

- The $E^1$-page hosts the “local information” of SPT phenomena.
- We should properly glue the local information together, which is partly done by the first differential $d^1_{p,-q} : E^1_{p,-q} \to E^1_{p-1,-q}$.
- The first differential $d^1$ can be “physically” understood and is computable. (cf. [Song-Huang-Fu-Hermele 16, Po-Watanabe-Jian-Zaletel 17])

<table>
<thead>
<tr>
<th>$S^0$</th>
<th>$q = 0$</th>
<th>$E^1_{0,0}$</th>
<th>$E^1_{1,0}$</th>
<th>$E^1_{2,0}$</th>
<th>$E^1_{3,0}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S^1$</td>
<td>$q = 1$</td>
<td>$E^1_{0,-1}$</td>
<td>$E^1_{1,-1}$</td>
<td>$E^1_{2,-1}$</td>
<td>$E^1_{3,-1}$</td>
</tr>
<tr>
<td>$S^2$</td>
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<td>$E^1_{0,-2}$</td>
<td>$E^1_{1,-2}$</td>
<td>$E^1_{2,-2}$</td>
<td>$E^1_{3,-2}$</td>
</tr>
<tr>
<td>$S^3$</td>
<td>$q = 3$</td>
<td>$E^1_{0,-3}$</td>
<td>$E^1_{1,-3}$</td>
<td>$E^1_{2,-3}$</td>
<td>$E^1_{3,-3}$</td>
</tr>
<tr>
<td>$E^1_{p,-q}$</td>
<td>$p = 0$</td>
<td>$p = 1$</td>
<td>$p = 2$</td>
<td>$p = 3$</td>
<td></td>
</tr>
</tbody>
</table>

Anomalous edge states of a 1d SPT phase over a 1-cell
The adiabatic creation of a 1d SPT phase over a 2-cell

Adia. pumps
SPT phases
Anomalies

Homology

0-cell 1-cell 2-cell 3-cell
The second differential $d^2$

- The $E^2$-page hosts the “local information” of SPT phenomena which are glued together over the 1-skeleton $X_1$.
- We should further compute the compatibility over 2-cells, which is represented by the second differential $d^2_{p,-q} : E^2_{p,-q} \rightarrow E^2_{p-2,-q+1}$.
- The second differential $d^2$ is also "physically" understood and computed.

<table>
<thead>
<tr>
<th>SPT$^q$</th>
<th>$q$</th>
<th>$E^2_{0,-q}$</th>
<th>$E^2_{1,-q}$</th>
<th>$E^2_{2,-q}$</th>
<th>$E^2_{3,-q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SPT$^0$</td>
<td>$q=0$</td>
<td>$E^2_{0,0}$</td>
<td>$E^2_{1,0}$</td>
<td>$E^2_{2,0}$</td>
<td>$E^2_{3,0}$</td>
</tr>
<tr>
<td>SPT$^1$</td>
<td>$q=1$</td>
<td>$E^2_{0,-1}$</td>
<td>$E^2_{1,-1}$</td>
<td>$E^2_{2,-1}$</td>
<td>$E^2_{3,-1}$</td>
</tr>
<tr>
<td>SPT$^2$</td>
<td>$q=2$</td>
<td>$E^2_{0,-2}$</td>
<td>$E^2_{1,-2}$</td>
<td>$E^2_{2,-2}$</td>
<td>$E^2_{3,-2}$</td>
</tr>
<tr>
<td>SPT$^3$</td>
<td>$q=3$</td>
<td>$E^2_{0,-3}$</td>
<td>$E^2_{1,-3}$</td>
<td>$E^2_{2,-3}$</td>
<td>$E^2_{3,-3}$</td>
</tr>
</tbody>
</table>

$E^2_{p,-q} : p = 0, 1, 2, 3$

A 2d SPT phase over a 2-cell may have an anomaly at a 0-cell

$$d^2_{p,-q} : E^2_{p,-q} \rightarrow E^2_{p-2,-q+1}$$

Homology
The third differential $d^3$

- In the same way, we have the third differential $d_{p,-q}^3 : E_{p,-q}^3 \rightarrow E_{p-3,-q+2}^3$.
- The third differential $d^3$ is also “physically” understood and computed.

![Diagram](https://via.placeholder.com/150)

<table>
<thead>
<tr>
<th>SPT$^0$</th>
<th>SPT$^1$</th>
<th>SPT$^2$</th>
<th>SPT$^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q = 0$</td>
<td>$E_{0,0}^3$</td>
<td>$E_{1,0}^3$</td>
<td>$E_{2,0}^3$</td>
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<tr>
<td>$q = 1$</td>
<td>$E_{0,-1}^3$</td>
<td>$E_{1,-1}^3$</td>
<td>$E_{2,-1}^3$</td>
</tr>
<tr>
<td>$q = 2$</td>
<td>$E_{0,-2}^3$</td>
<td>$E_{1,-2}^3$</td>
<td>$E_{2,-2}^3$</td>
</tr>
<tr>
<td>$q = 3$</td>
<td>$E_{0,-3}^3$</td>
<td>$E_{1,-3}^3$</td>
<td>$E_{2,-3}^3$</td>
</tr>
</tbody>
</table>

$d_{p,-q}^3 : E_{p,-q}^3 \rightarrow E_{p-3,-q+2}^3$

Homology

$E^\infty$-page

Adia. pumps

SPT phases

anomalies

The adiabatic creation of a 2d SPT phase over a 3-cell may be equivalent to an SPT phase over a 0-cell.
Filtration of the homology group

- $E^\infty$-page itself does not provides the classification of SPT phenomena.
- Introduce the following subgroups of $h^n_G(X, Y)$,  
  \[ F_p h_n := \text{Im} [h^n_G(X_p, X_p \cap Y) \to h^n_G(X, Y)], \quad p = 0, 1, \ldots. \]
- This has the clear physical meaning. For instance, $F_p h_0$ is the classification of SPT phases over the $p$-skeleton $X_p$ which persists after being embedded in the whole space $X$.
- We have a filtration of the homology group
  \[ 0 \subset F_0 h_n \subset F_1 h_n \subset \cdots \subset F_d h_n = h^n_G(X, Y), \]
  where $d$ is the space dimension of $X$.
- The following relation connects the $E^\infty$-page and the homology group. 
  \[ F_p h_n / F_{p-1} h_n \cong E^\infty_{p, n-p}. \]
- The $E^\infty$-page has good physical meanings.
Higher-order SPT phases

- $E_{p,-p}^\infty$: The classification of $(d - p + 1)$th-order SPT phases. (cf. Huang-Song-Huang-Hermele)
- Ex: 3d with point group symmetry (without translation symmetry):

  $F_0h_0 \subset F_1h_0 \subset F_2h_0 \subset F_3h_0$

- This unifies the terminology of “strong” and “weak” SPT phases and higher-order SPT phases.
Ex: the classification of higher-order TIs with magnetic point group symmetry via the AHSS \cite{Okuma-Sato-KS, cf. Cornfeld-Chapman, KS}

| $G$ | 2' | 2'/m | 2'/m' | 2'2' | m'm2' | 4' | m'm'm | m'm'm' | 4'22' | 42'2' | 42'm' | 42'm' |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_0^G(\mathbb{R}^3)$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^2$ | $\mathbb{Z} \oplus \mathbb{Z}_2$ |
| $E_{0,0}^\infty$ | 0 | 0 | $\mathbb{Z}_2$ | 0 | 0 | 0 | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^2$ |
| $E_{2,-2}^\infty$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |

2nd-order TI

<table>
<thead>
<tr>
<th>$G$</th>
<th>4'mm'</th>
<th>4'/m</th>
<th>4'/m'</th>
<th>4'/mmm'</th>
<th>4/mm'm'</th>
<th>4'/mm'm'</th>
<th>4/m'm'm</th>
<th>4'/m'm'm'</th>
<th>32'</th>
<th>$\bar{3}m'$</th>
<th>$\bar{3}m'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_0^G(\mathbb{R}^3)$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2^4$</td>
<td>$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z} \oplus \mathbb{Z}_2$</td>
<td></td>
</tr>
<tr>
<td>$E_{0,0}^\infty$</td>
<td>0</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2$</td>
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<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2^3$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$E_{2,-2}^\infty$</td>
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<td>$\mathbb{Z}$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| $G$ | 4'32' | m$\bar{3}$m' | m$\bar{3}$'m | 6' | 6'/m' | 6'/m | $\bar{6}$' | $\bar{6}$m'2' | $\bar{6}$m'2' | 6'm'm | 6'm'm | 6'2'2 | 62'2' |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_0^G(\mathbb{R}^3)$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^4$ | $\mathbb{Z}_2^4 \oplus \mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2^2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^6$ | $\mathbb{Z}_2^6$ |
| $E_{0,0}^\infty$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^3$ | $\mathbb{Z}_2^3 \oplus \mathbb{Z}_2$ | 0 | $\mathbb{Z}_2^2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2^2$ | 0 | $\mathbb{Z}_2^5$ | $\mathbb{Z}_2^5$ |
| $E_{2,-2}^\infty$ | $\mathbb{Z}_2$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ |

<table>
<thead>
<tr>
<th>$G$</th>
<th>6'/m'm'm</th>
<th>6/mm'm'</th>
<th>6'/mmm'</th>
<th>6/m'm'm</th>
<th>Others$^\dagger$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_0^G(\mathbb{R}^3)$</td>
<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2^6$</td>
<td>$\mathbb{Z}_2^3$</td>
<td>$\mathbb{Z}_2^2 \oplus \mathbb{Z}_2^2$</td>
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<tr>
<td>$E_{0,0}^\infty$</td>
<td>$\mathbb{Z}_2^2$</td>
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<td>0</td>
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<tr>
<td>$E_{2,-2}^\infty$</td>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>$\mathbb{Z}_2^2$</td>
<td>0</td>
</tr>
</tbody>
</table>
LSM type theorems

- LSM-type theorems forbid the system with a sort of dof having a unique symmetric gapped ground state in the presence of translation symmetry and others. [Chen-Gu-Wen 11, Watanabe-Po-Vishwanath-Zaletel 15]
- The group $E_{0,-1}^\infty$ is the classification of the LSM theorem with crystalline $G$ symmetry. (cf. Po-Watanabe-Jian-Zaletel 17, )
- “A LSM theorem as a boundary of an SPT phase” [Metlitski-Thorngren, ...]
- Using the AHSS, one can systematically classify the LSM-type theorems for a given space group and onsite symmetry. Many symmetry classes are remain unclassified.
The AHSS gives us a useful tool to study the SPT phases and LSM theorems with crystalline symmetry with respect to high-symmetry regions in the real space.

The differentials of the AHSS can be physically understood, thus they are computable from physical arguments. See KS-Xiong-Gomi for various worked examples of higher-differentials.

The $E^\infty$-page itself has a physical meaning. It represents the classification of higher-order SPT phases, anomalies, and adiabatic pumps. In particular, $E^\infty_{-1,0}$ is the classification of LSM theorems.