

# Higher-form symmetries and 3-group in axion electrodynamics

Ryo Yokokura (KEK)

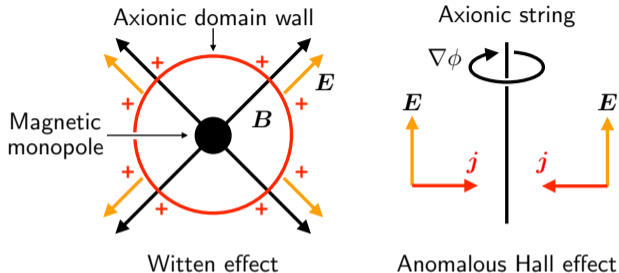
2020. 12. 15

KEK Theory Workshop 2020, online

Based on Y. Hidaka, M. Nitta, RY, Phys. Lett. B **808** (2020) 135672 [2006.12532]

See also Y. Hidaka, M. Nitta, RY, 2009.14368, to be published in JHEP

## What we did



- Axion electrodynamics in 4D possesses a **3-group** structure.
- The **Witten effect** and the **anomalous Hall effect** can be understood as 3-group transformations.

Technically, we have shown

- higher-form symmetries in axion electrodynamics
- the 3-group structure by correlation functions of symmetry generators of the higher-form symmetries

# Contents

- 1** Introduction
- 2** Higher-form symmetries in axion electrodynamics
- 3** Witten effect and anomalous Hall effect from higher-form symmetries
- 4** 3-group in axion electrodynamics

# Axion electrodynamics: axion $\phi$ + photon $a_\mu$ + topological coupling

[Wilczek '87]

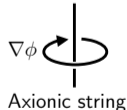
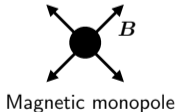
$$\frac{N}{4\pi^2} \phi \mathbf{E} \cdot \mathbf{B} = \frac{N}{32\pi^2} \phi \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = \frac{N}{8\pi^2} \phi da \wedge da \quad (N: \text{integer})$$

## Features

1. Topological coupling: determined by chiral anomaly (stable against higher-order corrections)



2. extended objects (temporally or spatially)



3. Ubiquitous in modern physics: QCD axion,  $\pi^0$  meson, axion insulator,...

# Axion electrodynamics: axion $\phi$ + photon $a_\mu$ + topological coupling

[Wilczek '87]

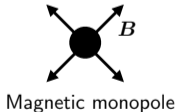
$$\frac{N}{4\pi^2} \phi \mathbf{E} \cdot \mathbf{B} = \frac{N}{32\pi^2} \phi \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} = \frac{N}{8\pi^2} \phi da \wedge da \quad (N: \text{integer})$$

## Features

1. Topological coupling: determined by chiral anomaly (stable against higher-order corrections)



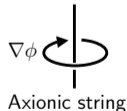
2. extended objects (temporally or spatially)



Magnetic monopole



Axionic domain wall

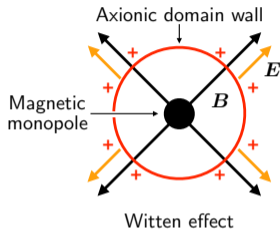


Axionic string

3. Ubiquitous in modern physics: QCD axion,  $\pi^0$  meson, axion insulator,...

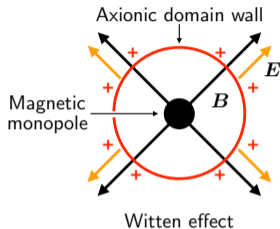
$$\frac{N}{4\pi^2} \phi \mathbf{E} \cdot \mathbf{B} + \text{extended objects} \rightarrow \text{rich and peculiar phenomena}$$

## Witten effect for axionic domain wall



- Magnetic monopole + axionic domain wall  $\rightarrow$  electric charge [Witten '79; Sikivie '84; Kogan '92]

## Witten effect for axionic domain wall



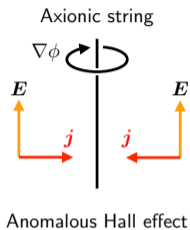
- Magnetic monopole + axionic domain wall  $\rightarrow$  electric charge [Witten '79; Sikivie '84; Kogan '92]
- Due to modification of elec. Gauss law

$$\nabla \cdot \mathbf{E} = \frac{N}{4\pi^2} \nabla \phi \cdot \mathbf{B}$$

(Original Witten effect: monopole becomes dyon in the presence of  $\theta \mathbf{E} \cdot \mathbf{B}$ )

# Anomalous Hall effect for axionic string

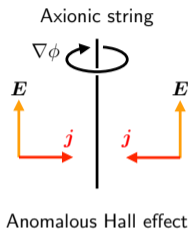
Anomalous Hall effect = Hall effect without external magnetic field



- Axionic string + electric field  $\rightarrow$  electric current [Sikivie '84; Wilczek '87; Qi, et al. '08; Teo & Kane '10]

# Anomalous Hall effect for axionic string

Anomalous Hall effect = Hall effect without external magnetic field



- Axionic string + electric field  $\rightarrow$  electric current [Sikivie '84; Wilczek '87; Qi, et al. '08; Teo & Kane '10]
- Due to modification of Maxwell-Ampère law

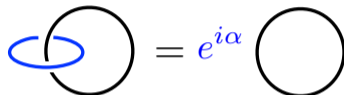
$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \frac{N}{4\pi^2} \nabla \phi \times \mathbf{E}$$

Q: Mathematical structure behind the phenomena for extended objects?

## Candidate: Higher-form global symmetries [Gaiotto et al. '14]

Symmetries under transf. of  $p$ -dimensional extended objects

(Ordinary symmetries are 0-form symmetries: acting on local 0-dim. objects)


$$\text{Circle with blue loop} = e^{i\alpha} \text{Circle}$$

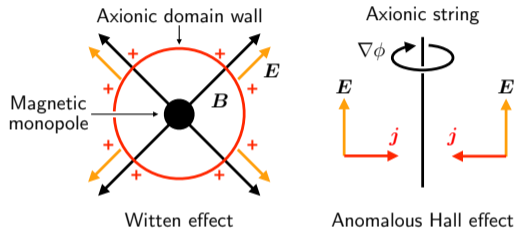
Understanding physics of extended objects by symmetries

e.g., Aharonov-Bohm effect: 1-form symmetry transf. (charged object = Wilson loop)

Classifying phases of matter beyond ordinary symmetries

- Maxwell theory in 4D: photon = NG boson
- Superconductor (s-wave)  $\neq$  charge 1 Abelian Higgs model [cf. Hansson, et al. '04]
- (mixed) 't Hooft anomalies and their matching

## Purpose of this talk



- We show higher-form symmetries and their group structure in axion electrodynamics.
- We discuss the Witten effect and anomalous Hall effect from the viewpoint of higher-form symmetries.

# Contents

- 1 Introduction
- 2 Higher-form symmetries in axion electrodynamics
- 3 Witten effect and anomalous Hall effect from higher-form symmetries
- 4 3-group in axion electrodynamics

# Higher-form symmetries in axion electrodynamics

Y. Hidaka, M. Nitta, RY, Phys. Lett. B **808** (2020) 135672

## Message

There are 4 kinds of higher-form symmetries in axion electrodynamics.

	conservation law	Group	Charged object
0-form	EOM of $\phi$	$\mathbb{Z}_N$	axion
Elec. 1-form	EOM of $a_\mu$	$\mathbb{Z}_N$	Wilson loop
Mag. 1-form	Bianchi id. of $a_\mu$	$U(1)$	't Hooft loop
2-form	Bianchi id. of $\phi$	$U(1)$	worldsurface of axionic string

## Set up [Wilczek '87]

### Action

$$S = - \int \left( \frac{v^2}{2} |d\phi|^2 + \frac{1}{2e^2} |da|^2 - \frac{N}{8\pi^2} \phi da \wedge da \right)$$

$v$ : decay constant,  $e$ : coupling constant

- Axion  $\phi$ : periodic pseudo-scalar  $\phi + 2\pi \sim \phi$  (NG boson  $e^{i\phi}$ )
- Photon  $a = a_\mu dx^\mu$ :  $U(1)$  gauge field,  $\int_S da \in 2\pi\mathbb{Z}$  ( $S$ : closed surface, e.g.,  $S^2$ )
- $N$ : integer (number of massive Dirac fermions in UV)

Higher-form symmetries can be found by EOM and Bianchi identities

---

$$S = - \int d^4x \left( \frac{v^2}{2} |\partial_\mu \phi|^2 + \frac{1}{2e^2} |f_{\mu\nu}|^2 - \frac{N}{32\pi^2} \phi \epsilon^{\mu\nu\rho\sigma} f_{\mu\nu} f_{\rho\sigma} \right), f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$$

---

---

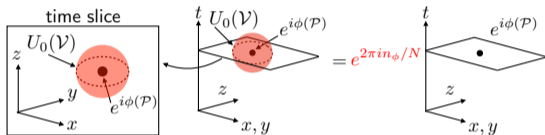
$$S_{\text{UV}} = - \int \left( \frac{1}{2e^2} |da|^2 + \frac{v^2}{2} |d\phi|^2 + i\bar{\psi}_i (\not{\partial} - i\phi)\psi_i + yv\bar{\psi}_i e^{i\gamma_5\phi} \psi_i \right), \psi_i: \text{Dirac fermions } i = 1, \dots, N$$

---

# $\mathbb{Z}_N$ 0-form symmetry: shift symmetry of axion $e^{i\phi(\mathcal{P})}$

0-form transformation [derivation]

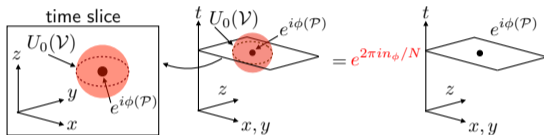
$$\langle U_0(\mathcal{V})e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi / N} \langle e^{i\phi(\mathcal{P})} \rangle$$



$\mathbb{Z}_N$  0-form symmetry: shift symmetry of axion  $e^{i\phi(\mathcal{P})}$

0-form transformation [derivation]

$$\langle U_0(\mathcal{V})e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi / N} \langle e^{i\phi(\mathcal{P})} \rangle$$

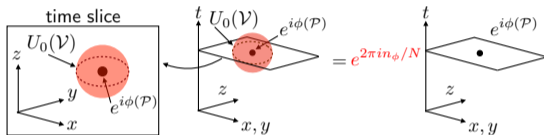


- Conservation law: EOM of axion  $d(v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da) = 0$  ( $v^2 \square \phi + \frac{N}{4\pi^2} \mathbf{E} \cdot \mathbf{B} = 0$ )

# $\mathbb{Z}_N$ 0-form symmetry: shift symmetry of axion $e^{i\phi(\mathcal{P})}$

0-form transformation [derivation]

$$\langle U_0(\mathcal{V})e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi / N} \langle e^{i\phi(\mathcal{P})} \rangle$$

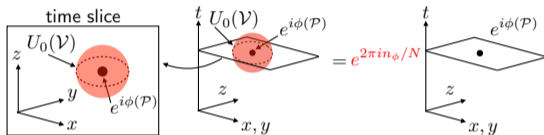


- Conservation law: EOM of axion  $d(v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da) = 0$  ( $v^2 \square \phi + \frac{N}{4\pi^2} \mathbf{E} \cdot \mathbf{B} = 0$ )
  1. Symmetry generator (3D):  $U_0(e^{2\pi i n_\phi / N}, \mathcal{V}) = \exp\left(\frac{2\pi i n_\phi}{N} \int_{\mathcal{V}} (-v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da)\right)$  ( $\mathcal{V}$ : 3D worldvolume)

# $\mathbb{Z}_N$ 0-form symmetry: shift symmetry of axion $e^{i\phi(\mathcal{P})}$

0-form transformation [derivation]

$$\langle U_0(\mathcal{V})e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi / N} \langle e^{i\phi(\mathcal{P})} \rangle$$

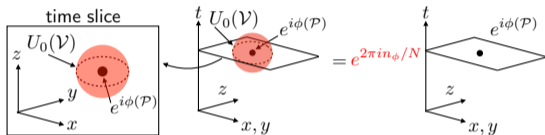


- Conservation law: EOM of axion  $d(v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da) = 0$  ( $v^2 \square \phi + \frac{N}{4\pi^2} \mathbf{E} \cdot \mathbf{B} = 0$ )
  1. Symmetry generator (3D):  $U_0(e^{2\pi i n_\phi / N}, \mathcal{V}) = \exp\left(\frac{2\pi i n_\phi}{N} \int_{\mathcal{V}} (-v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da)\right)$  ( $\mathcal{V}$ : 3D worldvolume)
  2. Symmetry group:  $\mathbb{Z}_N$  (Not  $U(1)$  due to chiral anomaly) [detail]

# $\mathbb{Z}_N$ 0-form symmetry: shift symmetry of axion $e^{i\phi(\mathcal{P})}$

0-form transformation [derivation]

$$\langle U_0(\mathcal{V})e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi / N} \langle e^{i\phi(\mathcal{P})} \rangle$$

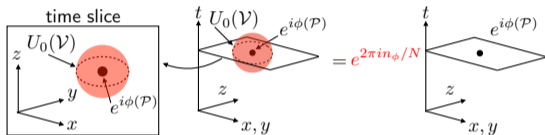


- Conservation law: EOM of axion  $d(v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da) = 0$  ( $v^2 \square \phi + \frac{N}{4\pi^2} \mathbf{E} \cdot \mathbf{B} = 0$ )
  1. Symmetry generator (3D):  $U_0(e^{2\pi i n_\phi / N}, \mathcal{V}) = \exp\left(\frac{2\pi i n_\phi}{N} \int_{\mathcal{V}} (-v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da)\right)$  ( $\mathcal{V}$ : 3D worldvolume)
  2. Symmetry group:  $\mathbb{Z}_N$  (Not  $U(1)$  due to chiral anomaly) [detail]
  3. Charged object (0D):  $e^{i\phi(\mathcal{P})}$  (invariant under  $\phi + 2\pi \sim \phi$ )

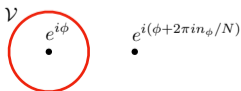
# $\mathbb{Z}_N$ 0-form symmetry: shift symmetry of axion $e^{i\phi(\mathcal{P})}$

0-form transformation [derivation]

$$\langle U_0(\mathcal{V})e^{i\phi(\mathcal{P})} \rangle = e^{2\pi i n_\phi / N} \langle e^{i\phi(\mathcal{P})} \rangle$$



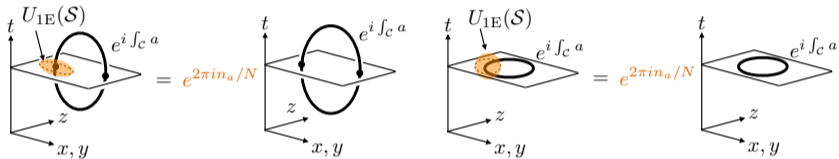
- Conservation law: EOM of axion  $d(v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da) = 0$  ( $v^2 \square \phi + \frac{N}{4\pi^2} \mathbf{E} \cdot \mathbf{B} = 0$ )
  1. Symmetry generator (3D):  $U_0(e^{2\pi i n_\phi / N}, \mathcal{V}) = \exp\left(\frac{2\pi i n_\phi}{N} \int_{\mathcal{V}} (-v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da)\right)$  ( $\mathcal{V}$ : 3D worldvolume)
  2. Symmetry group:  $\mathbb{Z}_N$  (Not  $U(1)$  due to chiral anomaly) [detail]
  3. Charged object (0D):  $e^{i\phi(\mathcal{P})}$  (invariant under  $\phi + 2\pi \sim \phi$ )
- $U_0 =$  (external) axionic domain wall ( $\phi(\mathcal{P})$  inside  $\mathcal{V}$  differs from  $\phi(\mathcal{P})$  outside  $\mathcal{V}$  by  $2\pi n_\phi / N$ )



# Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop

1-form symmetry transformation [derivation]

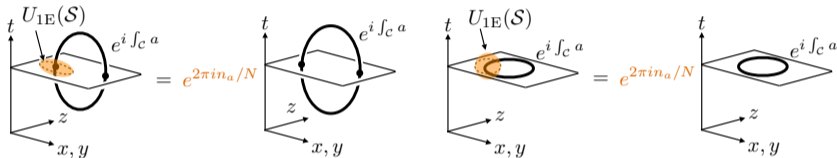
$$\langle U_{1E}(\mathcal{S}) e^{i \int_C a} \rangle = e^{2\pi i n_a / N} \langle e^{i \int_C a} \rangle$$



# Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop

1-form symmetry transformation [derivation]

$$\langle U_{1E}(\mathcal{S}) e^{i \int_C a} \rangle = e^{2\pi i n_a / N} \langle e^{i \int_C a} \rangle$$

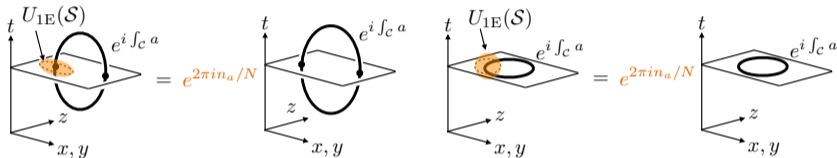


- Cons. law: elec. Gauss law & Maxwell-Ampère law (EOM)  $d(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da) = 0$

# Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop

1-form symmetry transformation [derivation]

$$\langle U_{1E}(\mathcal{S}) e^{i \int_C a} \rangle = e^{2\pi i n_a / N} \langle e^{i \int_C a} \rangle$$



- Cons. law: elec. Gauss law & Maxwell-Ampère law (EOM)  $d(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da) = 0$

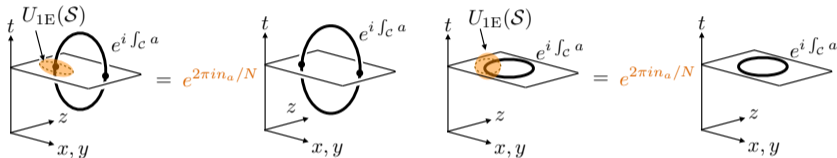
1. Symmetry generator: surface integral of elec. flux or line integral of mag. field

$$U_{1E}(e^{2\pi i n_a / N}, \mathcal{S}) = \exp\left(\frac{2\pi i n_a}{N} \int_{\mathcal{S}} \left(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da\right)\right)$$

# Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop

1-form symmetry transformation [derivation]

$$\langle U_{1E}(\mathcal{S}) e^{i \int_C a} \rangle = e^{2\pi i n_a / N} \langle e^{i \int_C a} \rangle$$



- Cons. law: elec. Gauss law & Maxwell-Ampère law (EOM)  $d(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da) = 0$

1. Symmetry generator: surface integral of elec. flux or line integral of mag. field

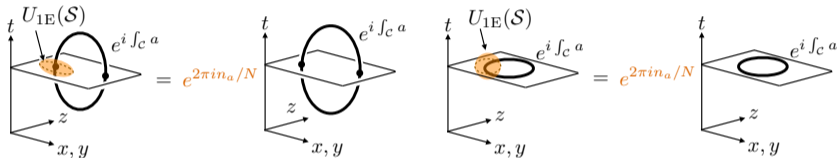
$$U_{1E}(e^{2\pi i n_a / N}, \mathcal{S}) = \exp\left(\frac{2\pi i n_a}{N} \int_{\mathcal{S}} \left(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da\right)\right)$$

2. Symmetry group:  $\mathbb{Z}_N$  (due to anomaly)

# Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop

1-form symmetry transformation [derivation]

$$\langle U_{1E}(\mathcal{S}) e^{i \int_C a} \rangle = e^{2\pi i n_a / N} \langle e^{i \int_C a} \rangle$$



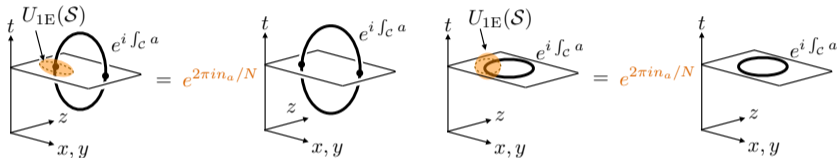
- Cons. law: elec. Gauss law & Maxwell-Ampère law (EOM)  $d(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da) = 0$ 
  1. Symmetry generator: surface integral of elec. flux or line integral of mag. field  

$$U_{1E}(e^{2\pi i n_a / N}, \mathcal{S}) = \exp\left(\frac{2\pi i n_a}{N} \int_{\mathcal{S}} (\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da)\right)$$
  2. Symmetry group:  $\mathbb{Z}_N$  (due to anomaly)
  3. Charged object: Wilson loop  $e^{i \int_C a}$  (worldline of test particle or instantaneous current)

# Electric $\mathbb{Z}_N$ 1-form symmetry: phase rotation of Wilson loop

1-form symmetry transformation [derivation]

$$\langle U_{1E}(\mathcal{S}) e^{i \int_C a} \rangle = e^{2\pi i n_a / N} \langle e^{i \int_C a} \rangle$$



- Cons. law: elec. Gauss law & Maxwell-Ampère law (EOM)  $d(\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da) = 0$ 
  1. Symmetry generator: surface integral of elec. flux or line integral of mag. field  

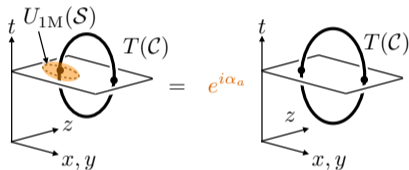
$$U_{1E}(e^{2\pi i n_a / N}, \mathcal{S}) = \exp\left(\frac{2\pi i n_a}{N} \int_{\mathcal{S}} (\frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da)\right)$$
  2. Symmetry group:  $\mathbb{Z}_N$  (due to anomaly)
  3. Charged object: Wilson loop  $e^{i \int_C a}$  (worldline of test particle or instantaneous current)
- $U_{1E}$ : external elec. field localized on a surface  $\mathcal{S}$



## Magnetic $U(1)$ 1-form symmetry: phase rotation of 't Hooft loop

### 1-form symmetry transformation

$$\langle U_{1M}(\mathcal{S})T(\mathcal{C}) \rangle = e^{i\alpha_a} \langle T(\mathcal{C}) \rangle$$



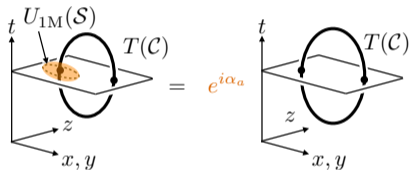
Cons. law: mag. Gauss law & Faraday law (Bianchi id.)  $dda = 0$

1. Symmetry generator: surface integral of mag. flux  $U_{1M}(e^{i\alpha_a}, \mathcal{S}) = \exp\left(\frac{i\alpha_a}{2\pi} \int_{\mathcal{S}} da\right)$

## Magnetic $U(1)$ 1-form symmetry: phase rotation of 't Hooft loop

### 1-form symmetry transformation

$$\langle U_{1M}(\mathcal{S})T(\mathcal{C}) \rangle = e^{i\alpha_a} \langle T(\mathcal{C}) \rangle$$



Cons. law: mag. Gauss law & Faraday law (Bianchi id.)  $dda = 0$

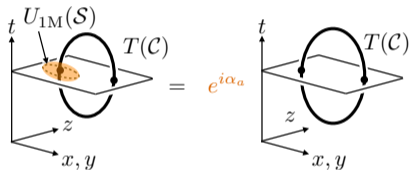
1. Symmetry generator: surface integral of mag. flux  $U_{1M}(e^{i\alpha_a}, \mathcal{S}) = \exp\left(\frac{i\alpha_a}{2\pi} \int_{\mathcal{S}} da\right)$
2. Charged object: 't Hooft loop  $T(\mathcal{C})$  (Worldline of mag. monopole)

$$\int_{\mathcal{S}} da = 2\pi \text{ in the presence of monopole}$$

## Magnetic $U(1)$ 1-form symmetry: phase rotation of 't Hooft loop

### 1-form symmetry transformation

$$\langle U_{1M}(\mathcal{S})T(\mathcal{C}) \rangle = e^{i\alpha_a} \langle T(\mathcal{C}) \rangle$$



Cons. law: mag. Gauss law & Faraday law (Bianchi id.)  $dda = 0$

1. Symmetry generator: surface integral of mag. flux  $U_{1M}(e^{i\alpha_a}, \mathcal{S}) = \exp\left(\frac{i\alpha_a}{2\pi} \int_{\mathcal{S}} da\right)$

2. Charged object: 't Hooft loop  $T(\mathcal{C})$  (Worldline of mag. monopole)

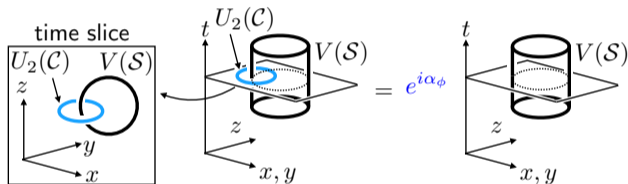
$$\int_{\mathcal{S}} da = 2\pi \text{ in the presence of monopole}$$

3. Symmetry group:  $U(1)$  (magnetic flux is quantized)

# $U(1)$ 2-form symmetry: phase rotation of axionic string

## 2-form symmetry transformation

$$\langle U_2(\mathcal{C})V(\mathcal{S}) \rangle = e^{i\alpha\phi} \langle V(\mathcal{S}) \rangle$$



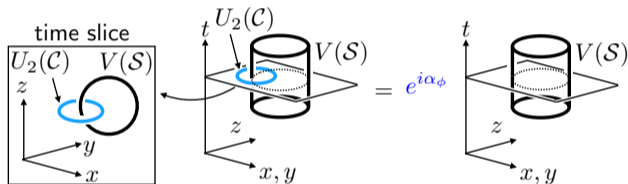
Cons. law: Bianchi identity of axion  $dd\phi = 0$

1. Symmetry generator: winding number of axion  $U_2(e^{i\alpha\phi}, \mathcal{C}) = \exp\left(\frac{i\alpha\phi}{2\pi} \int_{\mathcal{C}} d\phi\right)$

# $U(1)$ 2-form symmetry: phase rotation of axionic string

## 2-form symmetry transformation

$$\langle U_2(\mathcal{C})V(\mathcal{S}) \rangle = e^{i\alpha\phi} \langle V(\mathcal{S}) \rangle$$



Cons. law: Bianchi identity of axion  $dd\phi = 0$

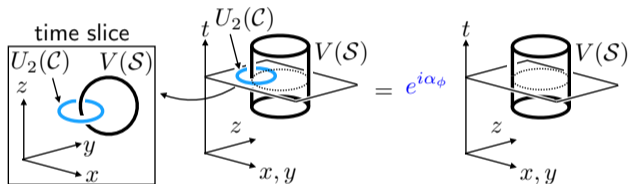
1. Symmetry generator: winding number of axion  $U_2(e^{i\alpha\phi}, \mathcal{C}) = \exp\left(\frac{i\alpha\phi}{2\pi} \int_{\mathcal{C}} d\phi\right)$
2. Charged object: worldsheet of axionic string  $V(\mathcal{S})$

$$\int_{\mathcal{C}} d\phi = 2\pi \text{ in the presence of axionic string}$$

# $U(1)$ 2-form symmetry: phase rotation of axionic string

## 2-form symmetry transformation

$$\langle U_2(\mathcal{C})V(\mathcal{S}) \rangle = e^{i\alpha\phi} \langle V(\mathcal{S}) \rangle$$



Cons. law: Bianchi identity of axion  $dd\phi = 0$

1. Symmetry generator: winding number of axion  $U_2(e^{i\alpha\phi}, \mathcal{C}) = \exp\left(\frac{i\alpha\phi}{2\pi} \int_{\mathcal{C}} d\phi\right)$

2. Charged object: worldsheet of axionic string  $V(\mathcal{S})$

$$\int_{\mathcal{C}} d\phi = 2\pi \text{ in the presence of axionic string}$$

3. Symmetry group:  $U(1)$  (winding number is quantized)

## Summary of higher-form symmetries

There are 4 kinds of higher-form symmetries

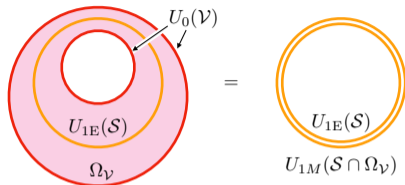
	group	transf. law	
0-form	$\mathbb{Z}_N$	shift of axion	$e^{i\phi(\mathcal{P})} \rightarrow e^{i(\phi(\mathcal{P})+2\pi/N)}$
Elec. 1-form	$\mathbb{Z}_N$	phase rot. of Wilson loop	$e^{i \int_{\mathcal{C}} a} \rightarrow e^{2\pi i/N} e^{i \int_{\mathcal{C}} a}$
Mag. 1-form	$U(1)$	phase rot. of 't Hooft loop	$T(\mathcal{C}) \rightarrow e^{i\alpha_1} T(\mathcal{C})$
2-form	$U(1)$	phase rot. of axionic string	$V(\mathcal{S}) \rightarrow e^{i\alpha_2} V(\mathcal{S})$

# Witten effect and anomalous Hall effect from higher-form symmetries

Q: What is mathematical structure behind the phenomena?

Method: evaluating correlation functions of symmetry generators (extension of current algebra)

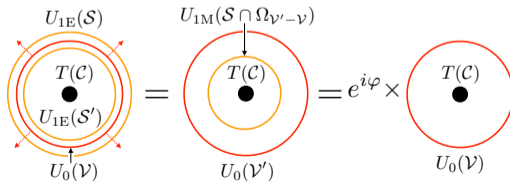
Witten effect: 0-form  $\times$  elec. 1-form = mag. 1-form



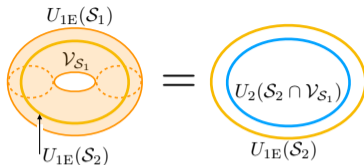
$$\langle U_0(e^{2\pi i n \phi / N}, \mathcal{V}) U_{1E}(e^{2\pi i n a / N}, \mathcal{S}) \rangle = \langle U_{1M}(e^{-2\pi i n \phi n a / N}, \Omega_{\mathcal{V}} \cap \mathcal{S}) U_{1E}(e^{2\pi i n a / N}, \mathcal{S}) \rangle$$

(Shifting  $\phi$  of  $U_{1E} \sim \exp(i \int_{\mathcal{S}} \phi da)$  by  $U_0$ ) [detail]

We can describe the Witten effect (Counting electric flux induced from axionic domain wall)



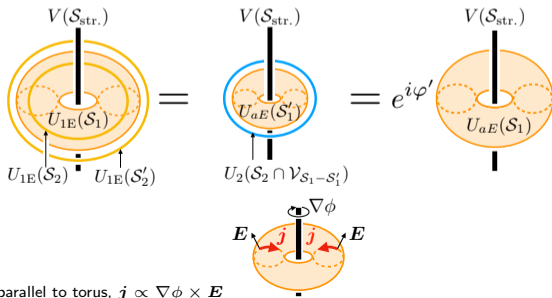
# Anomalous Hall effect: elec. 1-form $\times$ elec. 1-form = 2-form



$$\langle U_{1E}(e^{2\pi i n_a / N}, S) U_{1E}(e^{2\pi i n'_a / N}, S') \rangle = \langle U_2(e^{-2\pi i n_a n'_a / N}, V_S \cap S') U_{1E}(e^{2\pi i n'_a / N}, S') \rangle$$

(Shifting  $a$  of  $U_{1E} \sim \exp(\int_S \phi da)$  by  $U_{1E}$ ) [detail]

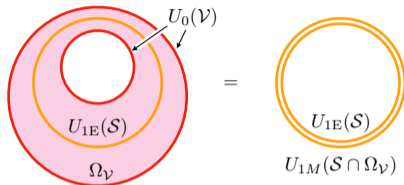
We can describe the anomalous Hall effect (counting magnetic flux by Ampère law)



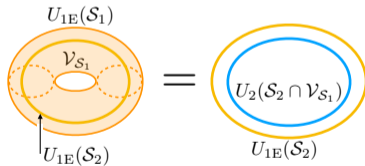
Note: Direction of Hall current = parallel to torus,  $j \propto \nabla\phi \times E$

Q: What is mathematical structure behind the phenomena?

- Witten effect: 0-form  $\times$  1-form = 1-form



- Anomalous Hall effect: 1-form  $\times$  1-form = 2-form



They are different from ordinary current algebra based on ordinary group theory: 0-form  $\times$  0-form = 0-form.

We need a group structure beyond ordinary group theory

3-group in axion electrodynamics

## 3-group? Mathematically... [Conduché '84; Matrins & Picken '09]

It is called semistrict 3-group or 2-crossed module

Axioms (which we will use)

- 3 kinds of groups  $G, H, L$
- Action  $\triangleright$  of  $G$  on  $G, H, L$ : (extension of adjoint representation)

$$g \triangleright g' = gg'g^{-1}, g \triangleright h, g \triangleright l$$

- Peiffer lifting: map from  $h, h' \in H$  to  $L$

$$\{h, h'\} \in L$$

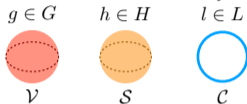
- ...

But the axioms seem abstract... Can we understand them physically?

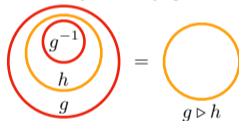
# 3-group? Physically...

We can describe the 3-group in terms of higher-form symmetries

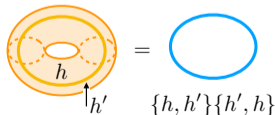
- Elements of  $G$ ,  $H$ ,  $L$ : Symmetry generators for 0-, 1-, 2-form symmetries (they are the unitary representations, precisely)



- Action of  $G \triangleright$ : enclosing of elements by 0-form symmetry generators



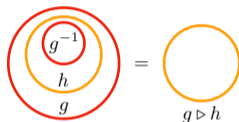
- Peiffer lifting  $\{h, h'\}$ : “braid” and then “link” of 1-form sym. generators (surface link) [Carter, et al '01]



The Witten effect & the anomalous Hall effect can be described!

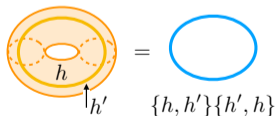
### 3-group in axion electrodynamics

- 3 kinds of groups:  $G = \mathbb{Z}_N$  (0-form),  $H = \mathbb{Z}_N \times U(1)$  (1-form),  $L = U(1)$  (2-form)
- Witten effect = action of  $G$ :



$$e^{2\pi i n_\phi / N} \triangleright (e^{2\pi i n_a / N}, e^{i\alpha_a}) = (e^{2\pi i n_a}, e^{-2\pi i n_\phi n_a / N} e^{i\alpha_a})$$

- Anomalous Hall effect = Peiffer lifting:



$$\{(e^{2\pi i n_a / N}, e^{i\alpha_a}), (e^{2\pi i n'_a / N}, e^{i\alpha'_a})\}^2 = e^{-2\pi i n_a n'_a / N} \text{ (we may need a spin structure)}$$

## Summary



- Axion electrodynamics in 4D possesses a 3-group structure.
- The Witten effect and the anomalous Hall effect can be understood as 3-group transformations.
- Future work: relation to anomaly inflow, (including DOF on defects), higher-form symmetries for massive axions, etc.

## Appendix

Group of 0-form symmetry is  $\mathbb{Z}_N$  (1/4)

Large gauge invariance of  $\int_{\mathcal{V}} a \wedge da$  is crucial

1. Symmetry group seems continuous, i.e.  $U(1)$  because conserved current exists

$$j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da, \quad dj_{\phi 3} = 0$$

Group of 0-form symmetry is  $\mathbb{Z}_N$  (1/4)

Large gauge invariance of  $\int_{\mathcal{V}} a \wedge da$  is crucial

1. Symmetry group seems continuous, i.e.  $U(1)$  because conserved current exists

$$j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da, \quad dj_{\phi 3} = 0$$

2. However, the conserved current is not gauge invariant under  $a \rightarrow a + d\lambda$ .

Group of 0-form symmetry is  $\mathbb{Z}_N$  (1/4)

Large gauge invariance of  $\int_{\mathcal{V}} a \wedge da$  is crucial

1. Symmetry group seems continuous, i.e.  $U(1)$  because conserved current exists

$$j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da, \quad dj_{\phi 3} = 0$$

2. However, the conserved current is not gauge invariant under  $a \rightarrow a + d\lambda$ .
3. The integral of the current,  $e^{i\alpha_0} \int_{\mathcal{V}} j_{\phi 3}$  with  $e^{i\alpha_0} \in U(1)$ , seems gauge invariant, since the gauge transf. is a total derivative.

Group of 0-form symmetry is  $\mathbb{Z}_N$  (1/4)

Large gauge invariance of  $\int_{\mathcal{V}} a \wedge da$  is crucial

1. Symmetry group seems continuous, i.e.  $U(1)$  because conserved current exists

$$j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da, \quad dj_{\phi 3} = 0$$

2. However, the conserved current is not gauge invariant under  $a \rightarrow a + d\lambda$ .
3. The integral of the current,  $e^{i\alpha_0} \int_{\mathcal{V}} j_{\phi 3}$  with  $e^{i\alpha_0} \in U(1)$ , seems gauge invariant, since the gauge transf. is a total derivative.
4. However, the integral is not large gauge invariant.

Group of 0-form symmetry is  $\mathbb{Z}_N$  (1/4)

Large gauge invariance of  $\int_{\mathcal{V}} a \wedge da$  is crucial

1. Symmetry group seems continuous, i.e.  $U(1)$  because conserved current exists

$$j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da, \quad dj_{\phi 3} = 0$$

2. However, the conserved current is not gauge invariant under  $a \rightarrow a + d\lambda$ .
3. The integral of the current,  $e^{i\alpha_0} \int_{\mathcal{V}} j_{\phi 3}$  with  $e^{i\alpha_0} \in U(1)$ , seems gauge invariant, since the gauge transf. is a total derivative.
4. However, the integral is not large gauge invariant.

Further, the shift symmetry of axion should be broken to  $\mathbb{Z}_N$  by chiral anomaly from UV viewpoint.

## Group of 0-form symmetry is $\mathbb{Z}_N$ (2/4)

In the following, we consider the topological unitary operator

$$U_0(e^{i\alpha_0}, \mathcal{V}) = e^{-i\alpha_0 \int_{\mathcal{V}} (v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da)},$$

and show that the large gauge invariance of  $U_0(e^{i\alpha_0}, \mathcal{V})$  requires  $e^{i\alpha_0} \in \mathbb{Z}_N$ .

**Note:** It is essentially the same as quantization of the level of Chern-Simons term in (2 + 1)

**dim.** [Henneaux & Teitelboim '86]

1. Since  $d\phi$  is gauge invariant, we focus on the term  $e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\mathcal{V}} a \wedge da}$ .

## Group of 0-form symmetry is $\mathbb{Z}_N$ (2/4)

In the following, we consider the topological unitary operator

$$U_0(e^{i\alpha_0}, \mathcal{V}) = e^{-i\alpha_0 \int_{\mathcal{V}} (v^2 * d\phi + \frac{N}{8\pi^2} a \wedge da)},$$

and show that the large gauge invariance of  $U_0(e^{i\alpha_0}, \mathcal{V})$  requires  $e^{i\alpha_0} \in \mathbb{Z}_N$ .

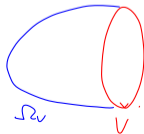
**Note:** It is essentially the same as quantization of the level of Chern-Simons term in (2 + 1)

dim. [Henneaux & Teitelboim '86]

1. Since  $d\phi$  is gauge invariant, we focus on the term  $e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\mathcal{V}} a \wedge da}$ .
2. In order to make the integrand be manifestly gauge invariant, we define

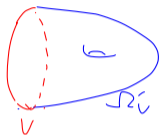
$$e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\mathcal{V}} a \wedge da} := e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\Omega_{\mathcal{V}}} da \wedge da}$$

on a 4d space  $\Omega_{\mathcal{V}}$  satisfying  $\partial\Omega_{\mathcal{V}} = \mathcal{V}$ .



Group of 0-form symmetry is  $\mathbb{Z}_N$  (3/4)

3. However, we have chosen a redundant space  $\Omega_{\mathcal{V}}$ , which may be replaced by another 4d space  $\Omega'_{\mathcal{V}}$ ,



4. The redundancy of the choice is absent if

$$e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\Omega_{\mathcal{V}}} da \wedge da} = e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\Omega'_{\mathcal{V}}} da \wedge da}, \quad \text{i.e.} \quad e^{-i\alpha_0 \frac{N}{8\pi^2} \int_{\Omega} da \wedge da} = 1$$

where  $\Omega = \Omega_{\mathcal{V}} - \Omega'_{\mathcal{V}}$  is a closed 4d space. ( $\Omega$  satisfies  $\partial\Omega = \mathcal{V} - \mathcal{V} = 0$ )



Group of 0-form symmetry is  $\mathbb{Z}_N$  (4/4)

5. Since  $\Omega$  is closed subspace  $\partial\Omega = 0$ , the Dirac quantization condition requires  $\int_{\Omega} da \wedge da \in 2 \cdot (2\pi)^2 \mathbb{Z}$
6. Therefore, we have the condition  $\alpha_0 \in \frac{2\pi}{N} \mathbb{Z}$ , which means

$$e^{i\alpha_0} \in \mathbb{Z}_N.$$

[back]

## Derivation of 0-form transformation 1/4

We evaluate the correlation function

$$\langle U_0(e^{2\pi i n \phi / N}, \mathcal{V}) e^{i\phi(\mathcal{P})} \rangle = \int \mathcal{D}[\phi, a] e^{iS + \frac{2\pi i n \phi}{N} \int_{\mathcal{V}} j_{\phi 3} + i\phi(\mathcal{P})}$$



Let us integrate out  $j_{\phi 3} = -v^2 * d\phi - \frac{N}{8\pi^2} a \wedge da$

1. local operators  $\rightarrow$  spacetime integral

- $e^{i\phi(\mathcal{P})} = e^{i \int \phi \delta_4(\mathcal{P})}$ , where  $\delta_4(\mathcal{P}) = \delta^4(x - \mathcal{P}) dx^0 \wedge \dots \wedge dx^3$

- $\int_{\mathcal{V}} j_{\phi 3} = \int_{\partial\Omega_{\mathcal{V}}} j_{\phi 3} = \int_{\Omega_{\mathcal{V}}} dj_{\phi 3} = \int dj_{\phi 3} \delta_0(\Omega_{\mathcal{V}})$ ,

where  $\Omega_{\mathcal{V}}$  is a 4d subspace satisfying  $\partial\Omega_{\mathcal{V}} = \mathcal{V}$ ,

$$\delta_0(\Omega_{\mathcal{V}}) = \int_{\Omega_{\mathcal{V}}} \delta^4(x - y) dy^0 \wedge \dots \wedge dy^3.$$

## Derivation of 0-form transformation 2/4

2. Action + symmetry generator can be rewritten by

### Completing the square

$$\begin{aligned} S[\phi, a] + \frac{2\pi n\phi}{N} \int dj_{\phi 3} \delta_0(\Omega_{\mathcal{V}}) \\ = S\left[\phi - \frac{2\pi n\phi}{N} \delta_0(\Omega_{\mathcal{V}}), a\right] + \frac{v^2}{2} \left(\frac{2\pi n\phi}{N}\right)^2 \int \delta_1(\mathcal{V}) \wedge * \delta_1(\mathcal{V}). \end{aligned}$$

Here, we have defined/used

- $\delta_1(\mathcal{V}) = \frac{\epsilon^{\mu\nu\rho\sigma}}{3!} dx^\sigma \int_{\mathcal{V}} \delta^4(x-y) dy^\nu \wedge dy^\rho \wedge dy^\sigma$
- $d\delta_0(\Omega_{\mathcal{V}}) = \delta_1(\partial\Omega_{\mathcal{V}}) = \delta_1(\mathcal{V})$
- $\int \omega_3 \wedge d\delta_0(\Omega_{\mathcal{V}}) = \int d\omega_3 \delta_0(\Omega_{\mathcal{V}}) = \int_{\Omega_{\mathcal{V}}} d\omega_3 = \int_{\partial\Omega_{\mathcal{V}}} \omega_3 = \int_{\mathcal{V}} \omega_3 = \int \omega_3 \wedge \delta_1(\mathcal{V})$

---

$$S[\phi, a] = - \int \left( \frac{v^2}{2} d\phi \wedge *d\phi + \frac{1}{2e^2} da \wedge *da - \frac{N}{8\pi^2} \phi da \wedge da \right), \quad j_{\phi 3} = -v^2 *d\phi - \frac{N}{8\pi^2} a \wedge da$$

---

## Derivation of 0-form transformation 3/4

3. By the redefinition  $\phi - \frac{2\pi n\phi}{N}\delta_0(\Omega_{\mathcal{V}}) \rightarrow \phi$ , we have

Integrating out  $U_0$

$$\begin{aligned}\langle U_0(e^{2\pi i n\phi/N}, \mathcal{V})e^{i\phi(\mathcal{P})} \rangle &= \int \mathcal{D}[\phi, a] e^{iS[\phi - \frac{2\pi n\phi}{N}\delta_0(\Omega_{\mathcal{V}}), a] + i \int \phi \delta_4(\mathcal{P})} \\ &= e^{\frac{2\pi i n\phi}{N} \int \delta_0(\Omega_{\mathcal{V}})\delta_4(\mathcal{P})} \langle e^{i\phi(\mathcal{P})} \rangle,\end{aligned}$$

( $\int \delta_1(\mathcal{V}) \wedge * \delta_1(\mathcal{V})$  has been regularized by local counter term.)

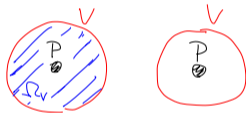
What is the phase factor  $\int \delta_0(\Omega_{\mathcal{V}})\delta_4(\mathcal{P})$ ?

## Derivation of 0-form transformation 4/4

### Linking number

$$\int \delta_0(\Omega_{\mathcal{V}}) \delta_4(\mathcal{P}) = \int_{\Omega_{\mathcal{V}}} \delta_4(\mathcal{P}) = \text{Link}(\mathcal{V}, \mathcal{P}) \in \mathbb{Z}$$

Intersection of  $\Omega_{\mathcal{V}}$  and  $\mathcal{P}$  = link of  $\mathcal{V}$  and  $\mathcal{P}$



Therefore, we obtain

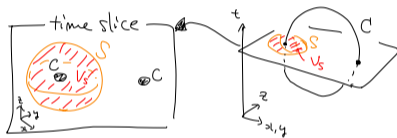
### $\mathbb{Z}_N$ 0-form transformation

$$\langle U_0(e^{2\pi i n \phi / N}, \mathcal{V}) e^{i\phi(\mathcal{P})} \rangle = e^{\frac{2\pi i n \phi}{N} \text{Link}(\mathcal{V}, \mathcal{P})} \langle e^{i\phi(\mathcal{P})} \rangle$$

## Derivation of 1-form transformation 1/4

We evaluate the correlation function

$$\langle U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) e^{i \int_C a} \rangle = \int \mathcal{D}[\phi, a] e^{iS + \frac{2\pi i n_a}{N} \int_S j_{a2} + i \int_C a}$$



Let us integrate out  $j_{a2} = \frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da$

1. local operators  $\rightarrow$  spacetime integral:

- $e^{i \int_C a} = e^{i \int a \wedge \delta_3(C)}$ , where  $\delta_3(C) = \frac{\epsilon^{\mu\nu\rho\sigma}}{3!} dx^\nu \wedge \dots \wedge dx^\sigma \int_C \delta^4(x-y) dy^\mu$
- $\int_S j_{a2} = \int_{\partial \mathcal{V}_S} j_{a2} = \int_{\mathcal{V}_S} dj_{a2} = \int dj_{a2} \wedge \delta_1(\mathcal{V}_S)$ ,

where  $\mathcal{V}_S$  is a 3d subspace satisfying  $\partial \mathcal{V}_S = S$ .

## Derivation of 1-form transformation 2/4

2. Action + symmetry generator can be rewritten by

### Completing the square

$$\begin{aligned} S[\phi, a] + \frac{2\pi n\phi}{N} \int dj_{a2} \wedge \delta_1(\mathcal{V}_S) \\ = S[\phi, a + \frac{2\pi n\phi}{N} \delta_1(\mathcal{V}_S)] + \frac{1}{2e^2} \left( \frac{2\pi n_a}{N} \right)^2 \int \delta_2(\mathcal{S}) \wedge * \delta_2(\mathcal{S}). \end{aligned}$$

Here, we have defined/used

- $\delta_2(\mathcal{S}) = \frac{\epsilon^{\mu\nu\rho\sigma}}{2! \cdot 2!} dx^\rho \wedge dx^\sigma \int_{\mathcal{S}} \delta^4(x-y) dy^\mu \wedge dy^\nu$
- $d\delta_1(\mathcal{V}_S) = -\delta_2(\partial\mathcal{V}_S) = -\delta_2(\mathcal{S})$
- $-\int \omega_2 \wedge d\delta_1(\mathcal{V}_S) = \int d\omega_2 \wedge \delta_1(\mathcal{V}_S) = \int_{\mathcal{V}_S} d\omega_2 = \int_{\partial\mathcal{V}_S} \omega_2 = \int_{\mathcal{S}} \omega_2 = \int \omega_2 \wedge \delta_2(\mathcal{S})$

---

$$S[\phi, a] = - \int \left( \frac{v^2}{2} d\phi \wedge *d\phi + \frac{1}{2e^2} da \wedge *da - \frac{N}{8\pi^2} \phi da \wedge da \right), \quad j_{a2} = \frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da$$

---

## Derivation of 1-form transformation 3/4

3. By the redefinition  $a + \frac{2\pi n\phi}{N} \delta_1(\mathcal{V}_S) \rightarrow a$ , we have

Integrating out  $U_{1E}$

$$\begin{aligned}\langle U_{1E}(e^{2\pi i n\phi/N}, \mathcal{V}) e^{i\phi(\mathcal{P})} \rangle &= \int \mathcal{D}[\phi, a] e^{iS[\phi, a + \frac{2\pi n\phi}{N} \delta_1(\mathcal{V}_S)] + i \int a \wedge \delta_3(\mathcal{C})} \\ &= e^{\frac{2\pi i n a}{N} \int \delta_3(\mathcal{C}) \wedge \delta_1(\mathcal{V}_S)} \langle e^{i \int \mathcal{C} a} \rangle,\end{aligned}$$

( $\int \delta_2(\mathcal{S}) \wedge * \delta_2(\mathcal{S})$  has been regularized by local counter term.)

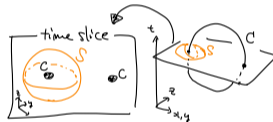
What is the phase factor  $\int \delta_3(\mathcal{C}) \wedge \delta_1(\mathcal{V}_S)$ ?

## Derivation of 1-form transformation 4/4

### Linking number

$$\int \delta_3(\mathcal{C}) \wedge \delta_1(\mathcal{V}_S) = \int_{\mathcal{V}_S} \delta_3(\mathcal{C}) = \text{Link}(\mathcal{S}, \mathcal{C}) \in \mathbb{Z}$$

Intersection of  $\mathcal{V}_S$  and  $\mathcal{C} = \text{link of } \mathcal{S} \text{ and } \mathcal{C}$



Therefore, we obtain

### $\mathbb{Z}_N$ 1-form transformation

$$\langle U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) e^{i \int_C a} \rangle = e^{\frac{2\pi i n_a}{N} \text{Link}(\mathcal{S}, \mathcal{C})} \langle e^{i \int_C a} \rangle$$

Correlation function  $\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) \rangle$

We evaluate

$$\langle U_0(e^{2\pi i n_\phi/N}, \mathcal{V}) U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) \rangle = \int \mathcal{D}[\phi, a] e^{iS[\phi, a] + \frac{2\pi i n_\phi}{N} \int_{\mathcal{V}} j_{\phi 3} + \frac{2\pi i n_a}{N} \int_{\mathcal{S}} j_{a 2}}$$

- Integrating out  $j_{\phi 3}$  can be done by the shift  $\phi \rightarrow \phi + \frac{2\pi n_\phi}{N} \delta_0(\Omega_{\mathcal{V}})$
- $U_{1E}(e^{2\pi i n_a/N}, \mathcal{S})$  is shifted as

$$\begin{aligned} U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) &\rightarrow U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) e^{-\frac{2\pi i n_\phi n_a}{N} \cdot \frac{1}{2\pi} \int_{\mathcal{S}} \delta_0(\Omega_{\mathcal{V}}) da} \\ &= U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1M}(e^{-2\pi i n_\phi n_a/N}, \Omega_{\mathcal{V}} \cap \mathcal{S}) \end{aligned}$$

[back]

---

$$U_{1E}(e^{2\pi i n_a/N}, \mathcal{S}) = e^{\frac{2\pi i n_a}{N} \int_{\mathcal{S}} j_{a 2}}, \quad j_{a 2} = \frac{1}{e^2} * da - \frac{N}{4\pi^2} \phi da$$

---

Correlation function  $\langle U_{1\mathbb{E}}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') \rangle$

We evaluate

$$\langle U_{1\mathbb{E}}(e^{2\pi i n_a/N}, \mathcal{S}) U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') \rangle = \int \mathcal{D}[\phi, a] e^{iS[\phi, a] + \frac{2\pi i n_a}{N} \int_{\mathcal{S}} j_{a2} + \frac{2\pi i n'_a}{N} \int_{\mathcal{S}'} j_{a2}}$$

- Integrating out  $U_{1\mathbb{E}}(e^{2\pi i n_a/N}, \mathcal{S})$  can be done by the shift  $a \rightarrow a - \frac{2\pi n_a}{N} \delta_1(\mathcal{V}_{\mathcal{S}})$
- $U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}')$  is shifted as  $(\int_{\mathcal{S}'} * \delta_2(\mathcal{S})$  has been regularized)

$$\begin{aligned} U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') &\rightarrow U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') e^{\frac{2\pi i n_a n'_a}{N} \cdot \frac{1}{2\pi} \int_{\mathcal{S}'} \phi d\delta_1(\mathcal{V}_{\mathcal{S}})} \\ &= U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') e^{\frac{-2\pi i n_a n'_a}{N} \cdot \frac{1}{2\pi} \int_{\mathcal{S}'} d\phi \wedge \delta_1(\mathcal{V}_{\mathcal{S}})} \\ &= U_{1\mathbb{E}}(e^{2\pi i n'_a/N}, \mathcal{S}') U_2(e^{-2\pi i n_a n'_a/N}, \mathcal{V}_{\mathcal{S}} \cap \mathcal{S}') \end{aligned}$$

## Axioms of 3-group 1/3

A semistrict 3-group (2-crossed module) is a set  $(L \xrightarrow{\partial_2} H \xrightarrow{\partial_1} G, \triangleright, \{-, -\})$  satisfying the following axioms. Here,  $G$ ,  $H$ , and  $L$  are groups.

1. The maps

$$\partial_1 : H \rightarrow G, \quad \partial_2 : L \rightarrow H$$

are group homomorphisms  $\partial_1(h_1 h_2) = (\partial_1 h_1)(\partial_1 h_2)$  and  $\partial_2(l_1 l_2) = (\partial_2 l_1)(\partial_2 l_2)$  for  $h_{1,2} \in H$  and  $l_{1,2} \in L$ , respectively. They satisfy

$$\partial_1 \circ \partial_2(l) = 1$$

for  $l \in L$

## Axioms of 3-group 2/3

2.  $\triangleright$  is an action of  $g \in G$  on  $g' \in G$ ,  $h \in H$ , and  $l \in L$  by automorphisms,  $g \triangleright g' \in G$ ,  $g \triangleright h \in H$ , and  $g \triangleright l \in L$ . The action  $g \triangleright g'$  is defined by conjugation,

$$g \triangleright g' := gg'g^{-1}.$$

3.  $\partial_{1,2}$  are  $G$ -equivalent, that is,

$$g \triangleright (\partial_1 h) = \partial_1(g \triangleright h), \quad g \triangleright (\partial_2 l) = \partial_1(g \triangleright l),$$

## Axioms of 3-group 3/3

4. The Peiffer lifting  $\{-, -\}$  is a morphism  $H \times H \rightarrow L$ . In terms of the elements  $h_{1,2} \in H$ ,

$$\{h_1, h_2\} \in L$$

For  $l_{1,2} \in L$ , the Peiffer lifting satisfies

$$\partial_2\{h_1, h_2\} = h_1 h_2 h_1^{-1} (\partial_1 h_1) \triangleright h_2^{-1},$$

$$g \triangleright \{h_1, h_2\} = \{g \triangleright h_1, g \triangleright h_2\},$$

$$\{\partial_2 l_1, \partial_2 l_2\} = l_1 l_2 l_1^{-1} l_2^{-1},$$

$$\{h_1 h_2, h_3\} = \{h_1, h_2 h_3 h_2^{-1}\} (\partial_1 h_1) \triangleright \{h_2, h_3\},$$

$$\{h_1, h_2 h_3\} = \{h_1, h_2\} \{h_1, h_3\} \{\partial_2 \{h_1, h_3\}\}^{-1}, (\partial_1 h_1) \triangleright h_2\},$$

$$\{\partial_2 l_1, h_2\} \{h_2, \partial_2 l_1\} = l_1 (\partial_1 h) \triangleright l_1^{-1}.$$

## Bibliography

# Bibliography - I

- [1] Y. Hidaka, M. Nitta, and R. Yokokura, "Higher-form symmetries and 3-group in axion electrodynamics," Phys. Lett. B **808** (2020) 135672, [arXiv:2006.12532 [hep-th]].  
(page 1).
- [2] Y. Hidaka, M. Nitta, and R. Yokokura, "Global 3-group symmetry and 't Hooft anomalies in axion electrodynamics," arXiv:2009.14368 [hep-th].  
(page 1).
- [3] F. Wilczek, "Two Applications of Axion Electrodynamics," Phys. Rev. Lett. **58** (1987) 1799.  
(pages 4, 5, 8, 9, 15).
- [4] E. Witten, "Dyons of Charge  $e\theta/2\pi$ ," Phys. Lett. B **86** (1979) 283–287.  
(pages 6, 7).
- [5] P. Sikivie, "On the Interaction of Magnetic Monopoles With Axionic Domain Walls," Phys. Lett. **137B** (1984) 353–356.  
(pages 6, 7, 8, 9).
- [6] I. I. Kogan, "Kaluza-Klein and axion domain walls: Induced charge and mass transmutation," Phys. Lett. B **299** (1993) 16–23.  
(pages 6, 7).
- [7] X.-L. Qi, T. Hughes, and S.-C. Zhang, "Topological Field Theory of Time-Reversal Invariant Insulators," Phys. Rev. **B78** (2008) 195424, [arXiv:0802.3537 [cond-mat.mes-hall]].  
(pages 8, 9).
- [8] J. C. Teo and C. Kane, "Topological Defects and Gapless Modes in Insulators and Superconductors," Phys. Rev. B **82** (2010) 115120, [arXiv:1006.0690 [cond-mat.mes-hall]].  
(pages 8, 9).
- [9] T. H. Hansson, V. Oganesyan, and S. L. Sondhi, "Superconductors are topologically ordered," Annals Phys. **313** (2004) no. 2, 497–538, [arXiv:cond-mat/0404327 [cond-mat.supr-con]].  
(page 10).

## Bibliography - II

- [10] D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, "Generalized Global Symmetries," JHEP **02** (2015) 172, [arXiv:1412.5148 [hep-th]].  
(page 10).
- [11] D. Conduché, "Modules croisés généralisés de longueur 2," J. Pure Appl. Alg. **34** (1984) no. 2, 155–178.  
(page 40).
- [12] J. F. Martins and R. Picken, "The fundamental Gray 3-groupoid of a smooth manifold and local 3-dimensional holonomy based on a 2-crossed module," Diff. Geom. Appl. **29** (2009) 179–206, [arXiv:0907.2566 [math.CT]].  
(page 40).
- [13] J. S. Carter, S. Kamada, M. Saito, and S. Satoh, "A theorem of sanderson on link bordisms in dimension 4," Algebr. Geom. Topol. **1** (2001) no. 1, 299–310.  
(page 41).
- [14] M. Henneaux and C. Teitelboim, "Quantization of Topological Mass in the Presence of a Magnetic Pole," Phys. Rev. Lett. **56** (1986) 689–692.  
(pages 50, 51).