

Progress in Heterotic Compactification and Phenomenology

Lara B. Anderson

Virginia Tech

Work done in collaboration with:

J. Gray (VT), M. Larfors, M. Magill, and R. Schneider (Uppsala)

(arXiv:2103.10454)

J. Gray (VT), M. Gerdis, S. Krippendorff (LMU), N. Raghuram (VT), and F. Ruehle (CERN)

(arXiv:2012.04656)

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Motivation

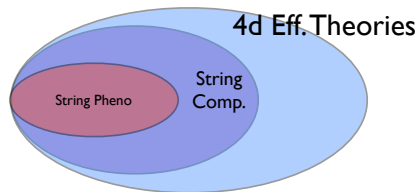
String theory is a powerful extension of quantum field theory, but extracting low-energy physics from string geometry is mathematically challenging...

Higher dimensional theory \rightarrow String Comp. \rightarrow 4d physics

- Need a good toolkit in any corner of string theory to extract the full low energy physics: (e.g. *the mass of the electron?*)

- Rules for “top down” model building?

Patterns/Constraints/Predictions?



What possible EFTs?



Which geometries?

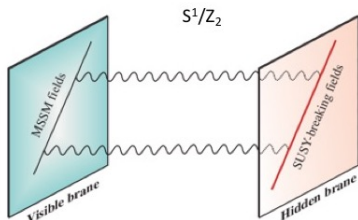
Today: Progress in $\mathcal{N} = 1$, 4D Heterotic String Compactifications...

Heterotic in a nutshell

- Heterotic compactifications amongst earliest approaches to string pheno (Candelas, Horowitz, Strominger, Witten).
- The basic idea: $E_8 \times E_8$ YM coupled to SUGRA in 10D
 \implies Solutions of the form $\mathbb{R}^{1,3} \times X$ with X a compact, 6-dimensional manifold (with gauge field vevs $\subset E_8$ over X)
 \implies Leads to a 4D GUT. (coupled to SUGRA).
- GUT group can be broken to Standard Model symmetry: E.g. $SU(5) \rightarrow SU(3) \times SU(2) \times U(1)$. (i.e. via Wilson lines, Hypercharge flux, etc., Higgs mechanism, etc)

A sketch of Heterotic pheno

- All matter descending from the adjoint of E_8 in 10D \Rightarrow possible to obtain exact charged matter spectrum of the standard model (or MSSM, etc). \Rightarrow Major effort in recent years to create Heterotic SM database ($\sim 10^7$ models). (LA, Gray, Lukas, Palti).
- Gravitationally mediated SUSY breaking (in ‘hidden sector’ E_8).
- Heterotic GUT features significant improvement on field theory GUTs \Rightarrow no problematic relationships on couplings, color doublet/triplet splitting, etc.



But...

- In broad brush strokes, this all looks promising! But... there are several major catches.
 - Don't fully know necessary ingredients to work out theory \Rightarrow Calabi-Yau (Ricci flat) metric and Hermitian-Yang-Mills bundle connection unknown analytically.
 - Need a compelling mechanism for stabilization of geometric moduli (this is coupled to the problem of SUSY breaking). Ideas: Bundle/manifold interplay, Fractional Chern Simons fluxes (Gukov, et al), etc. \Rightarrow **Incomplete.**

Summary of Heterotic Geometry:

- The geometric ingredients include:
 - A complex 3-fold, $X \Rightarrow$ **$SU(3)$ Structure Manifold** (solution of Hull-Strominger System).
 - Two principle H_i -bundles, $(\mathcal{V}_1, \mathcal{V}_2)$ on X (with structure group $H_i \subset E_8$).
Leading to a collection of holomorphic, Mumford (poly)-stable vector bundles: $V_i, V_i^\vee, \wedge^2 V_i, \dots$ etc.
- Compactifying on X leads to $\mathcal{N} = 1$ SUSY in $4D$, while V breaks $E_8 \rightarrow H \times G$ (H_i are the structure groups of V_i)
 - E.g. $H = SU(n)$, $n = 3, 4, 5$ leads to $G = E_6, SO(10), SU(5)$
- Anomaly Cancellation: $ch_2(TX) = ch_2(V_1) + ch_2(V_2) + [W]_{eff}$
- Matter and **Moduli**: Modes of V -twisted Dirac Operator: $\nabla_X \Psi = 0$
 - H -charged matter, $h^1(X, V), h^1(X, V^\vee), h^1(X, \wedge^2 V), \dots$
 - $X \Rightarrow h^{1,1}(X)$ (Kähler), $h^{2,1}(X)$ (Complex structure), $V \Rightarrow h^1(X, End_0(V))$ (Bundle).

Determining the 4D theory in string compactification

- Physical quantities in low energy string theory depend on the metric and gauge connections in the extra dimensions.
- For example:
 - Yukawa couplings in Heterotic string theory descend from a term in the 10-dimensional action of the form $\sim \int d^{10} \sqrt{-g} \bar{\psi} A \psi$. Normalization of fields and coefficients of the superpotential depend on g .
 - I.e. Defining the perturbative theory around a background:

$$\mathcal{A}_b = A_b^0 + \phi^I v_{Ib}^\times T_x + \dots$$

where v_I is the bundle valued **harmonic** 1-form on compact 6D space X which counts the multiplicity of the 4D fields ϕ^I .

- Superpotential tri-linear coupling $\lambda_{IJK} \phi^I \phi^J \phi^K$ is given by the integral

$$\lambda_{IJK} \sim \int_X v_I \wedge v_J \wedge v_K \wedge \Omega$$

Particle masses/couplings

- Interested in
 - Particle masses/couplings.
 - Textures/hierarchies: Standard model (Heavy top quark), SUSY “mu problem” (want to forbid a mass term allowed by gauge symmetry, i.e. $\mu H_d H_u$), Forbid rapid proton decay operators, etc...
- **Major obstacle:** Matter field Kahler potential unknown except for special cases.

$$G_{IJ} = \frac{1}{2(\text{Vol}(X))} \int_X v_I \wedge \bar{\star}_V v_J$$

- **Easiest** class of solutions for X a complex, Kahler manifold: Ricci flat metric \Rightarrow i.e. a Calabi-Yau manifold
- **How to determine the metric and the connection?**
- Only current viable approach via \Rightarrow numeric approximation.

The Donaldson Algorithm

- **Idea:** Use projective embeddings to generate simple metrics that can be parametrically tuned to the Ricci-flat solution.
- **Kodaira embedding:** Given an ample line bundle \mathcal{L} on X then an embedding

$$i_k : X \rightarrow \mathbb{P}^{n_k-1}, \quad (x_0, \dots, x_2) \mapsto [s_0(x) : \dots : s_{n_k-1}(x)]$$

exists for all $\mathcal{L}^k = \mathcal{L}^{\otimes k}$ with $k \geq k_0$ for some k_0 , where $s_\alpha \in H^0(X, \mathcal{L}^k)$.

- What do we know about metrics on \mathbb{P}^n ? **Fubini-Study:**

$$(g_{FS})_{i\bar{j}} = \frac{i}{2} \partial_i \bar{\partial}_{\bar{j}} K_{FS} \quad \text{where} \quad K_{FS} = \frac{1}{\pi} \ln \sum_{i\bar{j}} h^{i\bar{j}} z_i \bar{z}_{\bar{j}}$$

and $h^{i\bar{j}}$ is a non-singular, hermitian matrix.

- FS metric restricted to X is not Ricci-flat. But...

- Generalize: $K_{h,k} = \frac{1}{k\pi} \ln \sum_{\alpha, \bar{\beta}=0}^{n_k-1} h^{\alpha\bar{\beta}} s_{\alpha} \bar{s}_{\bar{\beta}} = \ln ||s||_{h,k}^2$
- $h^{\alpha\bar{\beta}}$ is a hermitian fiber metric on $\mathcal{L}^{\otimes k}$.
- Such Kähler potentials are dense in the moduli space (Tian)
- Fixed point of Donaldson's "T-operator" \leftrightarrow "balanced metric".

$$T(h)_{\alpha\bar{\beta}} = \frac{n_k}{\text{Vol}_{CY}(X)} \int_X \frac{s_{\alpha} \bar{s}_{\bar{\beta}}}{\sum_{\gamma, \bar{\delta}} h^{\gamma\bar{\delta}} s_{\gamma} \bar{s}_{\bar{\delta}}} d\text{Vol}_{CY}$$

Theorem (Donaldson)

For each $k \geq 1$, the balanced metric, h , on $\mathcal{L}^{\otimes k}$ exists and is unique. As $k \rightarrow \infty$, the sequence of metrics

$$g_{i\bar{j}}^{(k)} = \frac{1}{k\pi} \partial_i \bar{\partial}_{\bar{j}} \ln \sum_{\alpha, \bar{\beta}=0}^{n_k-1} h^{\alpha\bar{\beta}} s_{\alpha} \bar{s}_{\bar{\beta}}$$

on X converges to the unique Ricci-flat metric for the given Kähler class and complex structure.

A new approach

- Existing numeric implementations of Donaldson's algorithm (Douglas et al, Ovrut et al). Computationally intensive. (Accurate enough? Don't know...)
- Moduli dependence difficult to obtain.
- New Approach \Rightarrow Machine Learning. What we did:
 - 1 Supervised learning of moduli dependence of Calabi-Yau metrics using the Donaldson algorithm to generate training data.
 - 2 Direct learning of moduli dependent Calabi-Yau metrics both using the metric ansatz and without it.
 - 3 Direct learning of metrics associated to $SU(3)$ structures with torsion.
- I'll give a brief flavor of these results...(see also, (Douglas et al), (Jejjala, et al))(More recent: package by Ruehle, Lukas, et al).

Preliminaries

- One definition of a Calabi-Yau three-fold: A complex 3-fold admitting a nowhere vanishing real two-form, J , and a complex three-form, Ω , such that:

$$\begin{aligned} J \wedge \Omega &= 0 & J \wedge J \wedge J &= \frac{3i}{4} \Omega \wedge \bar{\Omega} \\ dJ &= 0 & d\Omega &= 0 \end{aligned}$$

- Metric is related to the two form as $ig_{a\bar{b}} = J_{a\bar{b}}$
- Example CY manifold: “Quintic” hypersurface: $X = \mathbb{P}^4[5]$
- e.g. $p(\vec{z}) = z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + \psi z_0 z_1 z_2 z_3 z_4 = 0$
- The holomorphic $(3,0)$ form can explicitly constructed for such manifolds (Candelas, et al).

$$\Omega = \frac{1}{\partial p_{\psi}(\vec{z})/\partial z_b} \bigwedge_{\substack{c=0,\dots,3, \\ c \neq a,b}} dz_c$$

Direct Learning of the Kahler Potential

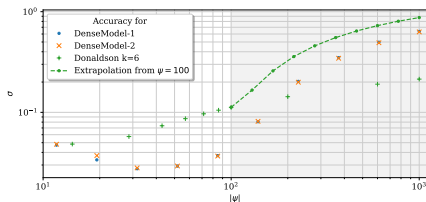
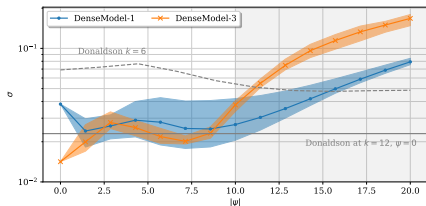
- The balanced metric output by Donaldson's algorithm at given finite k is not necessarily the most accurate approximation to the Ricci-flat metric - maybe we can do better?
- Can generate networks to find the parameters that are trained directly using a loss such as:

$$\mathcal{L}_{MA} = \left| 1 + \frac{4i}{3} \frac{J^3}{\Omega \wedge \overline{\Omega}} \right|$$

- C.f.: [Headrick and Nassar](#) (although note that we are obtaining moduli dependent results and using ML). Network Architecture:

Layer	Number of Nodes	Activation	Number of Parameters
input	17	–	–
hidden 1	100	leaky ReLU	1800
hidden 2	100	leaky ReLU	10 100
hidden 3	100	leaky ReLU	10 100
output	d^2	identity	$101 \ d^2$

- These networks were optimized for $0 \leq |\psi| \leq 10$



(shaded region denotes extrapolation of the networks).

- Note Donaldson algorithm with $k = 12$ takes order days to run even for the single case of $\psi = 0$. This network at $k = 6$ takes only minutes and gives comparable accuracy for a whole range of ψ .
- We do better than Donaldson Alg. at $k = 6$ and that this improvement extends up to $|\psi| \simeq 175$, nearly a factor of 2 beyond the regime used during training.

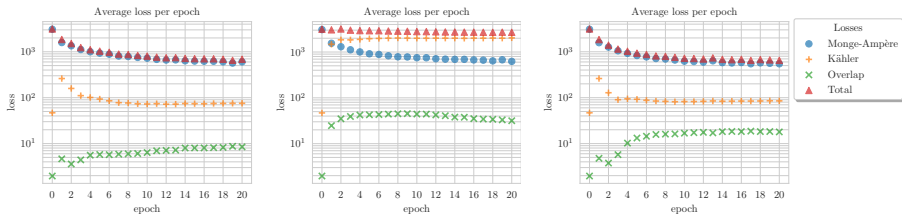
Direct learning of the metric

- Instead of learning parameters in an ansatz for the Kahler potential we can try to learn the CY metric directly.
- Why try?
 - Perhaps we can improve performance by not being tied to an ansatz at fixed k .
 - We will be able to generalize this approach to more complicated geometries.
- One disadvantage:
 - We now need loss functions to check that the metric is globally well defined and Kahler! We use $\mathcal{L} = \lambda_1 \mathcal{L}_{\text{MA}} + \lambda_2 \mathcal{L}_{\text{dJ}} + \lambda_3 \mathcal{L}_{\text{overlap}}$
 - Here \mathcal{L}_{MA} is the loss described before and we add to this

$$\mathcal{L}_{\text{dJ}} = \frac{1}{2} \|dJ\|_1$$

$$\mathcal{L}_{\text{overlap}} = \frac{1}{d} \sum_{k,j} \left\| g_{\text{NN}}^{(k)}(\vec{z}) - T_{jk}(\vec{z}) \cdot g_{\text{NN}}^{(j)}(\vec{z}) \cdot T_{jk}^\dagger(\vec{z}) \right\|_n$$

- Input: $Re(z_i)$, $Im(z_i)$ (homogeneous coords describing pt in CY)
 $Re(\psi)$, $Im(\psi)$. Output: d^2 real and imaginary parts of a metric at point.
- To give a concrete example: optimized at $\psi = 10$ on a data set of 10,000 points. We split the points according to train:test=90 : 10 and we train for 20 epochs.
- Accuracy reaches same level as Donaldson Alg. at $k = 5$ (we expect more points and better architecture will easily improve this).



Left: Optimizing the NN with all three losses. *Middle:* Optimizing the NN without Kähler loss (i.e. $\lambda_2 = 0$). *Right:* Optimizing the NN without overlap loss (i.e. $\lambda_3 = 0$).

Learning $SU(3)$ structures from an ansatz

- Important reason to directly learn the metric: Can be generalized to non-Kähler geometry!
- One important class of geometries for $\mathcal{N} = 1$ compactifications: $SU(3)$ structure manifolds
- These are six-manifolds with a nowhere vanishing two form J and three form Ω obeying the same algebraic properties as the Calabi-Yau threefold case:

$$J \wedge \Omega = 0 \qquad J \wedge J \wedge J = \frac{3i}{4} \Omega \wedge \bar{\Omega}$$

But with different differential properties...

- An $SU(3)$ structure can be classified by its torsion classes:

$$\begin{aligned} dJ &= -\frac{3}{2} \text{Im}(W_1 \bar{\Omega}) + W_4 \wedge J + W_3 \\ d\Omega &= W_1 J \wedge J + W_2 \wedge J + W_5 \wedge \Omega, \end{aligned}$$

- Where torsion classes are given the defining forms:

$$W_1 = -\frac{1}{6}i\Omega \lrcorner dJ = \frac{1}{12}J^2 \lrcorner d\Omega, \quad W_4 = \frac{1}{2}J \lrcorner dJ, \quad W_5 = -\frac{1}{2}\Omega_+ \lrcorner d\Omega_+$$

- Given string theories place different constraints on the torsion classes for there to be an associated solution to the theory of the type we want.
- E.g. **heterotic string theory**: $W_1 = W_2 = 0$, $W_4 = \frac{1}{2}W_5 = d\phi$, W_3 free.
- Note that a CY structure is a special case: $W_i = 0 \forall i = 1, \dots, 5$.
- **Need to start with some well-controlled/simple example.**
- **Observation:** Some CY manifolds admit not only Ricci-flat metrics, but other $SU(3)$ structures as well.
- E.g. (generalization of work by **Larfors, Lukas and Ruehle**)

$$J = \sum_{i=1}^{h^{1,1}(x)} a_i J_i \qquad \Omega = A_1 \Omega_0 + A_2 \bar{\Omega}_0$$

- The a_i are real functions and A_1 and A_2 are complex functions. CY taken to be a complete intersection in a product of projective spaces.

- Quintic E.g. (from Larfors, et al)

$$a_1 = \frac{1}{\pi^3} \frac{|\nabla p|^2}{\sigma^4} \quad , \quad A_1 = a_1^2 \quad , \quad A_2 = 0 \quad , \quad (\text{where } \sigma = \sum_{a=0}^4 |z_a|^2)$$

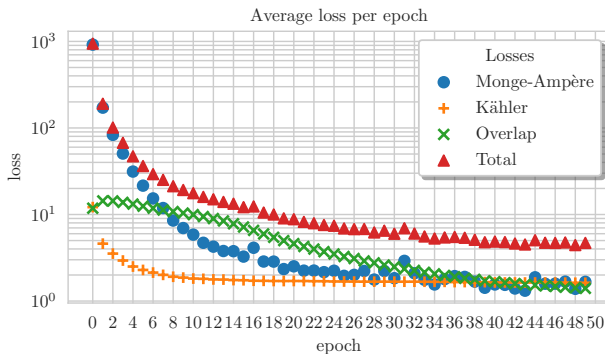
with p the defining equation of the hypersurface.

- This has torsion classes $W_1 = W_2 = W_3 = 0$, $W_5 = 2W_4 = 2d(\ln(a_1))$ and thus provides a solution to heterotic string theory.
- We aimed to reproduce this known analytic solution using direct learning of the metric.
- Such a [check is particularly important](#) in learning such metrics as we have no analogue of Yau's theorem to use to argue we are converging towards an exact/unique solution.

- Can use the same losses as in the Calabi-Yau case, then, with the exception of replacing the Kahler loss by the following.

$$\mathcal{L}_{W_4} = \|dJ - d\ln(a_1 \wedge J)\|_n$$

- We ran this for the $\psi = 10$ quintic, using multiplicative boosting from g_{FS} .



Topological Vanishings

- In general: to understand heterotic effective theories, we need
 - a) Good tools to effectively compute couplings/masses
 - b) Ways to study/predict general structure (which could lead to hierarchies like those above).
- **Observation:** Heterotic theories can exhibit “Topological Vanishings” (i.e. vanishing couplings that must be zero due to the structure of the geometric background (X, V) , rather than gauge invariance)
- Goals:
 - Understand what geometric effects can lead to topological vanishings
 - How can these geometric effects differ/be related?
 - How generic are such vanishings?
 - Hidden symmetries?

Algebraic vs. Differential Geometry Approaches

- How to actually compute?
- First E.g.s, $V = TX$ here some Yukawa couplings are simply triple intersection numbers ($d_{rst} = D_r \cap D_s \cap D_t$) and can use mirror symmetry (Strominger, Candelas,...)
- **Algebro-geometric approach:** Can show that integral form of λ_{IJK} is 1-1 with a triple product in cohomology (i.e. a Cup/Yoneda product). Eg. for 27^3 coupling in an E_6 -theory:

$$H^1(X, V) \times H^1(X, V) \times H^1(X, V) \rightarrow H^3(X, \wedge^3 V) \simeq \mathbb{C}$$

- Given algebraic representation of $H^1(X, V)$, can turn this into a problem in polynomial multiplication (i.e. Groebner basis calculation). (Distler, et al),(Ovrut, Donagi, Pantev, et al), (LA, Gray, Lukas-et al...)

Algebraic vs. Differential Geometry Approaches

- **Differential Geometry Approach** (Blesneag, et al):
- E.g. Suppose $X \subset \mathcal{A} = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_m}$ is defined via $p(x) = 0$ and the bundles/forms descend from objects on \mathcal{A} :

$$\lambda(v_1, v_2, v_3) \sim -\frac{1}{2i} \int_{\mathcal{A}} \hat{v}_1 \wedge \hat{v}_2 \wedge \hat{v}_3 \wedge \delta^2(p) dp \wedge d\bar{p}$$

with $\delta^2(p)$ is a delta-function current satisfying

$$\delta^2(p) d\bar{p} = \frac{1}{\pi} \bar{\partial} \left(\frac{1}{p} \right)$$

- Defining $\hat{\Omega} \wedge dp = \mu$ one has an explicit form

$$\lambda(v_1, v_2, v_3) = -\frac{1}{2\pi i} \int_{\mathcal{A}} \frac{\mu}{p} \wedge [\bar{\partial} \hat{v}_1 \wedge \hat{v}_2 \wedge \hat{v}_3 - \hat{v}_1 \wedge \bar{\partial} \hat{v}_2 \wedge \hat{v}_3 + \hat{v}_1 \wedge \hat{v}_2 \wedge \bar{\partial} \hat{v}_3]$$

note: \hat{v}_i is not necessarily closed. **Can easily see that some couplings must vanish.**

A homological algebra framework

- We proved a number of theorems, only give a flavor here:
- *Theorem : Given any resolution*

$$\dots \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow V_X \rightarrow 0$$

can define a notion of ‘type’ for elements of $H^1(V_X)$. Namely, a cohomology element has type $\tau = i$ if it descends from $H^i(\mathcal{F}_{i-1})$. Then if $H^3(\wedge^3 \mathcal{F}_0) = 0$, all trilinear couplings between $\tau = 1$ fields will vanish. If, in addition, $H^4(\mathcal{F}_1 \otimes \wedge^2 \mathcal{F}_0) = 0$, then all Yukawa couplings between one type $\tau = 2$ and two type $\tau = 1$ fields will also vanish.

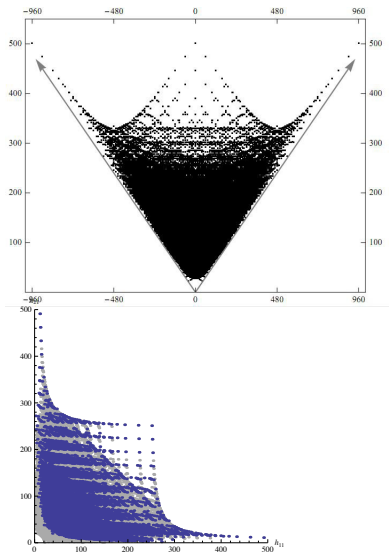
- Special case: (Generalization of work of [Blesneag, et al](#)): Koszul resolution for $X \subset \mathcal{A}$ (a complete intersection in co-dimension k):

$$0 \rightarrow \wedge^k \mathcal{N}^\vee \otimes \mathcal{V} \rightarrow \wedge^{k-1} \mathcal{N}^\vee \otimes \mathcal{V} \rightarrow \dots \rightarrow \mathcal{V} \rightarrow V_X \rightarrow 0$$

Geometric origins for topological vanishings

- There are three main classes of geometric structures in the literature that lead to vanishing criteria (not necessarily exhaustive)
- Broadly, constraints arise from either – **the manifold** – and/or – **the bundle**. These include constraints on couplings due to
 - ① The description of the CY manifold inside an ambient space (i.e. as a toric complete intersection) (Candelas, Lukas, et al)
 - ② The description of the CY manifold as a fibration (Braun, Pantev, Ovrut, Donagi, et al), (Bouchard, Cvetič, Donagi), etc
 - ③ The stability of the bundle and Kähler cone substructure (i.e. “stability walls”) (Watari et al), (LA, Gray, Ovrut)
- Homological algebra tools can also describe fibrations/stability walls.
- **Simple Question:** Given diverse geometric origins for topological vanishings of Yukawa couplings, must different descriptions agree? \Rightarrow No.

Genericity of vanishing criteria



(from Taylor 1205.0952)

- CY3 Fibrations are actively studied (LA,Gao,Gray,Lee), (Taylor-Huang)
- Observation 1: Almost all known CY 3-folds are fibered.
- Observation 2: generic manifolds do not admit just one elliptic fibration, they admit many (~ 10 s or 100 s).
- Each of these distinct fibrations can induce topological vanishings! **Overall Conclusion**: Heterotic couplings *generically* highly constrained (beyond any one basis).

Results and Future Work

- Progress is being made in characterizing generic features of heterotic CY vacua – E.g. when/why **topological vanishings** arise in heterotic couplings.
- Can phrase (all) such constraints (Koszul/fibrations/etc) in a common language (using homological algebra). **Constraints ubiquitous.**
- Open questions:
 - Higher order and non-perturbative contributions?
 - Dual theories? Hidden Symmetries? Link to Swampland conjectures?
- Control of the metric is necessary to specify the 4D theory in compactification \Rightarrow **ML techniques can provide a powerful new tool.**
- In particular, we have provided the first numeric approx. to **$SU(3)$ structure metrics.**
- Open questions: What next? \Rightarrow Combining model building with moduli stabilization.