Numerical studies on the IKKT matrix model using Lefschetz thimble method

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IKKT Matrix model

Problems in Perturbative Superstring Theory

・Existence of infinite individual vacua

・Singularity at the beginning of the universe

→ Non-perturbative approach

[Ishibashi-Kawai-Kitazawa-Tsuchiya('96)]

IKKT matrix model：promising candidate for non-perturbative string theory

$$S_b = -\frac{1}{4g^2}\mathrm{Tr}\left([\mathrm{A}^\mu, \mathrm{A}^\nu][\mathrm{A}_\mu, \mathrm{A}_\nu]\right)$$

$$S_f = -\frac{1}{2g^2}\mathrm{Tr}\left(\bar{\Psi}\Gamma^\mu[\mathrm{A}^\mu, \Psi]\right)$$

$N\times N$ Hermitian matrix

SO(9,1) Symmetry

$\mathrm{A}_\mu(\mu = 0, \cdots, 9)$：Lorentz vector

$\Psi_\alpha(\alpha = 1, \cdots, 16)$：Majorana Weyl Spinor

$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$

Space-time does not exist a priori, emerges from the degrees of freedom of the matrix

Sign problem in the IKKT matrix model

We can choose Lorentz or Euclidean metric when we define a theory

Euclidean model

complex

$$Z = \int dA e^{-S_E} \mathrm{Pf}\mathcal{M}_E(A)$$

$$S_E = \frac{N}{4}\left[2\mathrm{Tr}(F_{0i})^2 + \mathrm{Tr}(F_{ij})^2\right]$$

Lorentzian model

real

$$Z = \int dA e^{iS_L} \mathrm{Pf}\mathcal{M}_L(A)$$

$$S_L = -\frac{N}{4}\left[-2\mathrm{Tr}(F_{0i})^2 + \mathrm{Tr}(F_{ij})^2\right]$$

Sign problem (It happens when integrand of partition function becomes complex)

(c.f. Morning talks on Thu. 08/12, Fukuma, Kadoh and Nishimura)

Studies of IKKT matrix model by Complex Langevin method

One approach to solve a sign problem : Complex Langevin method(CLM) [Parisi('83),Klauder('84)]

- Euclidean model : SO(10)→SO(3) [Anagnostopoulos-Azuma-Ito-Nishimura-Okubo-Stratos ('20)]

  Effects of Pfaffian complex phase

- Lorentzian model [Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-Nishimura-Tsuchiya-Stratos(work in progress)]

Complex Langevin method has application limits [Nagata-Nishimura-Shimasaki('16)]

We applied the Lefschetz thimble method to the IKKT matrix model for the first time

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Lorentzian IKKT matrix model

Equivalence between Lorentzian and Euclidean model

$$A_0 = -ie^{i\frac{\pi}{8}} A_0 = e^{-i\frac{3}{8}\pi} \tilde{A}_0$$

$$A_i = e^{i\frac{\pi}{8}} \tilde{A}_i$$

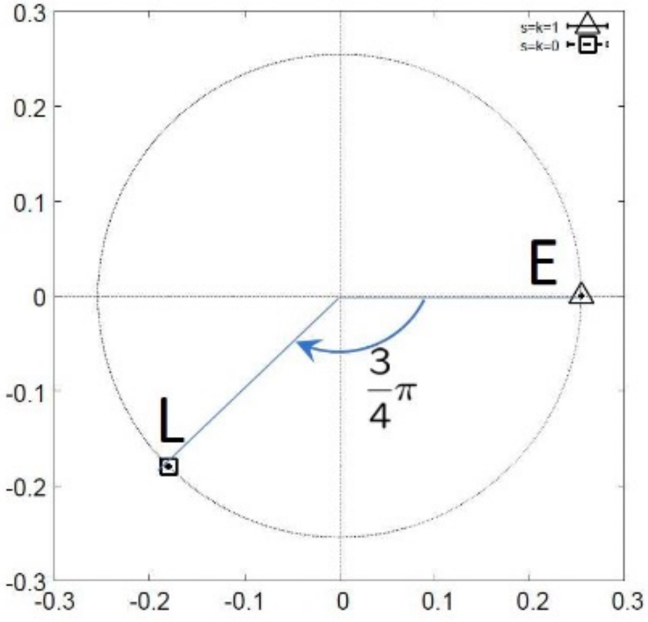
A : Lorentzian
$\tilde{A}$: Euclidean

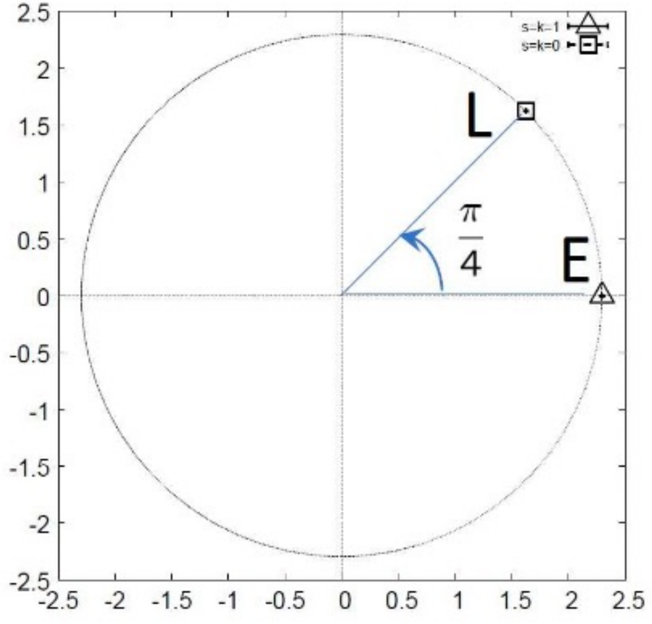
wick rotation of world sheet
wick rotation on target space

[Hatakeyama-Anagnostopoulos-Azuma-Hirasawa -Ito-Nishimura-Papadoudis-Tsuchiya('22)]

$$\left\langle \frac{1}{N} \mathrm{Tr}(A_0)^2 \right\rangle_L = e^{-i\frac{3\pi}{4}} \underbrace{\left\langle \frac{1}{N} \mathrm{Tr}(\tilde{A}_0)^2 \right\rangle_E}_{\text{real positive}}$$

$$\left\langle \frac{1}{N} \mathrm{Tr}(A_i)^2 \right\rangle_L = e^{i\frac{\pi}{4}} \underbrace{\left\langle \frac{1}{N} \mathrm{Tr}(\tilde{A}_i)^2 \right\rangle_E}_{\text{real positive}}$$





Lorentzian model and Euclidean model become equivalent by appropriate wick rotation
→Real space-time does not appear from Lorentzian model

Adding a Lorentz invariant mass term

We add a Lorentz invariant mass term for the purpose of emergence of real space-time

[Anagnostopoulos-Azuma-Hatakeyama-Hirasawa -Nishimura-Tsuchiya-Stratos(work in progress)]

$$Z = \int dA e^{-\tilde{S}} \qquad F_{\mu\nu} = i[A_\mu, A_\nu], \quad \gamma > 0$$

$$\tilde{S} = -\frac{i}{4} N \left[-2\mathrm{Tr}(F_{0i})^2 + \mathrm{Tr}(F_{ij})^2\right] - \frac{i}{2} N\gamma \left[\mathrm{Tr}(A_0)^2 - \mathrm{Tr}(A_i)^2\right]$$

wick rotation : $A_0 = e^{-i\frac{3}{8}\pi} \tilde{A}_0, \quad A_i = e^{i\frac{\pi}{8}} \tilde{A}_i$

$$\tilde{S} = \frac{1}{4} N \left[2\mathrm{Tr}(\tilde{F}_{0i})^2 + \mathrm{Tr}(\tilde{F}_{ij})^2\right] + \underline{\frac{1}{2} N\gamma e^{i\frac{3}{4}\pi} \left[\mathrm{Tr}(\tilde{A}_0)^2 + \mathrm{Tr}(\tilde{A}_i)^2\right]}$$

real part of mass term become negative

$\gamma > 0$:classical solution express expanding real space-time

[Hatakeyama-Anagnostopoulos-Azuma-Hirasawa-Ito-Nishimura-Papadoudis-Tsuchiya('22)]

Numerical studies of bosonic model using CLM

$$S = \frac{1}{4}N\left[-2\mathrm{Tr}(F_{0i})^2 + \mathrm{Tr}(F_{ij})^2\right] + \frac{1}{2}N\gamma\left[\mathrm{Tr}(A_0)^2 - \mathrm{Tr}(A_i)^2\right]$$

[Anagnostopoulos,etc.]

Expanding real space-time appears at some γ and space collapses to 1 dimension from bosonic action

→(3+1)-dimensional space-time is expected to emerge in the model includes fermions

(c.f. parallel session in Tue. 08/12 14:00 Hatakeyama)

CLM has an application limits

We applied a Lefschetz thimble method which have no application limits

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Lefschetz thimble method

[Alexandru-Basar-Bedaque-Ridgway-Warrington ('15)]

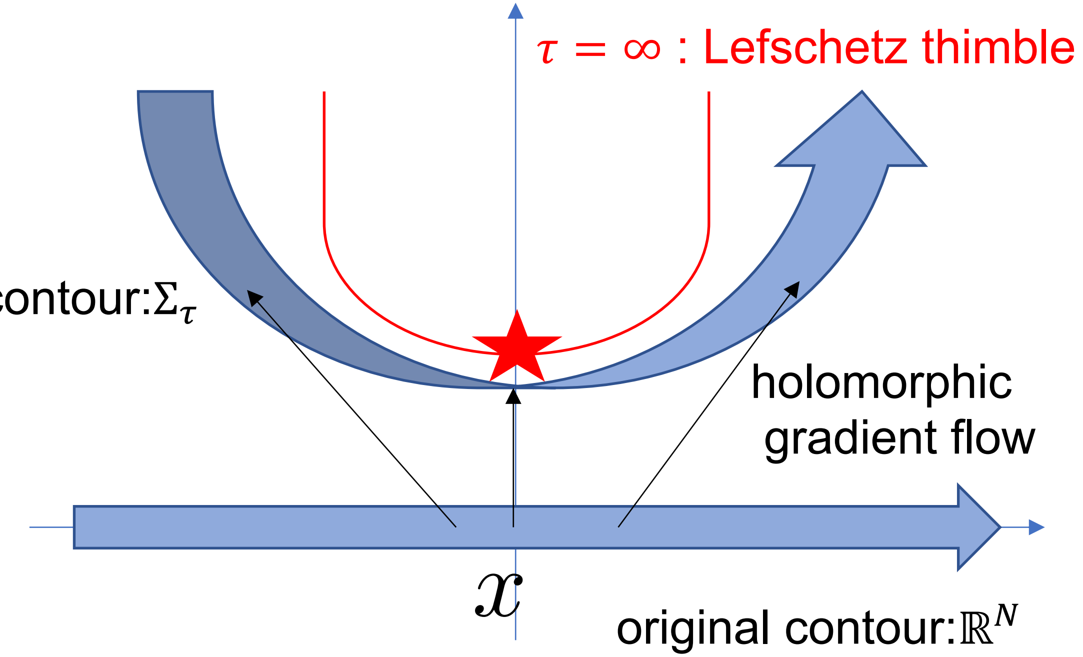
Lefschetz thimble method

$$Z = \int dx e^{-S(x)}, S(x) \in \mathbb{C}, x = (x_1, x_2, \cdots, x_N) \in \mathbb{R}^N$$

deform a integration contour by holomorphic gradient flow

$$\frac{\partial}{\partial \sigma} z_k(x,\sigma) = \overline{\frac{\partial S(z(x,\sigma))}{\partial z_k}}$$

solve form $\sigma = 0$ to $\sigma = \tau$ with the initial condition $z(x, 0) = x$



★ fix point

$$\frac{\partial S(z)}{\partial z} = 0$$

$$\frac{\partial}{\partial \sigma} S(z(x,\sigma)) = \left| \frac{\partial S(z(x,\sigma))}{\partial z_k} \right|^2 \geq 0$$

$$\begin{cases} ReS : \nearrow \\ ImS : const \end{cases}$$

The fluctuation of ImS becomes smaller on deformed contour

→sign problem is solved

Lefschetz thimble method

deformed contour : Σ_τ

$$Z = \int_{\mathbb{R}^N} dx e^{-S(x)}$$

Cauchy's theorem

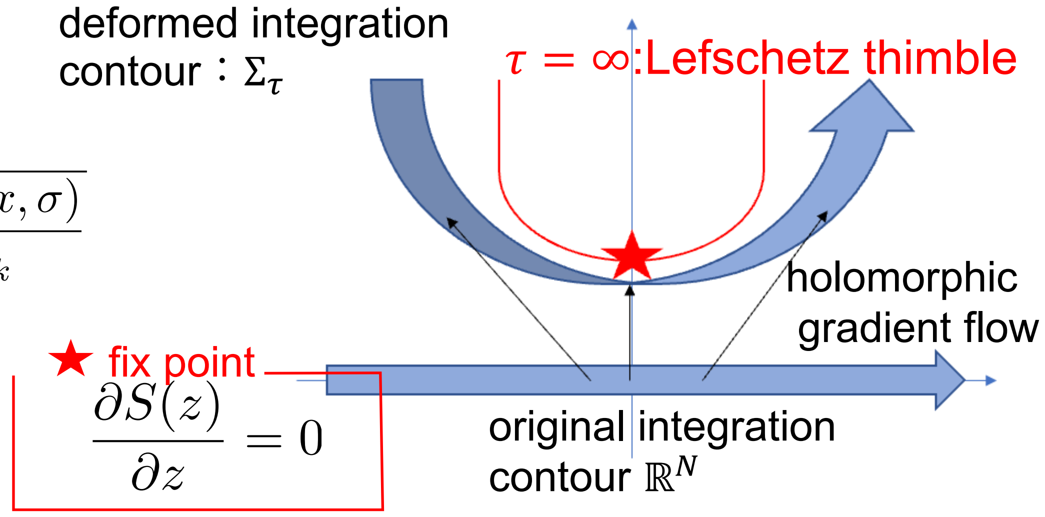
$$\frac{\partial}{\partial\sigma} z_k(x,\sigma) = \overline{\frac{\partial S(z(x,\sigma))}{\partial z_k}}$$

$$= \int_{\Sigma_\tau} dz e^{-S(z(x,\tau))}$$

[Fukuma-Matsumoto('12)]

$$= \int_{\mathbb{R}^N} dx e^{-S(z(x,\tau))} detJ(x,\tau)$$

Monte Carlo integration on real axis



$$\theta = -ImS(z) + arg(detJ(x,\tau))$$

The phase distribution becomes sharp
→sign problem is solved

Calculation of observable

$$Z_\tau = \int dx e^{-ReS(z(x,\tau))}$$

$$\langle O \rangle = \frac{\langle O(z(x,\tau)) e^{i\theta} |detJ(x,\tau)| \rangle_\tau}{\langle e^{i\theta} |detJ(x,\tau)| \rangle_\tau}$$

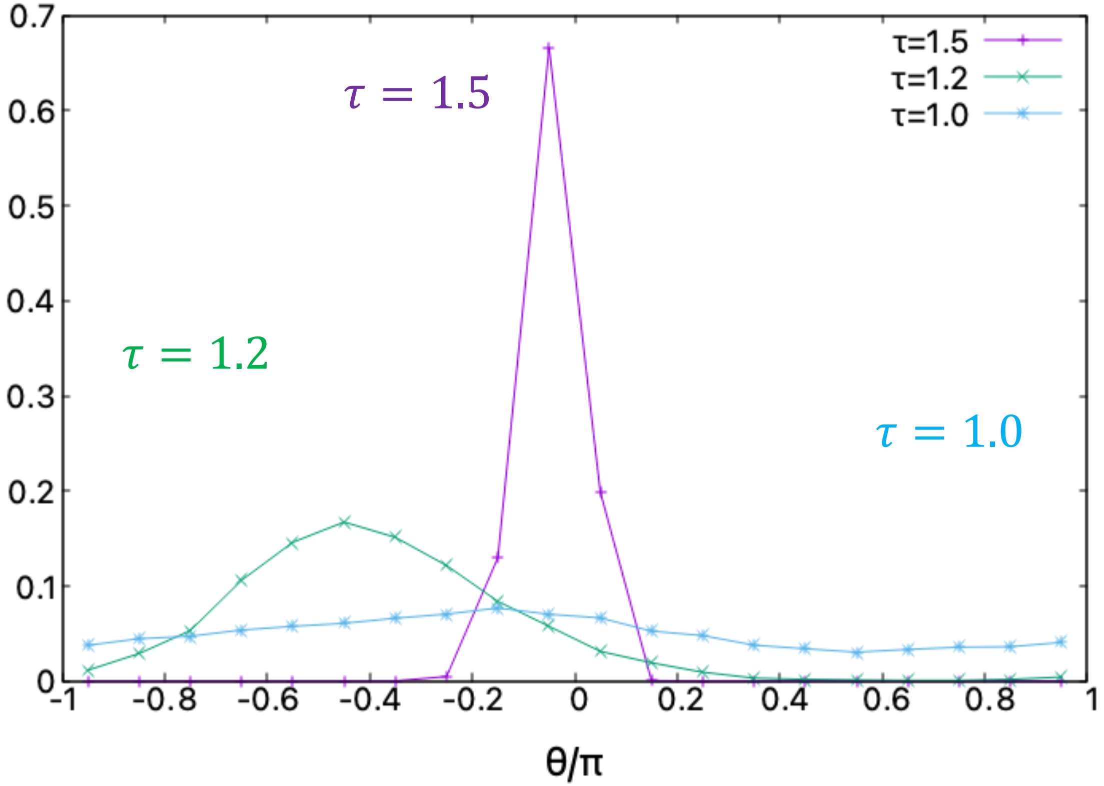
Overlap problem
$\log|detJ|$ fluctuate violently
→ cannot gather configuration effectively

Result

Solving a sign problem

γ : coefficient of Lorentz invariant mass term
τ : time to solve flow equation

$N = 2, \gamma = 1.5$



$$\langle O \rangle = \frac{\langle O(z(x,\tau)) e^{i\theta} |detJ(x,\tau)| \rangle_\tau}{\langle e^{i\theta} |detJ(x,\tau)| \rangle_\tau}$$

$$\theta = -ImS(z) + arg(detJ(x,\tau))$$

$\tau \lesssim 1.0$

The phase distribution is flat

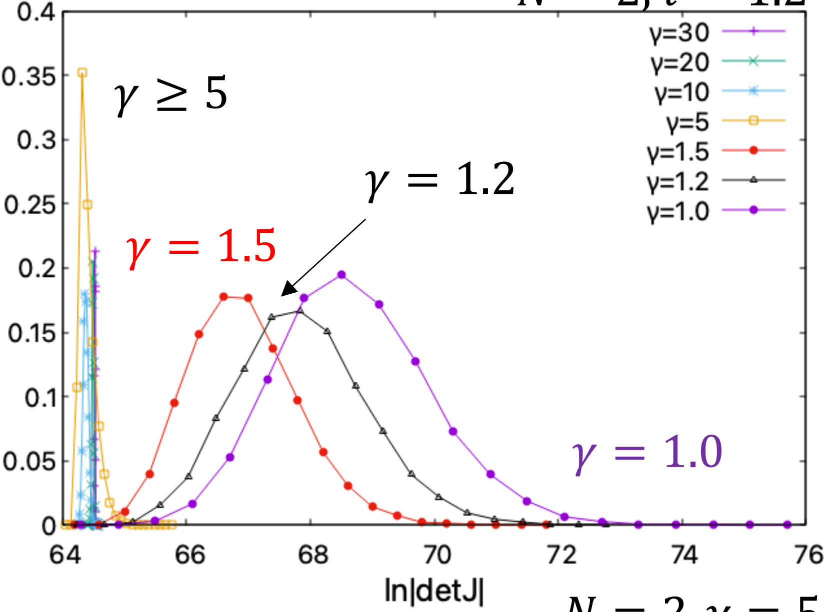
→ severe sign problem

The distribution becomes sharper as τ become large

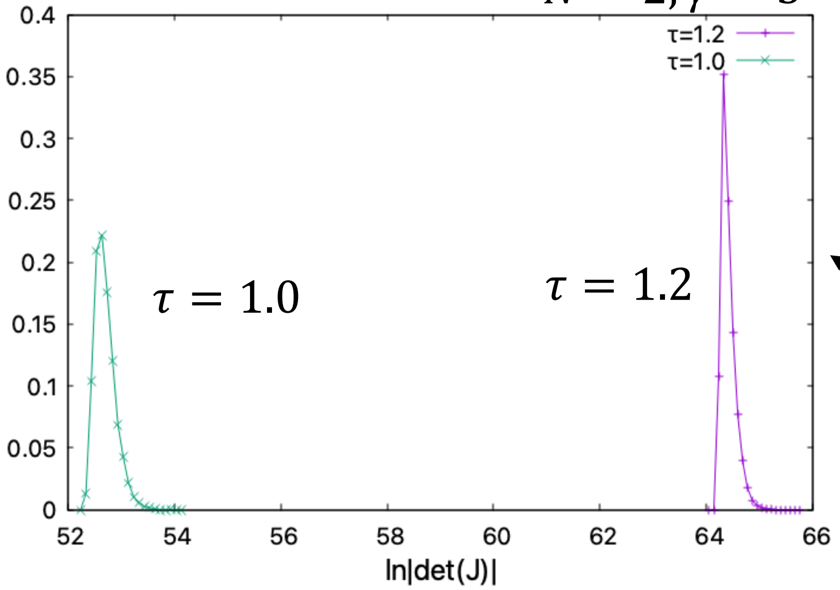
→sign problem becomes weaker

Overlap problem(1)

$N = 2, \tau = 1.2$



$N = 2, \gamma = 5$



$$\langle O \rangle = \frac{\left\langle O(z(x,\tau))e^{i\theta}\boxed{|detJ(x,\tau)|}\right\rangle_\tau}{\left\langle e^{i\theta}\boxed{|detJ(x,\tau)|}\right\rangle_\tau}$$

$\ln|detJ|$ differ from one configuration to another
→ Overlap problem
(The only configuration which has large $\ln|detJ|$ contributes to observables)

The distribution of $\ln|detJ|$

$\gamma \geq 5$: sharp

$\gamma < 1.5$: the distribution becomes wider (Overlap problem)

The distribution of $\ln|detJ|$ at $\tau = 1.0, 1.2 (\gamma = 5)$

There is a tendency for solving Overlap problem by increasing τ

SO(9,1) symmetry

$$S = \frac{1}{4}N\left[-2\mathrm{Tr}(F_{0i})^2 + \mathrm{Tr}(F_{ij})^2\right] + \frac{1}{2}N\gamma\left[\mathrm{Tr}(A_0)^2 - \mathrm{Tr}(A_i)^2\right]$$

If SO(9,1) symmetry is not broken spontaneously

$$\left(c \in \mathbb{C} \leftarrow Z = \int dA e^{iS(x)}\right)$$

$$\left\langle \frac{1}{N}\mathrm{Tr}(A_\mu A_\nu) \right\rangle = c\eta_{\mu\nu}$$

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \cdots, 1)$$

$$\Rightarrow \boxed{\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle = -\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle}$$

$$\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle = e^{i2\theta_t}\left|\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle\right|$$

$$\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle = e^{i2\theta_s}\left|\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle\right|$$

$$\begin{cases} \left|\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle\right| = \left|\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle\right| \\ e^{2i\theta_t} = -e^{2i\theta_s} = e^{i(2\theta_s+\pi)} \end{cases}$$

$$\theta_t = \theta_s + \pi/2$$

Euclidean signature

Real space-time

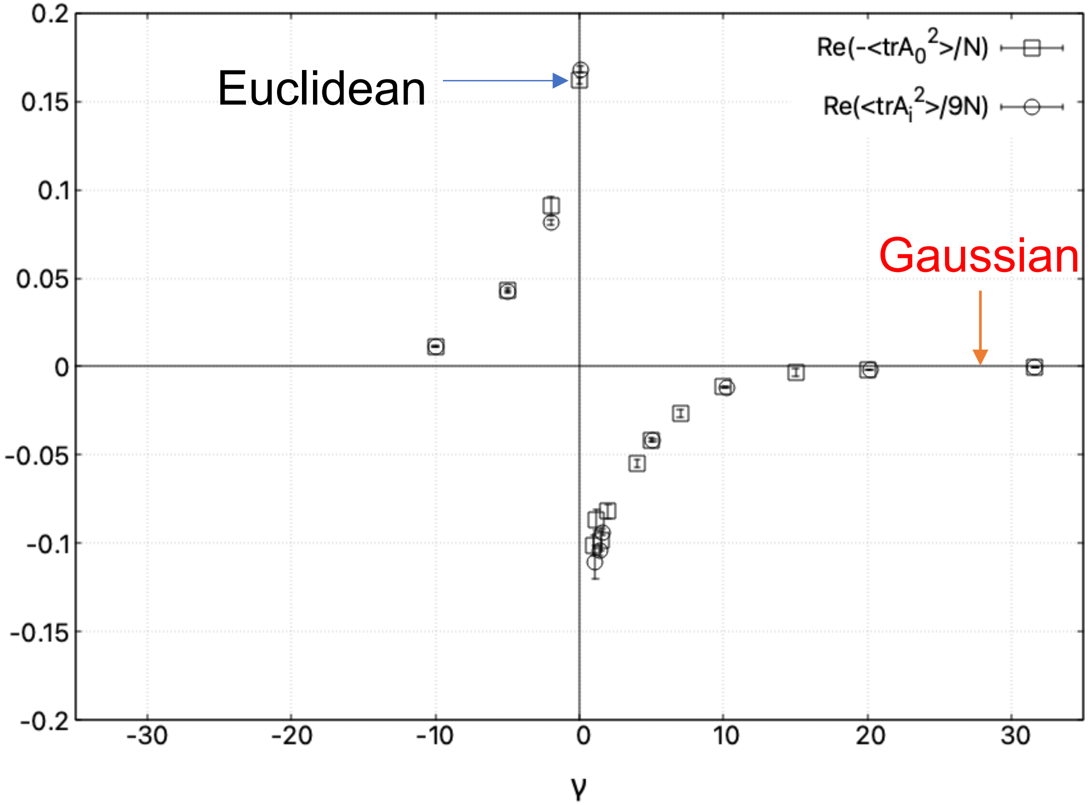
$$\theta_t = \theta_s (= 0)$$

Lorentzian signature

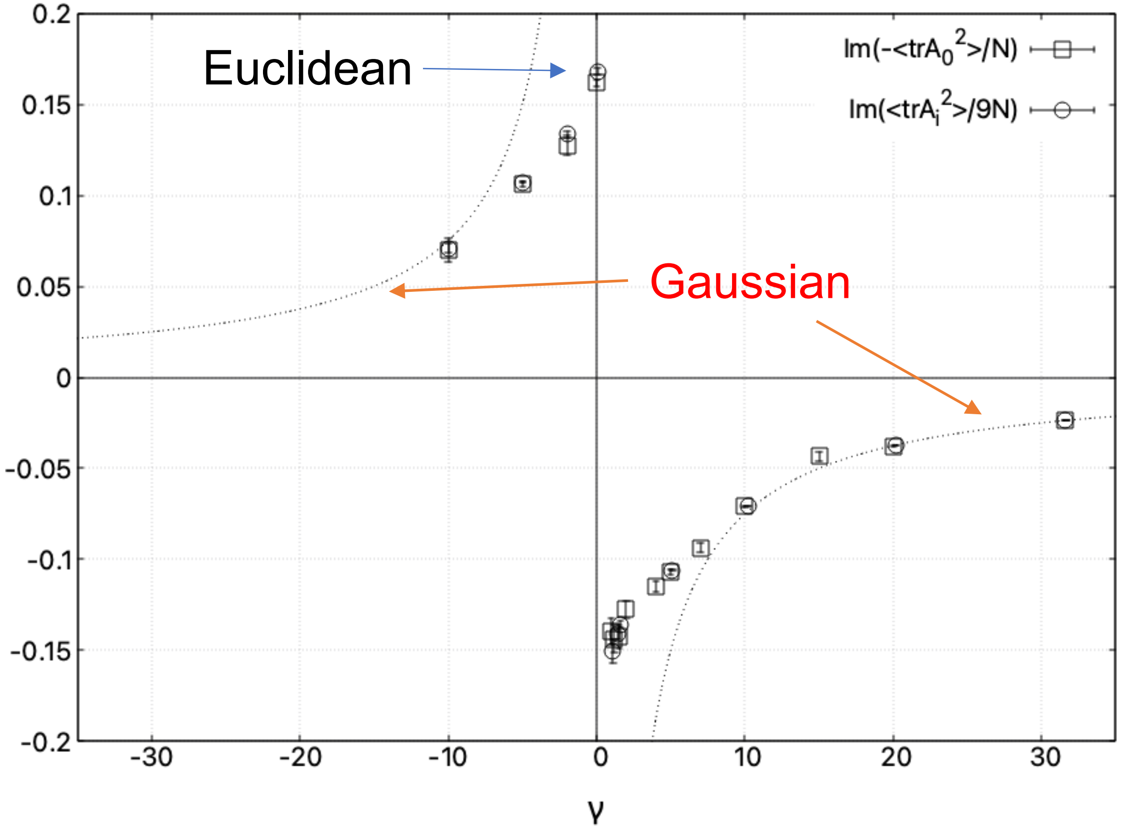
At least SO(9,1) → SO(9) SSB is required to realize real space-time

SO(9,1) symmetry

Real parts of $-\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle$ and $\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle$



Imaginary parts of $-\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle$ and $\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle$



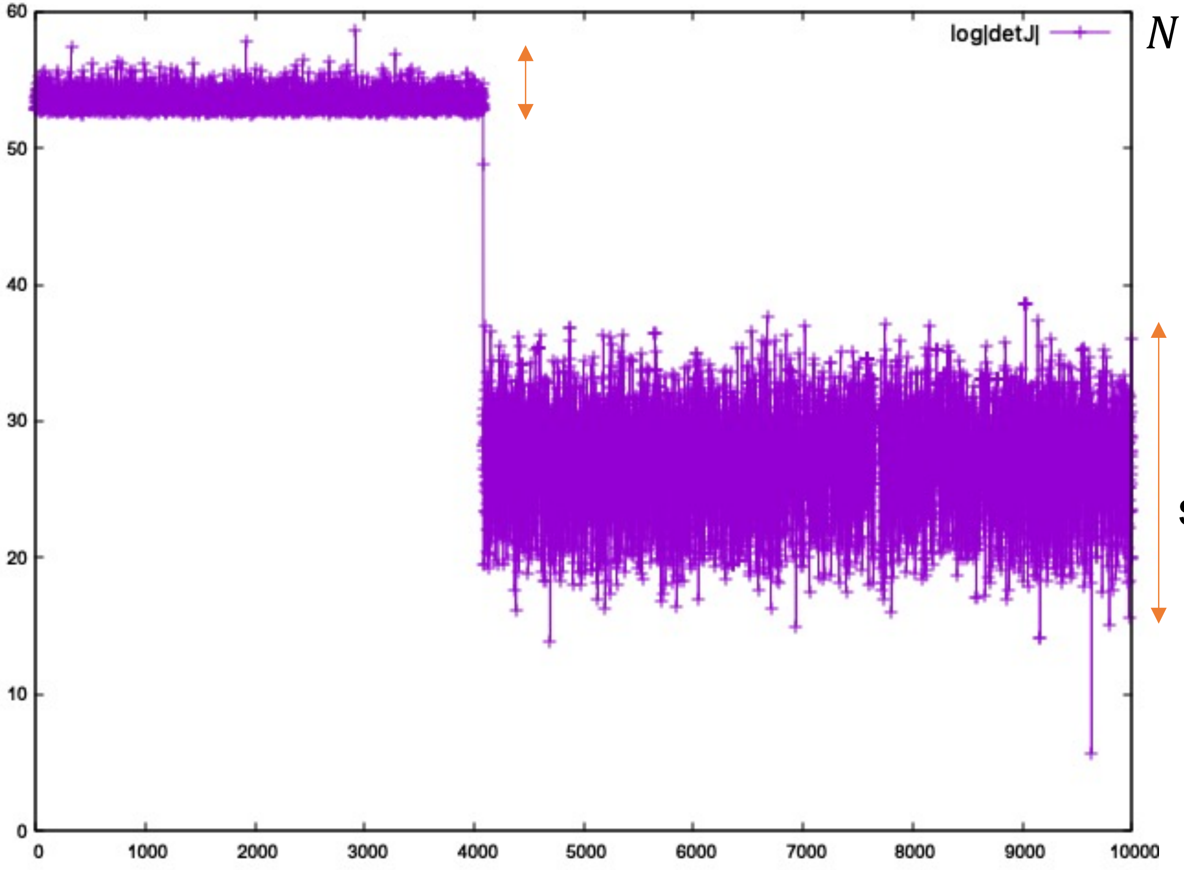
Gaussian

$$\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle = \frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle = \boxed{-\frac{i}{\gamma}}$$

$\frac{1}{9}\left\langle \frac{1}{N}\sum_{i=1}^{9}\mathrm{Tr}(A_i)^2 \right\rangle = -\left\langle \frac{1}{N}\mathrm{Tr}(A_0)^2 \right\rangle$ is satisfied→ SO(9,1) is not be broken

Phase transition between two thimbles

$\gamma < 2.0$ The phase transition happens



The history of the value of $\log|detJ|$

$N = 2, \gamma \sim 1.8$

$$\langle O \rangle = \frac{\left\langle O(z(x,\tau)) e^{i\theta} \boxed{|detJ(x,\tau)|} \right\rangle_\tau}{\left\langle e^{i\theta} \boxed{|detJ(x,\tau)|} \right\rangle_\tau}$$

severe Overlap problem occurs

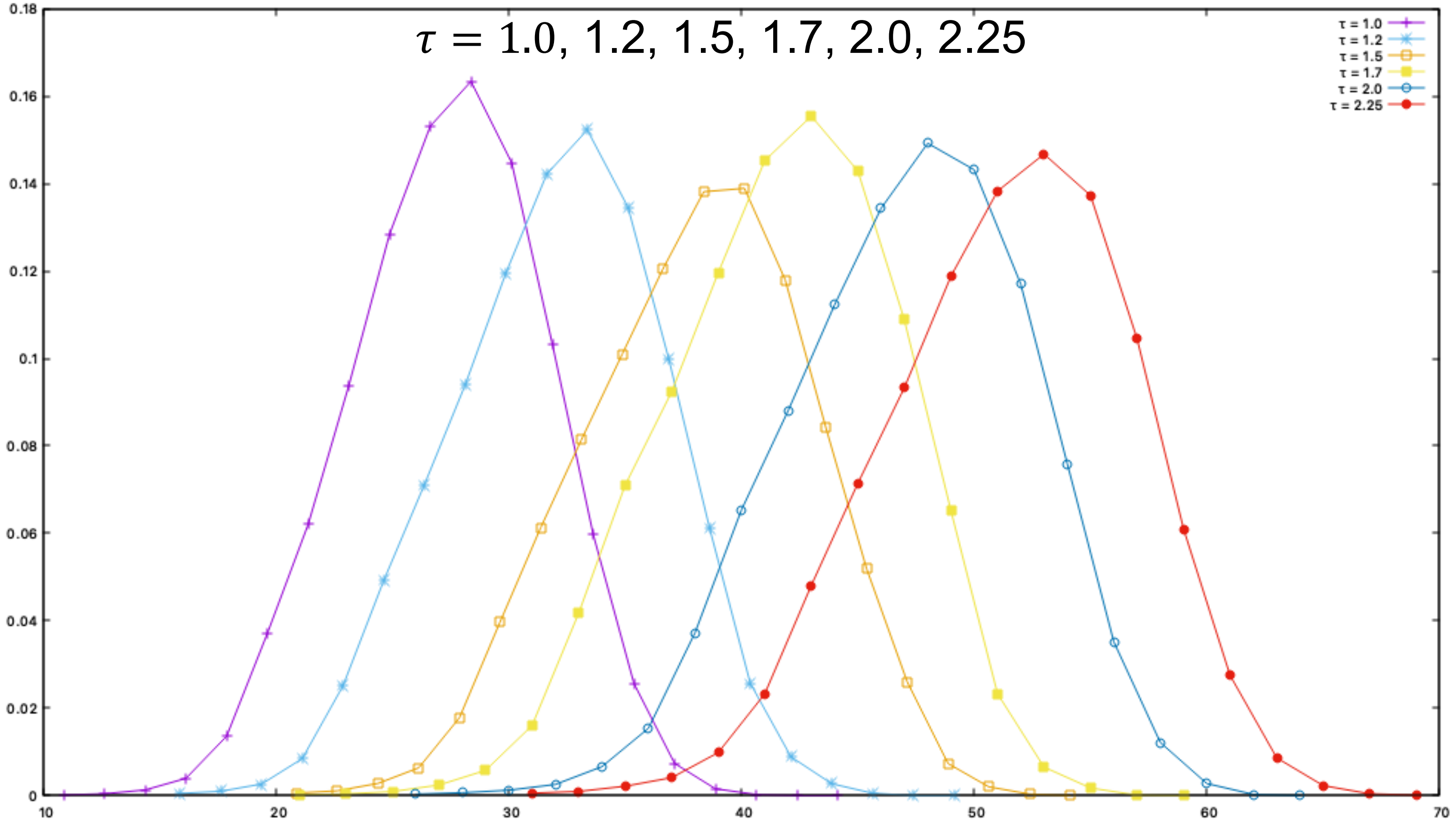
The thimble after transition may corresponds to the branch where SO(9,1) is broken spontaneously

However, the spread distribution of $\log|detJ|$ makes difficult to simulate on this thimble

Overlap problem (2)

After the phase transition, even thought we increase τ, the distribution of $\log|detJ|$ does not become sharp

$N = 2, \gamma = 1.5$



∴ Only increasing τ doe not help to solve Overlap problem on this thimble

- Optimized flow
- Worldvolume approach

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Summary and Future directions

Summary

- We applied a Lefschetz thimble method to the IKKT matrix model
  - the thimble method has almost no limit of application
- Adding a Lorentz invariant mass term for real space-time

- We can solve sign problem by increasing τ at $\gamma > 2.0$

- SO(9,1) symmetry may be broken on the thimble after phase transition
- We have to overcome Overlap problem to simulate after phase transition

Future directions

The problem based on Monte Carlo integration on real axis

・Overlap problem

→Optimized flow [Nishimura-Sakai-Yosprakob (to be published)]

or Worldvolume approach [Fukuma-Matsumoto('12)]

Emergence of real space-time : SO(9,1)→SO(9)

Emergence of three dimensional space : SO(9)→SO(3)

・Take large matrix size N

・Add up different thimbles

→ Integrating over the flow time [Fukuma-Matsumoto('12)]

(Compare with the analytical results, 1/D expansion)

(c.f. Next talk Worapat Piensuk)