

1/D expansion in the bosonic Lorentzian IKKT matrix model with mass term

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IKKT matrix model

Ishibashi-Kawai-Kitazawa-Tsuchiya ('96)

- Non-perturbative description of type IIB string theory, and hence string theories (string web dualities)
- The original IKKT matrix model: $Z = \int dA e^{iS}$ where $S = S_b + S_f$
 - Bosonic part: $S_b = -\frac{N}{4} \text{tr}[A_\mu, A_\nu][A^\mu, A^\nu]$ A_μ ($\mu = 0, \dots, 9$) (vector)
 - Fermionic part: $S_f = -\frac{N}{2} \text{tr} (\bar{\psi}_\alpha (\Gamma^\mu)_{\alpha\beta} [A_\mu, \psi_\beta])$ ψ_α (Majorana-Weyl spinors) : $N \times N$ Hermitian matrices
- Γ^μ : 10-dimensional gamma matrices
- Other than $\text{SO}(9,1)$ Lorentz symmetry, we also have $A_\mu \rightarrow A_\mu + \alpha_\mu \mathbf{1}$ = translation in SUSY algebra
- ⇒ Allows us to interpret the eigenvalues of A_μ as spacetime coordinates
spacetime dynamically emerges

IKKT matrix model

- Bosonic action in the Lorentzian model is unbounded due to the Lorentzian signature

$$S = \frac{1}{4}N \left[-2\text{Tr}(F_{0i})^2 + \text{Tr}(F_{ij})^2 \right] \quad F_{\mu\nu} = i[A_\mu, A_\nu] \quad \text{Hermitian matrices}$$

Positive Positive

- Many people focused on the well-defined Euclidean version.

$$Z = \int dA e^{-S_b^{(E)}} \text{Pf } \mathcal{M}^{(E)}(A) \quad S_E = \frac{N}{4} [2\text{Tr}(F_{0i})^2 + \text{Tr}(F_{ij})^2]$$

Generally complex \Rightarrow Sign problem (符号問題)

- There are many ways to avoid or suppress the sign problem

Phase Quenching (ignoring Pfaffian) Anagnostopoulos-Azuma-Nishimura ('16), etc.

Gaussian Expansion Method (GEM) Aoyama-Nishimura-Okubo ('11), Nishimura-Okubo-Sugino ('11), etc.

Complex Langevin Method (stochastic quantization) Hatakeyama-Anagnostopoulos-Azuma-Hirasawa-Ito-Nishimura-Papadoudis-Tsuchiya ('22), etc.

For recent review: Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-Ito-Nishimura-Papadoudis-Tsuchiya ('22)

IKKT matrix model

Hatakeyama-Anagnostopoulos-Azuma-Hirasawa-Ito-Nishimura-Papadoudis-Tsuchiya ('22)

- Bosonic actions of Lorentzian and Euclidean models are related by Wick rotations

$$\begin{array}{ccc}
 A_0 = e^{-i\frac{3\pi}{8}} \tilde{A}_0 & \xrightarrow{\hspace{1cm}} & \left\langle \frac{1}{N} \text{Tr} A_0^2 \right\rangle_L = e^{-i\frac{3\pi}{4}} \left\langle \frac{1}{N} \text{Tr} \tilde{A}_0^2 \right\rangle_E \\
 A_i = e^{i\frac{\pi}{8}} \tilde{A}_i & & \left\langle \frac{1}{N} \text{Tr} A_i^2 \right\rangle_L = e^{i\frac{\pi}{4}} \left\langle \frac{1}{N} \text{Tr} \tilde{A}_i^2 \right\rangle_E
 \end{array}$$

Spacetime in
Lorentzian model
is **Euclidean and
complex.**

- Add Lorentz-invariant mass term to realize real and Lorentzian spacetime

$$S = \frac{1}{4} N \left[-2 \text{Tr}(F_{0i})^2 + \text{Tr}(F_{ij})^2 \right] + \frac{1}{2} N \gamma \left[\text{Tr}(A_0)^2 - \text{Tr}(A_i)^2 \right]$$

IKKT matrix model

- Under the same Wick rotations, the corresponding Euclidean model is then

$$\tilde{S} = \frac{1}{4}N \left[2\text{Tr}(\tilde{F}_{0i})^2 + \text{Tr}(\tilde{F}_{ij})^2 \right] + \frac{1}{2}N\gamma e^{i\frac{3\pi}{4}} \left[\text{Tr}(\tilde{A}_0)^2 + \text{Tr}(\tilde{A}_i)^2 \right]$$

- The sign of γ is very crucial as

$\Rightarrow \gamma < 0$: Wick rotations connect Euclidean and Lorentzian model (keeping real part of S positive) \Rightarrow Lorentzian and Euclidean models are equivalent

$\Rightarrow \gamma > 0$: Cannot connect them while keeping real part of S positive \Rightarrow Not equivalent

1/D expansion

Hotta-Nishimura-Tsuchiya ('99)

- Allows analytic calculations of Euclidean bosonic IKKT model
- Consider effective action by introducing auxilliary variables h_{ab} such that

$$\begin{aligned}
 S &= -\frac{1}{4g^2} \text{tr}([A_\mu, A_\nu]^2) \\
 &= -\frac{1}{4g^2} \lambda^{abcd} A_\mu^a A_\mu^b A_\nu^c A_\nu^d
 \end{aligned}
 \quad \longrightarrow \quad
 \begin{aligned}
 S[A, h] &= \frac{1}{4g^2} \lambda^{abcd} (h_{ab} h_{cd} - A_\mu^a A_\mu^b h_{cd} - h_{ab} A_\nu^c A_\nu^d) \\
 &= \frac{1}{4g^2} \lambda^{abcd} h_{ab} h_{cd} + \frac{1}{2} \left(\frac{1}{g} K^{ab} \right) A_\mu^a A_\mu^b \quad K_{ab} = -\frac{1}{g} \lambda^{abcd} h_{cd}
 \end{aligned}$$

$\lambda^{abcd} = \frac{1}{2} \left[\text{tr} \left([t^a, t^d][t^b, t^c] \right) + \text{tr} \left([t^a, t^c][t^b, t^d] \right) \right]$
inverse propagator of A_μ^a

- Rescale $\tilde{h}_{ab} = \frac{1}{g\sqrt{D}} h_{ab}$ such that $S_{\text{eff}} = \frac{D}{2} \left\{ \text{Tr} \ln K + \frac{1}{2} \lambda^{abcd} \tilde{h}_{ab} \tilde{h}_{cd} \right\}$ $Z = \int dh_{ab} e^{-S_{\text{eff}}}$
- At large D ($D \rightarrow \infty$), dominant configurations satisfy **saddle-point equation** $\frac{\delta S_{\text{eff}}}{\delta h_{ab}} = 0$

1/D expansion

- Assuming SU(N)-invariant solution $\tilde{h}_{ab}^{(0)} = v\delta_{ab}$ $X_{ab} = \lambda^{abcd}\varphi_{cd}$
 - Expanding \tilde{h}_{ab} around the solution, $S_{\text{eff}} = \frac{1}{2}T^{abcd}\varphi_{ab}\varphi_{cd}$ ^{perturbation} $- \sum_{n=3}^{\infty} \frac{\epsilon^{n-2}}{n} \text{Tr}(X^n) + \text{const.}$
 - Recall inverse propagator of A_μ^a Expand K^{-1} and S_{eff} in terms of X_{ab}
- $\langle A_\mu^a A_\nu^b \rangle = g\delta_{\mu\nu}(K^{-1})_{ab}$ \rightarrow $\langle \text{tr}(A^2) \rangle = \frac{\int dh_{ab} g D(K^{-1})_{aa} e^{-S_{\text{eff}}}}{\int dh_{ab} e^{-S_{\text{eff}}}}$
- This gives consistent results for various observables compared to simulations.
 - There is no SSB at large N limit. with SU(N)-invariant ansatz
 - In this work, considering Lorentzian bosonic IKKT model with a mass term.

Outline

- Introduction
- Large-D limit and Relevant saddle points
- $1/D$ expansion: Justification and Predictions
- Conclusion and Future directions

Large-D limit and Relevant saddle points

- Lorentzian bosonic IKKT matrix model with a mass term: $Z = \int dA \exp(-S[A])$

$$S = \frac{1}{4g^2} i\eta^{\mu\rho}\eta^{\nu\sigma} \Lambda^{abcd} A_\rho^a A_\mu^b A_\sigma^c A_\nu^d + \frac{1}{2g^2} i\gamma\eta^{\mu\nu} \delta^{ab} A_\mu^a A_\nu^b$$

$\Lambda^{abcd} = \frac{1}{2} \left[\text{tr} \left([t^a, t^d][t^b, t^c] \right) + \text{tr} \left([t^a, t^c][t^b, t^d] \right) \right]$

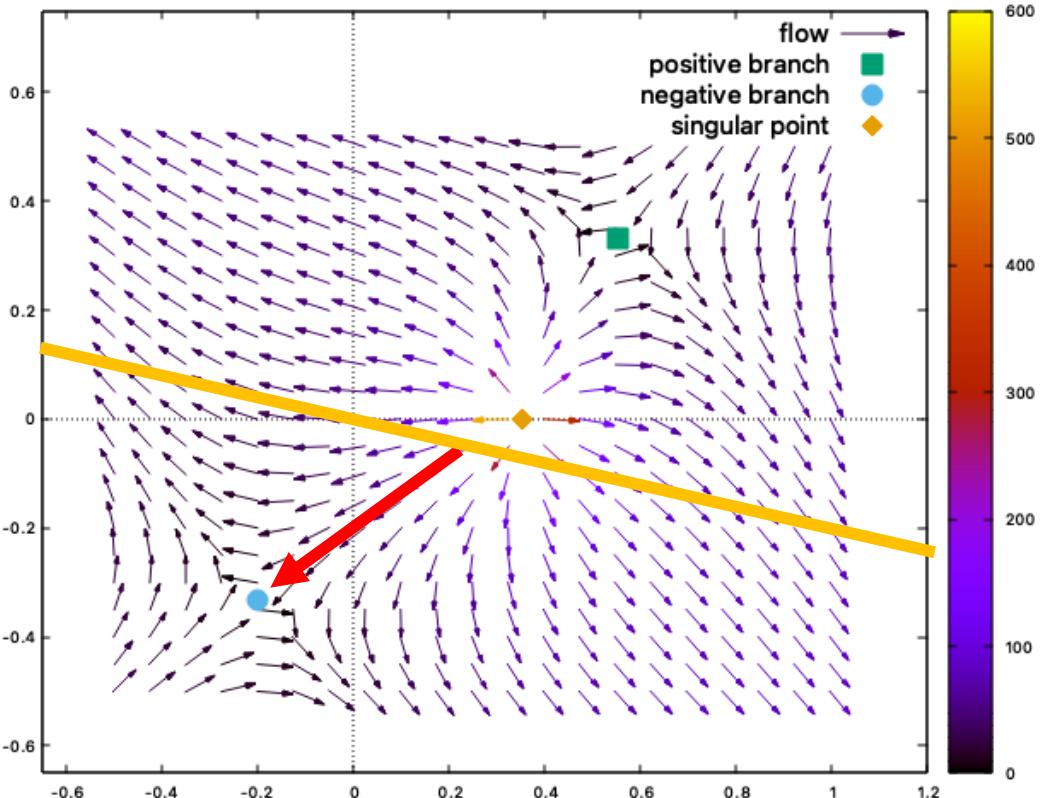
- Here we assume $\gamma \sim \tilde{\gamma} \sqrt{D}$ for the mass term to survive $D \rightarrow \infty$ limit.
- With SU(N)-invariant ansatz, $K = (-2Nv + \tilde{\gamma}/g) \mathbf{1}_{(N^2-1) \times (N^2-1)}$
appears in effective action
 $S_{eff}[h] \not\rightarrow S[A]$
- $\ln K$ has a pole on the real line! The presumed real line integration contour for h_{ab} is wrong
- For $\delta > 0$, we avoid this pole by rotating the contour by small δ : $\mathbb{R} \rightarrow e^{-i\delta} \mathbb{R}$
gives converging factor
positive/negative-branch
- Solutions to saddle-point equation: $v = v_\pm \equiv \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4\sqrt{N}}$
 - \Rightarrow Which one is relevant?
 - \Rightarrow Picard-Lefschetz theory

Large-D limit and Relevant saddle points

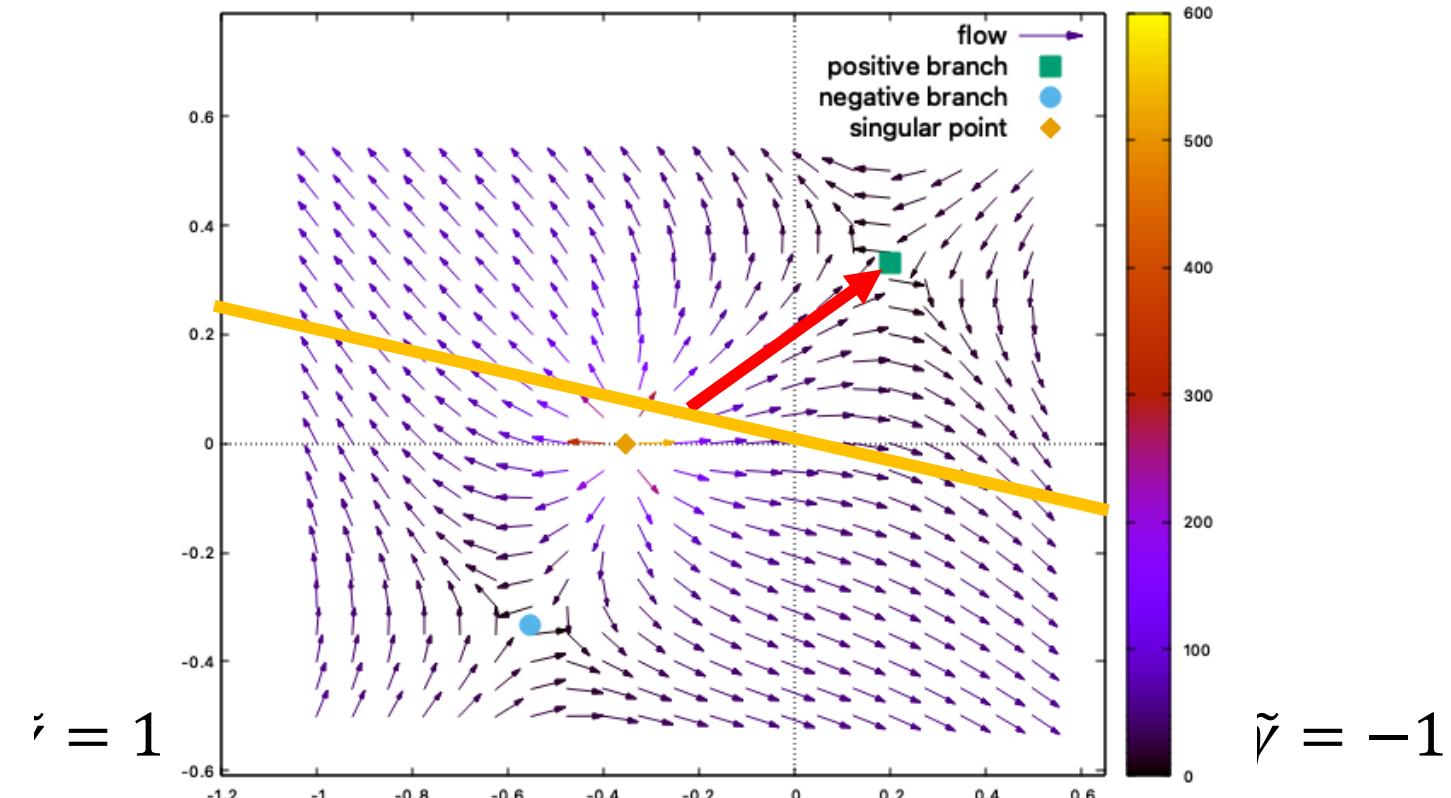
- Relevant saddle points are those reached by points on the original integration contour by means of flow equation (gradient descent)

$$\frac{dv}{d\sigma} = \frac{\overline{dS_{\text{eff}}}}{dv}$$

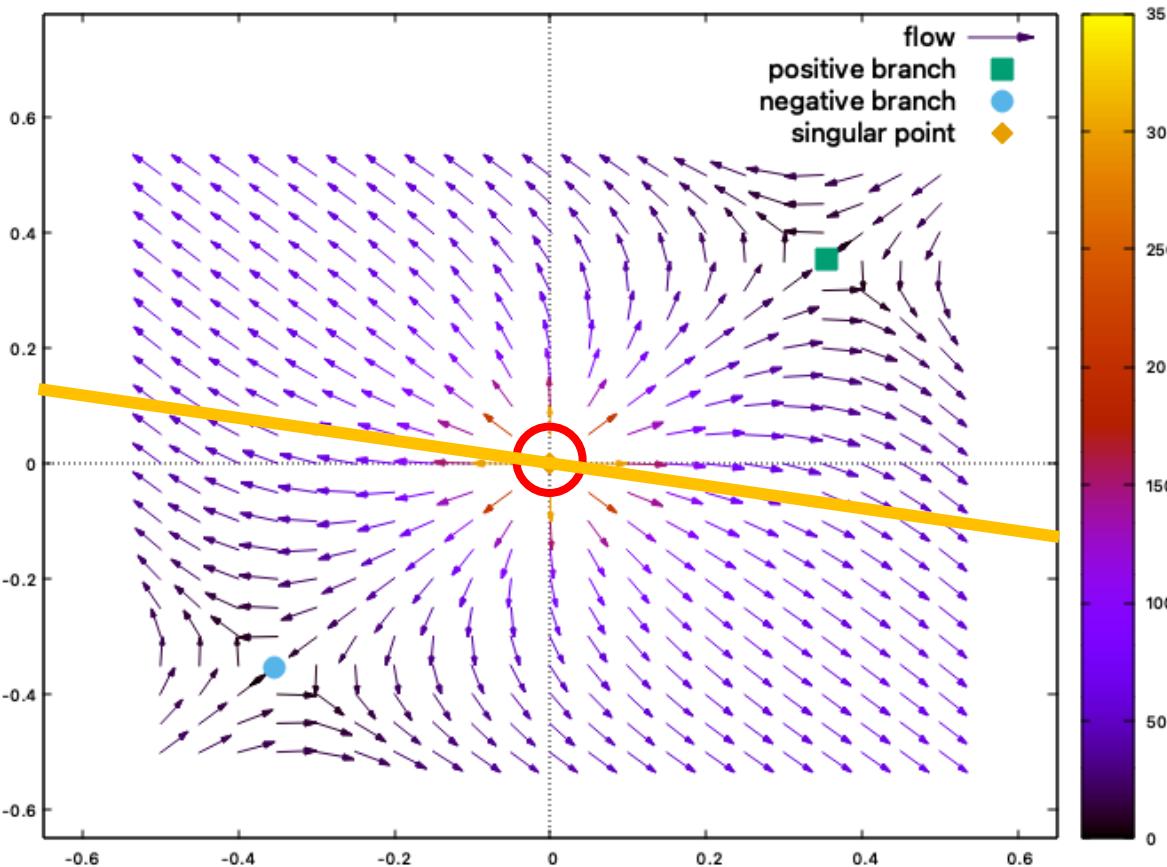
negative-branch solution: relevant



positive-branch solution: relevant



Large-D limit and Relevant saddle points



The positive-branch solution reduces to $e^{i\pi/4}/\sqrt{2N}$.

Hotta-Nishimura-Tsuchiya ('99)

At $\tilde{\gamma} = 0$, rotating contour does not help.



We define Lorentzian IKKT model without the mass term by taking limit $\tilde{\gamma} \rightarrow 0^-$ (where positive-branch solution is relevant).



At $\tilde{\gamma} = 0$, there is a sudden change in the relevant saddle points.



Stokes phenomenon

1/D expansion: Justification and Predictivity

- To compare simulation data with 1/D expansion, note that simulations use $U(N)$ as symmetry group, while 1/D expansion uses $SU(N)$ **since $U(1)$ decouples**

$$A_\mu \longrightarrow A_\mu + (\text{Tr}A_\mu)\mathbf{1} \quad \longrightarrow \quad \left\langle \frac{1}{N} \text{Tr}A^2 \right\rangle_{U(N)} = \left\langle \frac{1}{N} \text{Tr}A^2 \right\rangle_{SU(N)} + \left\langle \frac{1}{N} (\text{Tr}A_\mu)^2 \right\rangle_{U(1)}$$

- To justify the validity of 1/D expansion, consider schematically the results we obtained earlier

$$\left\langle \frac{1}{N} \text{Tr}A^2 \right\rangle_{SU(N)} = \sqrt{D} \left(c_0(\tilde{\gamma}) + \frac{c_1(\tilde{\gamma})}{D} + \frac{c_2(\tilde{\gamma})}{D^2} + \dots \right) \quad v = v_\pm \equiv \frac{\tilde{\gamma} \pm \sqrt{\tilde{\gamma}^2 + 8i}}{4\sqrt{N}}$$

If we write the LHS in terms of γ instead of $\tilde{\gamma}$, the D-dependence will be more non-trivial, and the analysis will be unnecessarily complicated.

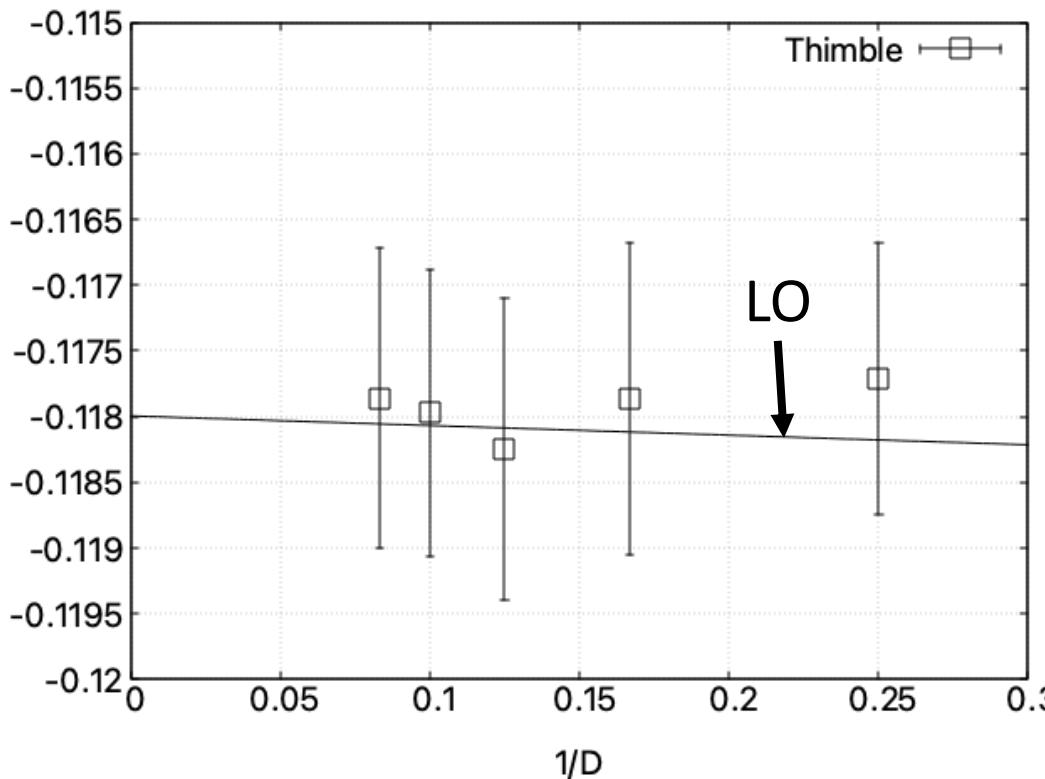
- Plotting $\frac{1}{\sqrt{D}} \left\langle \frac{1}{N} \text{Tr}A^2 \right\rangle$ from the simulations against 1/D, it should converge to c_0 as $D \rightarrow \infty$

1/D expansion: Justification and Predictivity

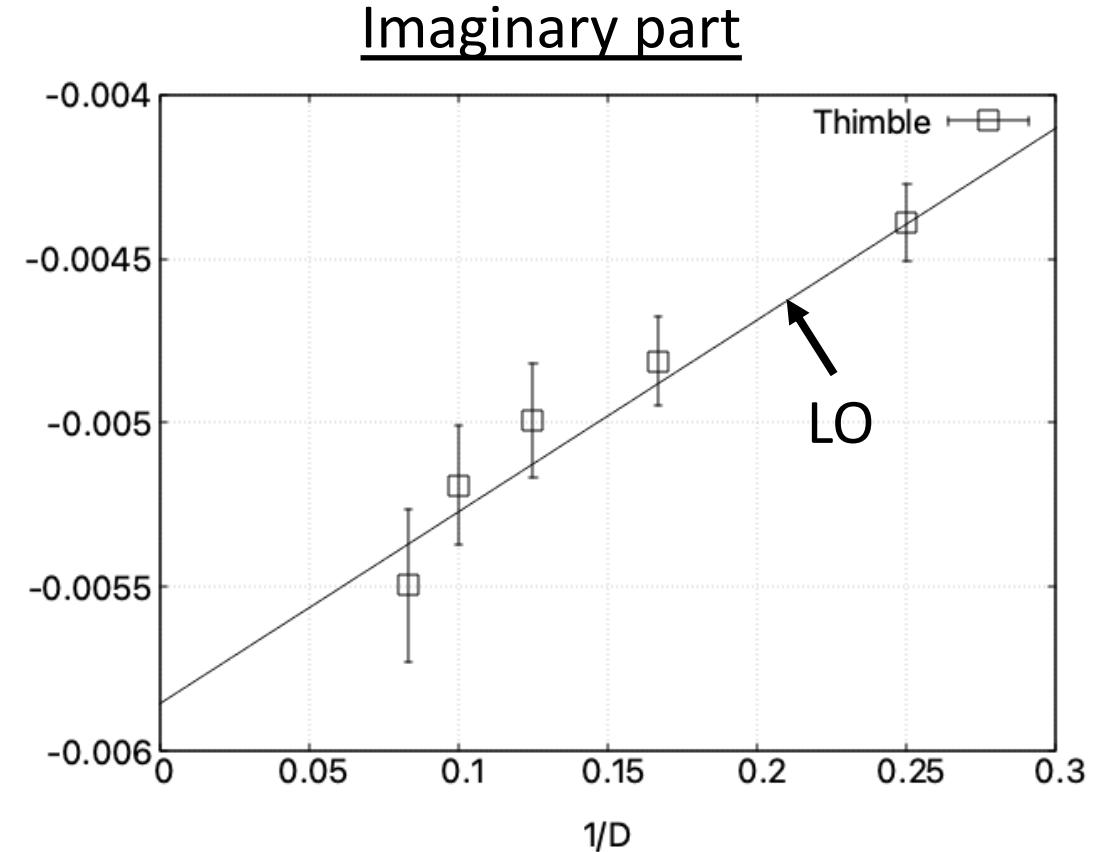
due to $SO(9,1)$ symmetry (N.Yamamori's talk parallel session B Dec. 7 14:20)

$$\sqrt{D} \left\langle -\frac{1}{N} \text{Tr}(A_0^2) \right\rangle_{\text{SU}(N)} \quad \text{at } \tilde{\gamma} = 6.325$$

Real part



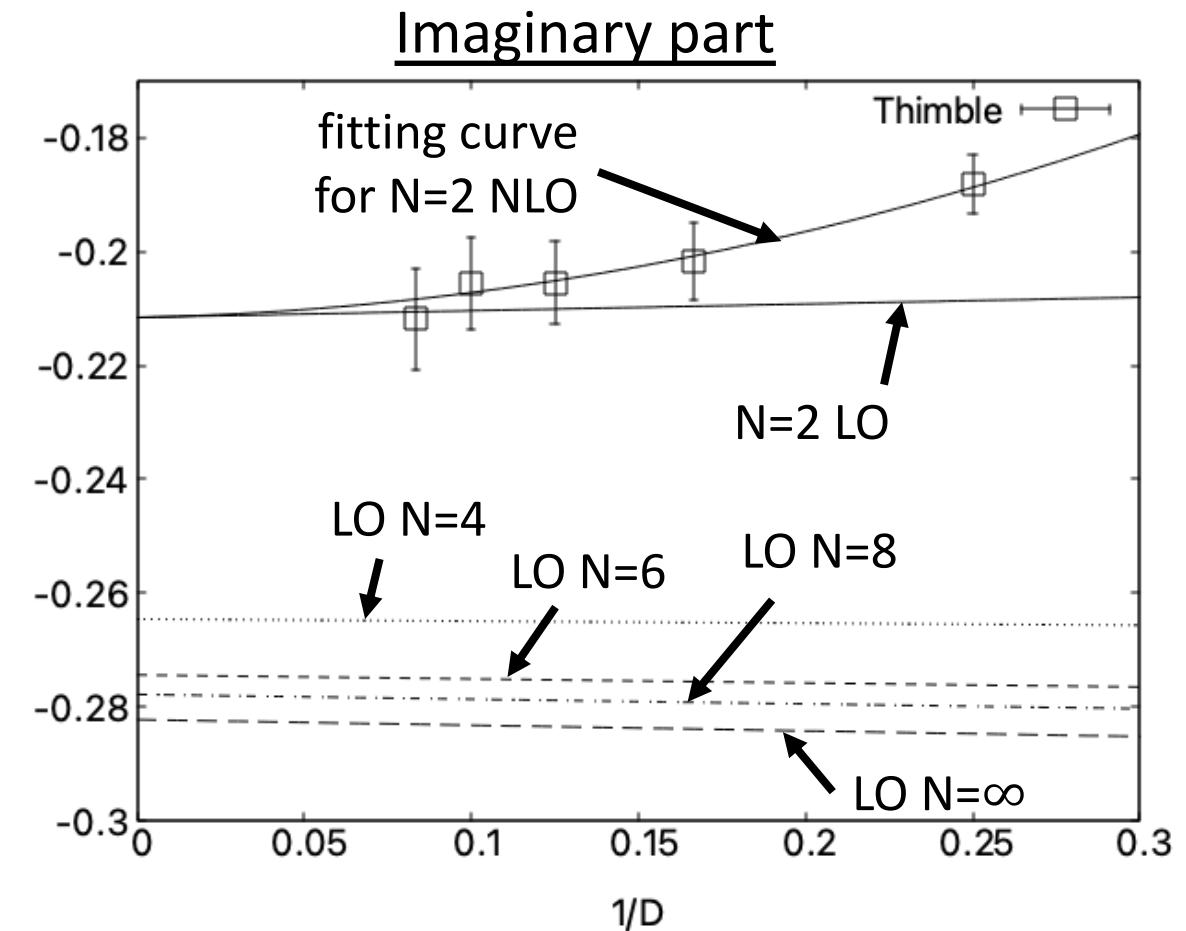
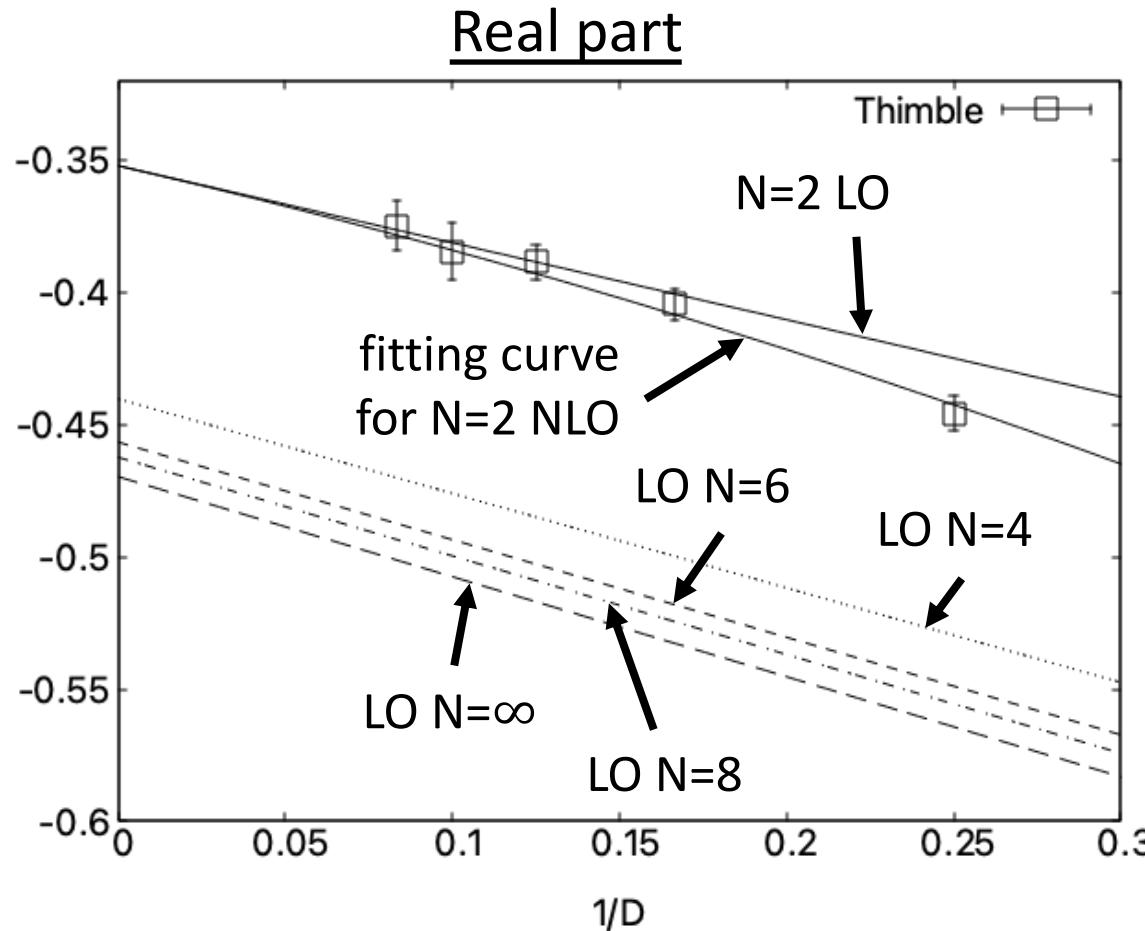
1/D expansion is valid!



1/D expansion: Justification and Predictivity

$$\sqrt{D} \left\langle -\frac{1}{N} \text{Tr}(A_0^2) \right\rangle_{\text{SU}(N)} \text{ at } \tilde{\gamma} = 1$$

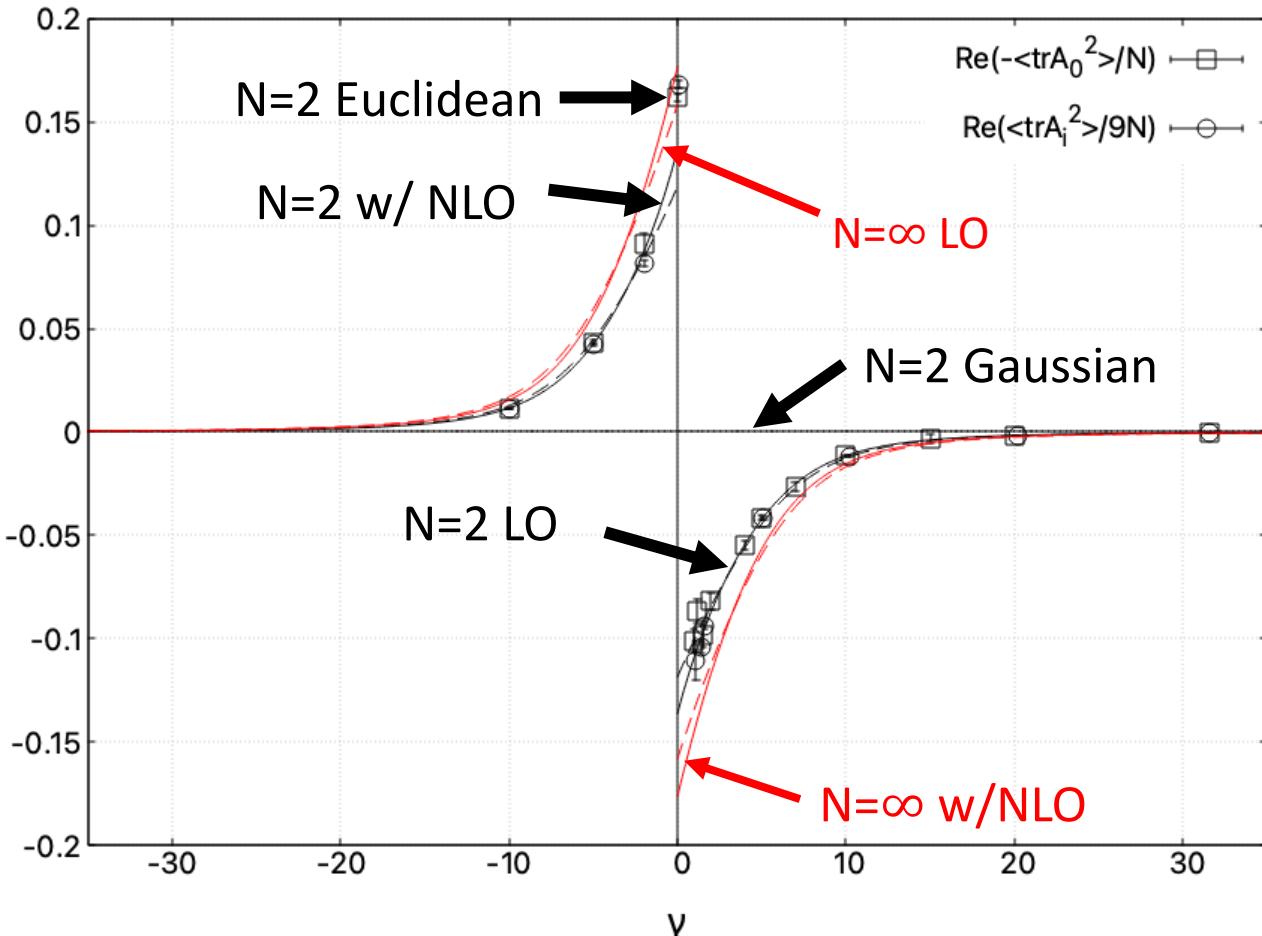
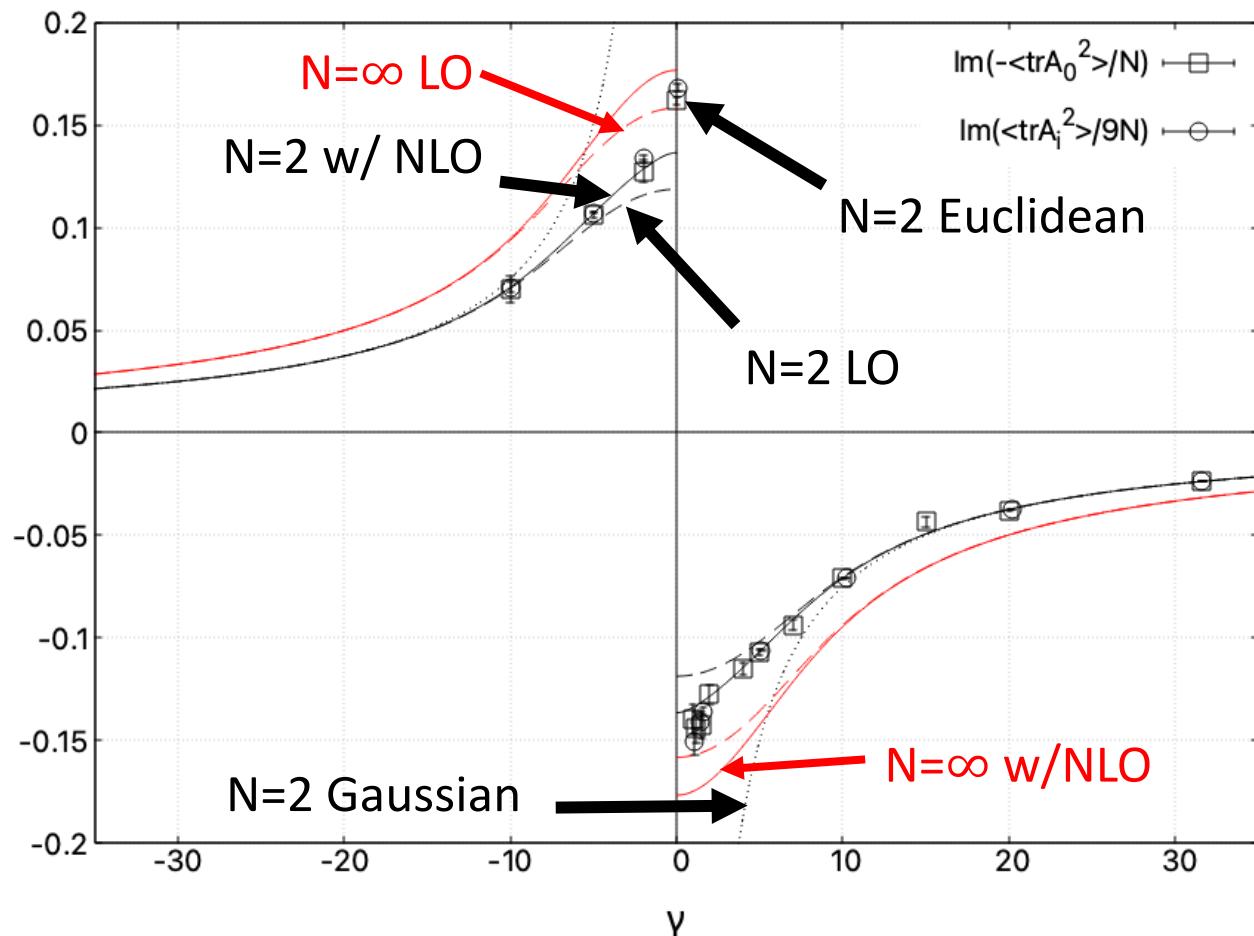
- 1/D expansion is valid!
- Higher-order terms gain importance at smaller $\tilde{\gamma}$
- Large-N limit is not so far (N=4,6,8 could be a good estimate)



1/D expansion: Justification and Predictivity

$$\frac{1}{D-1} \left\langle \frac{1}{N} \sum_{i=1}^{D-1} \text{Tr}(A_i)^2 \right\rangle_{SU(N)} = - \left\langle \frac{1}{N} \text{Tr}(A_0)^2 \right\rangle_{SU(N)} \quad \text{at } D = 10$$

- Approach Euclidean from $\gamma \rightarrow 0^-$
- N=2 are very close to N= ∞



Conclusion and Future directions

- There is a Stokes phenomenon occurring at $\tilde{\gamma} = 0$ causing a sudden change in the relevant saddle points.
- The 1/D expansion on Lorentzian IKKT matrix model with a mass term is valid and very predictive.
- The relevant SU(N)-invariant saddle points are the fixed points of the corresponding dominating thimbles which are SO(9,1)-symmetry-preserving.
- The results at (once thought to be too low) $N=2$ are very close to those of $N=\infty$
 \Rightarrow We do not have to go at large N ($N=16, 32, \dots$) to see large-N behaviour.
- There seems to be a 'reflection symmetry' between γ and $-\gamma$ such that

$$\langle \text{Tr}(A^2)/N \rangle_+ = -\langle \text{Tr}(A^2)/N \rangle_-$$

Understood by Wick rotating $A_\mu \rightarrow iA_\mu$
and the partition functions are the same
under the change $\gamma \rightarrow -\gamma$

Conclusion and Future directions

Hoppe ('21)

- There is a way to obtain more general solution to the saddle-point equation.
- It was shown that the eigenmatrices of Λ^{abcd} are think of a pair (ab) as one index

$$\Lambda^{abcd}\delta_{cd} = -2N\delta_{ab}$$

$$\Lambda^{abcd}(W_-^{\bar{i}})_{cd} = 2(W_-^{\bar{i}})_{ab}$$

$$\Lambda^{abcd}d^{cde} = -Nd^{abe}$$

$$\Lambda^{abcd}(W_+^i)_{cd} = -2(W_+^i)_{ab}$$

$$\stackrel{\bar{i}, \bar{i} = (cd)}{\longrightarrow}$$

$$(W_{\mp}^{cd})_{ab} = \frac{1}{2}\delta_{c(a}\delta_{b)d} \pm \frac{1}{4}\Lambda^{abcd}$$

$$+ \frac{-1 \pm N}{2(N^2 - 1)}\delta_{ab}\delta_{cd} + \frac{(-2 \pm N)N}{8(N^2 - 4)}d^{abe}d^{cde}$$

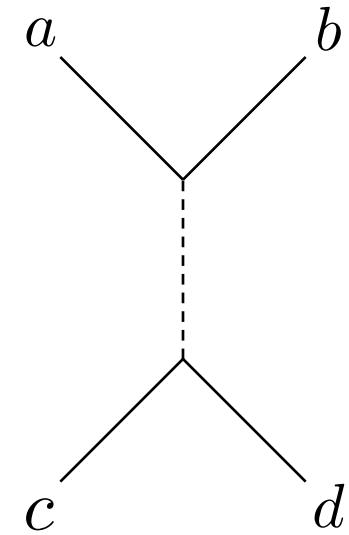
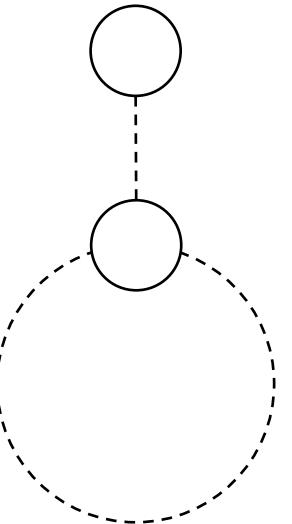
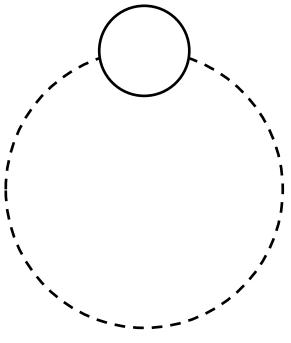
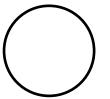
- Therefore, h_{ab} can be generally written as a linear combination of these 4 symmetric matrices

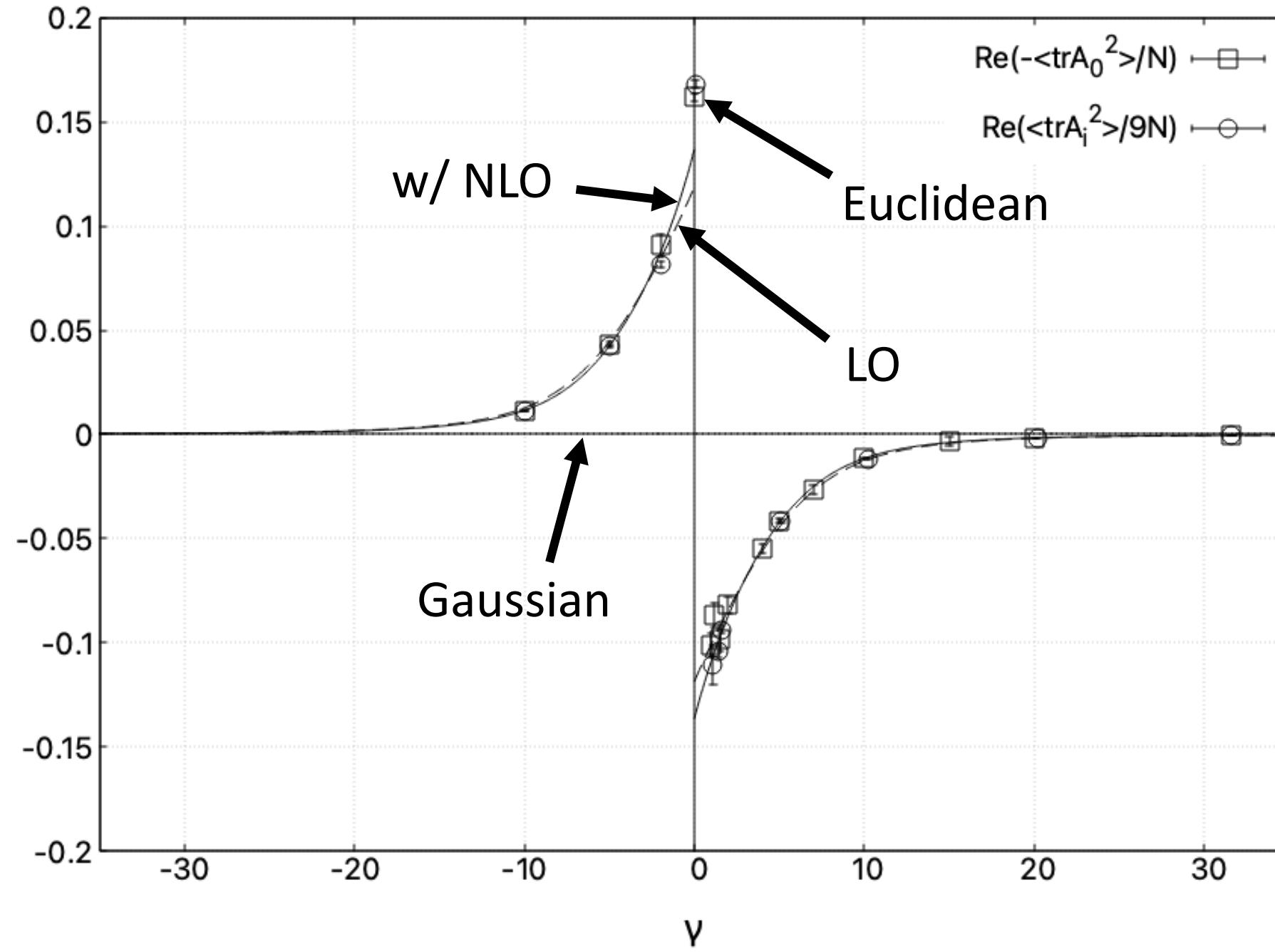
$$h_{ab} = v_0\delta^{ab} + v_{\text{Adj}}^c d^{abc} + v_-^{\bar{i}}(W_-^{\bar{i}})_{ab} + v_+^i(W_+^i)_{ab}$$

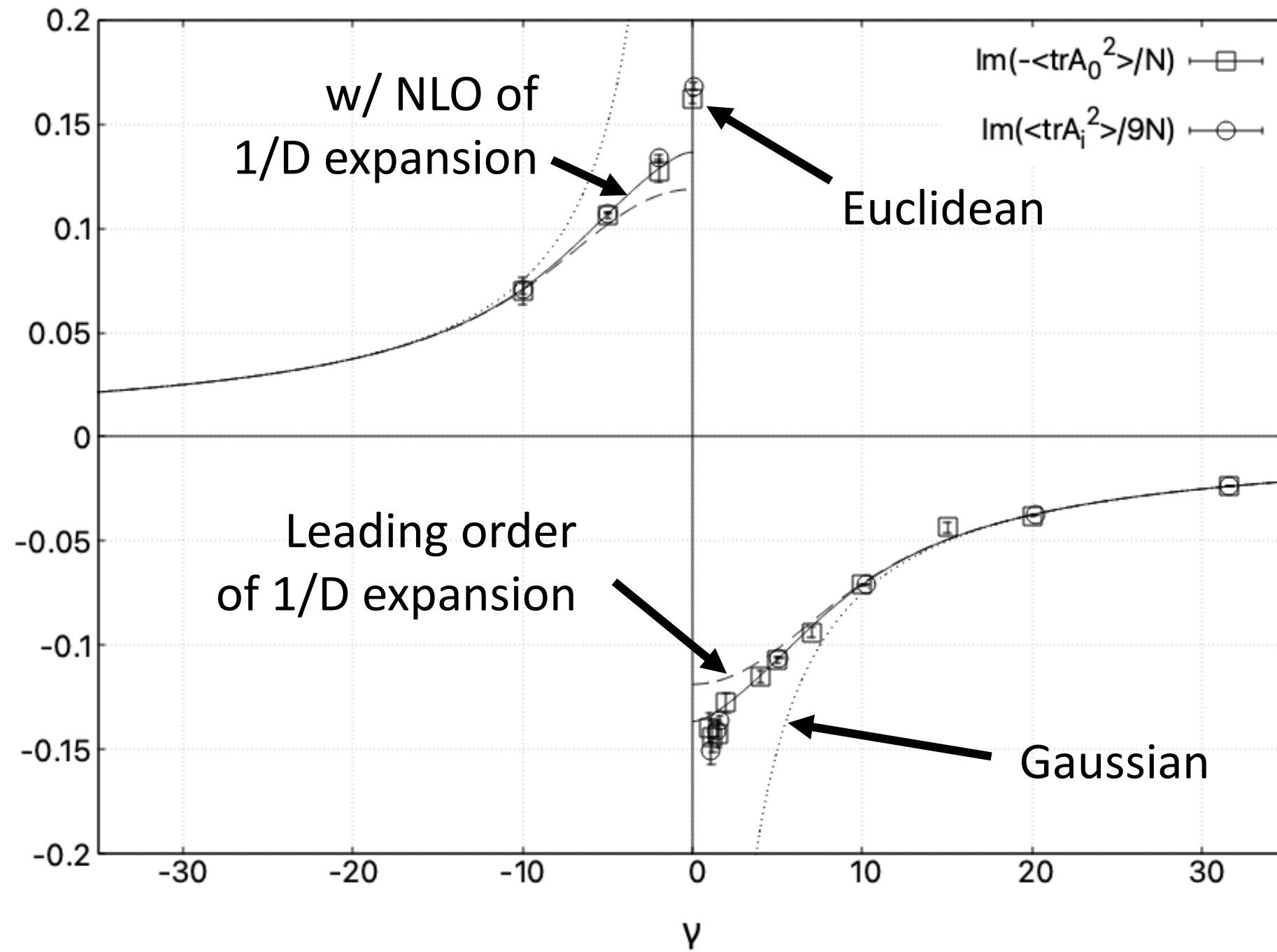
Asano-Nishimura-WP-Yamamori (work in progress)

Not only allowing us to study Lorentzian IKKT matrix model with mass term more generally, but also simplifies the bosonic part of the action since these matrices are eigenmatrices.

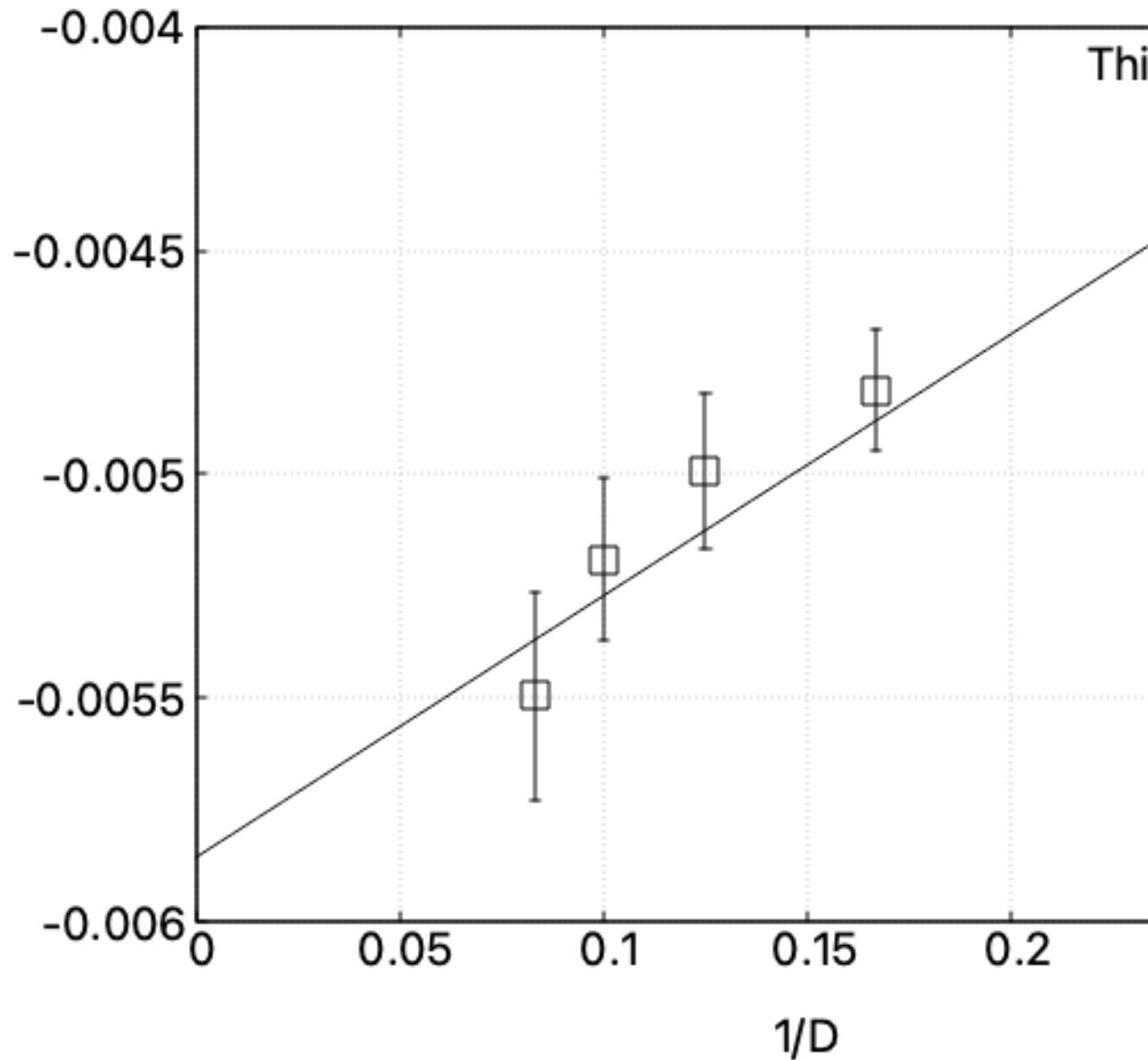
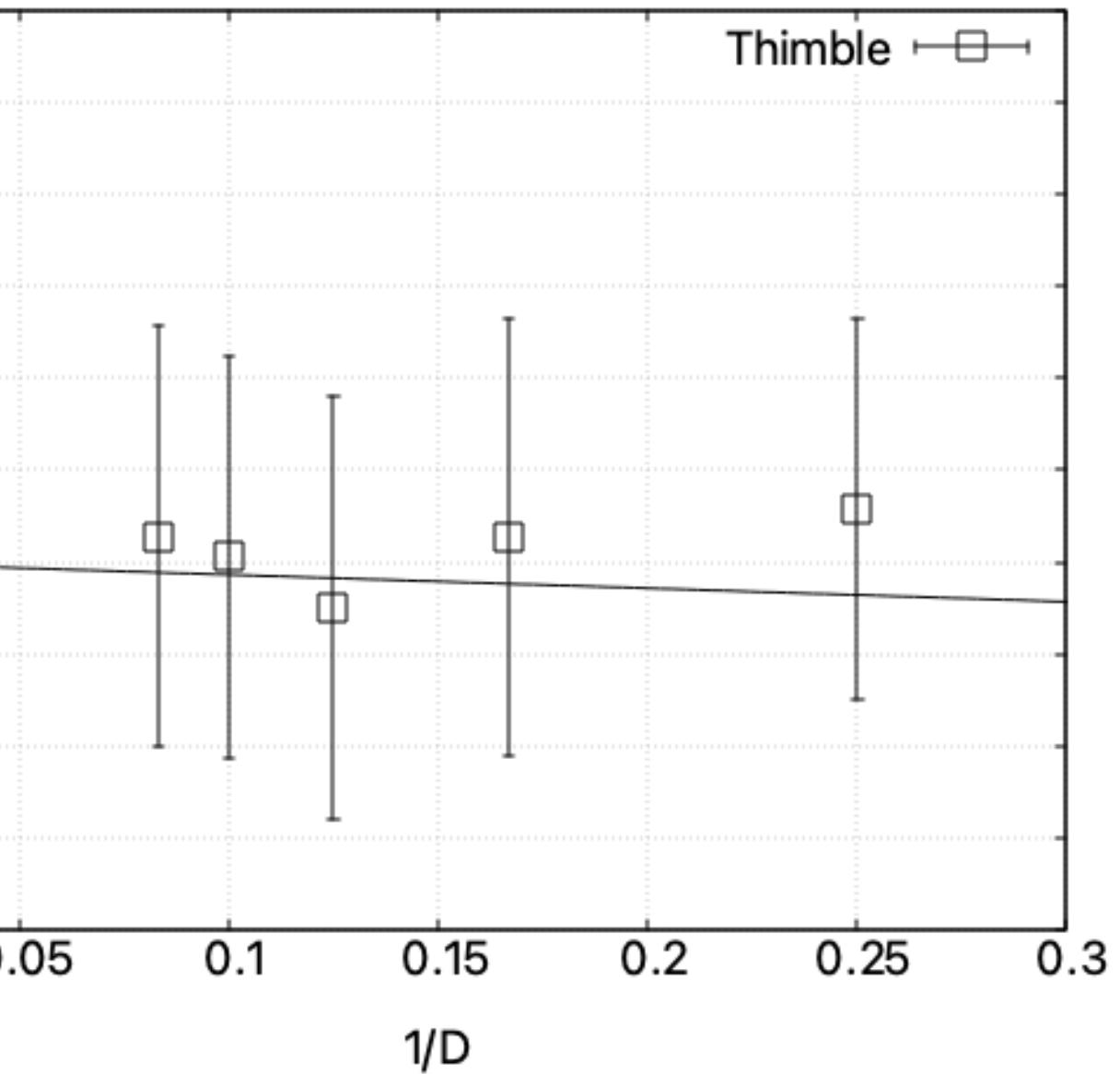
Backup





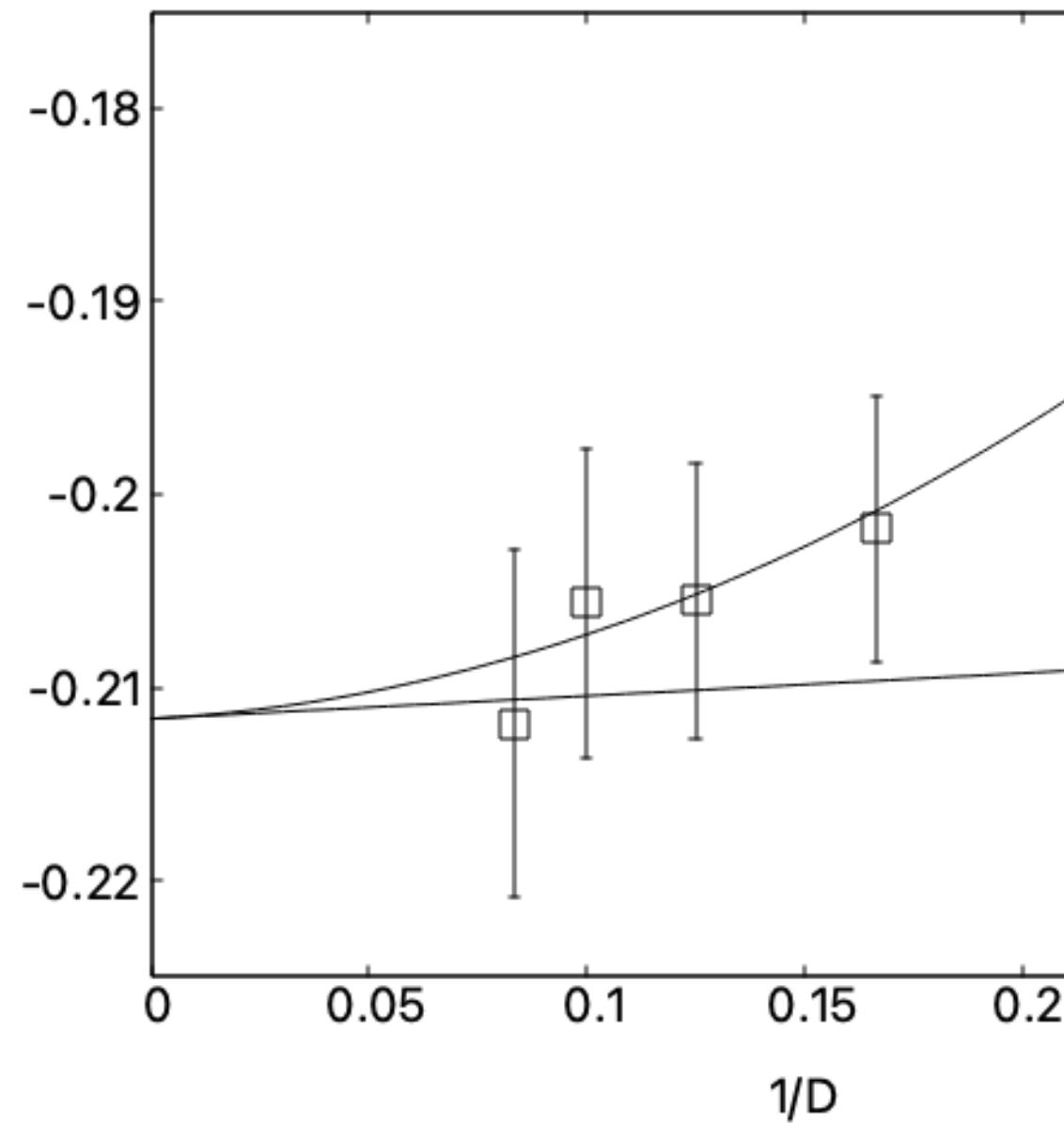
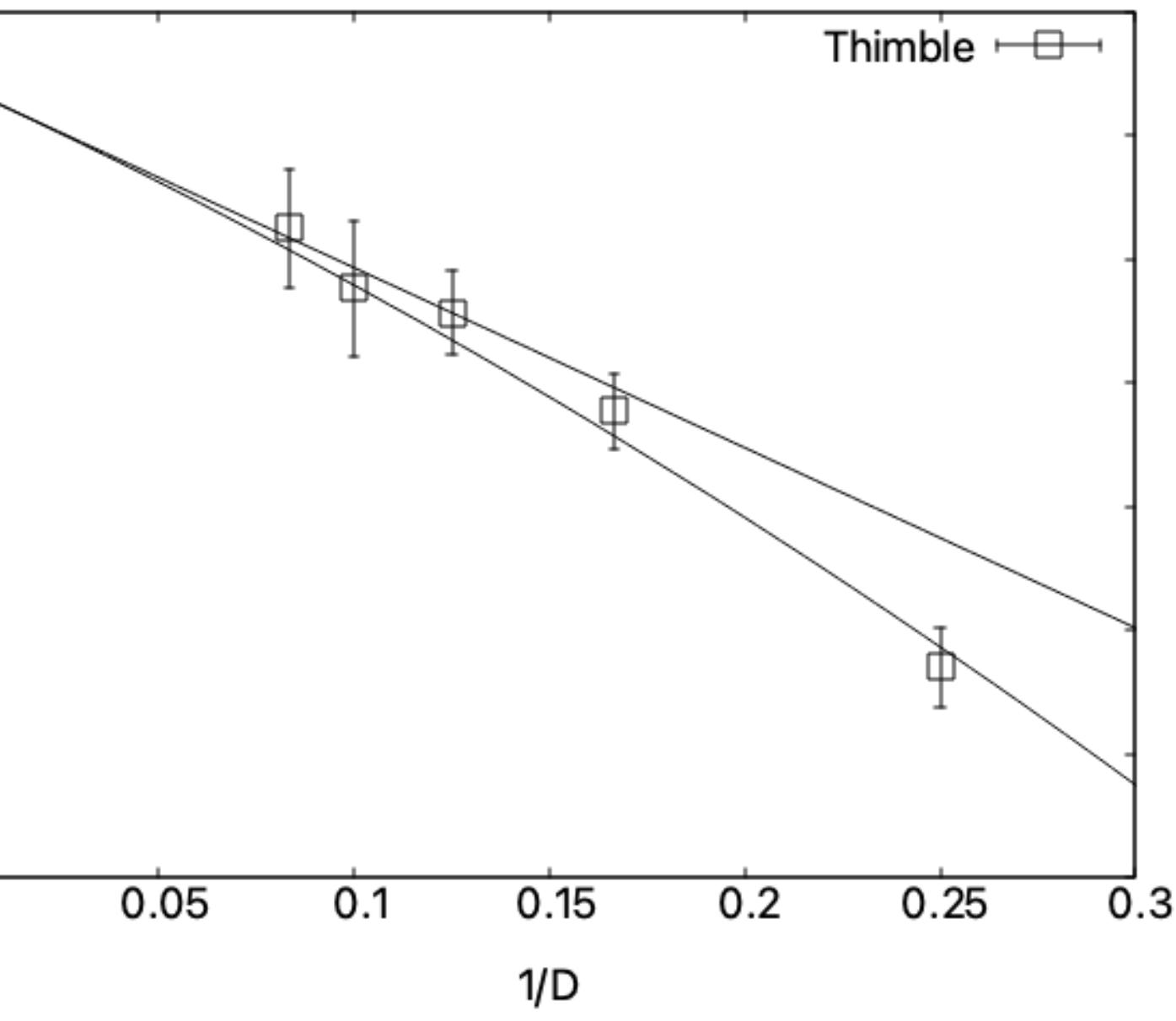


$$\tilde{\gamma} = 6.325$$



$\tilde{\gamma} = 1$

Thimble



Euclidean Model

$$Z = \int dA e^{-S_b^{(E)}} \text{Pf} \mathcal{M}^{(E)}(A)$$

Generally complex

Lorentzian Model

$$Z = \int dA e^{iS_b^{(L)}} \text{Pf} \mathcal{M}^{(L)}(A)$$

Complex

Sign problem (符号問題)

