Fractional Topological Charge in Lattice Abelian Gauge Theory

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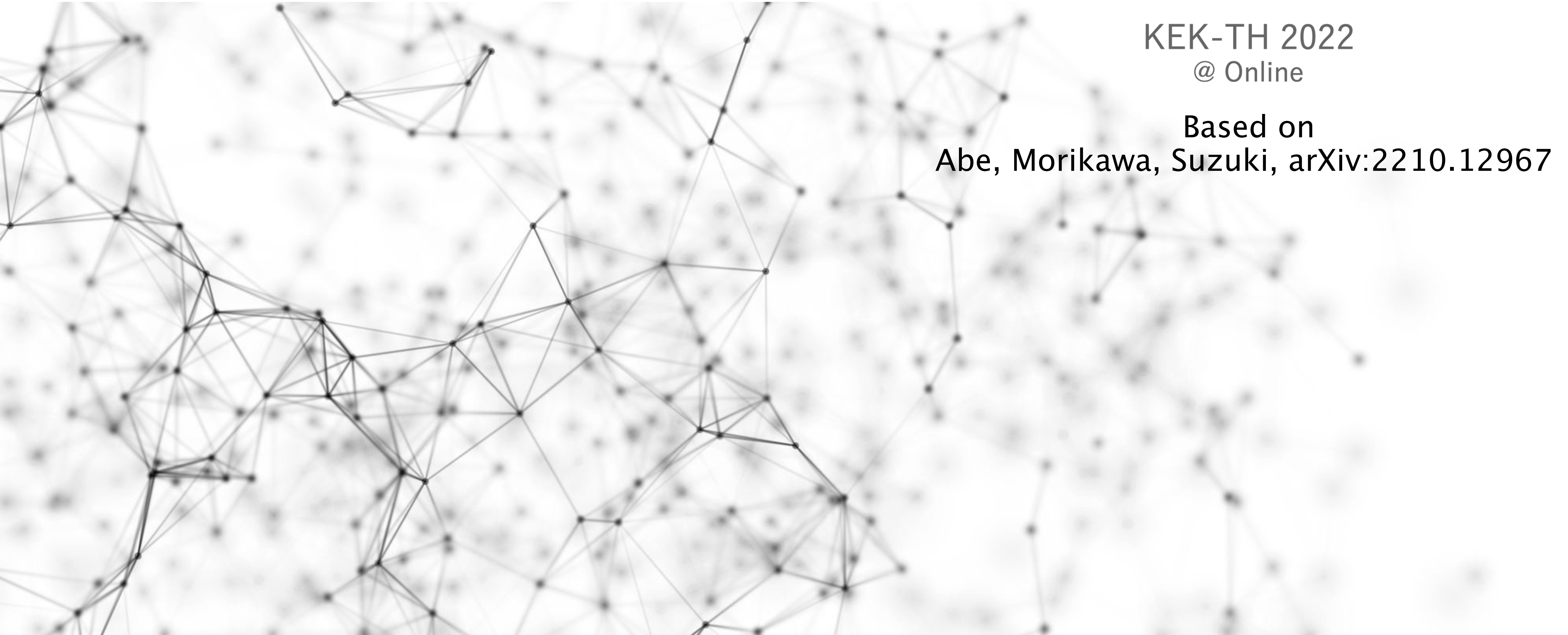

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Based on
Abe, Morikawa, Suzuki, arXiv:2210.12967

Recent Studies about ’t Hooft Anomaly

- Discussion of the low-energy dynamics of gauge theories based on the mixed 't Hooft anomaly between discrete and higher-form symmetries.
  （Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148
  Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501）
- This type of application of the anomaly has been studied vigorously.
  - ✓ Yamaguchi, arXiv:1811.09390
  - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389
  - ✓ Honda, Tanizaki, arXiv:2009.10183
  - ✓ etc
- Keywords: ’t Hooft anomaly, higher-form symmetry

$\mathbb{Z}_N$ One-form Gauge Symmetry

$\mathbb{Z}_N$ zero-form gauge symmetry

- A pair, $U(1)$ gauge field A_μ and scalar field ϕ, makes $\mathbb{Z}_N$ one-form gauge field.
- Constraint

$$NA_\mu = \partial_\mu \phi$$

- $\mathbb{Z}_N$ zero-form gauge transformation

$$\phi \mapsto \phi + N\lambda$$
$$A_\mu \mapsto A_\mu + \partial_\mu \lambda$$

$\mathbb{Z}_N$ one-form gauge symmetry

- A pair, $U(1)$ two-form gauge field $B_{\mu\nu}$ and $U(1)$gauge field C_μ, makes $\mathbb{Z}_N$ two-form gauge field.
- Constraint

$$NB_{\mu\nu} = \partial_{[\mu} C_{\nu]}$$

For short, we write
$NB = \mathrm{d}C$

- $\mathbb{Z}_N$ one-form gauge transformation

$$C_\mu \mapsto C_\mu + N\lambda_\mu$$
$$B_{\mu\nu} \mapsto B_{\mu\nu} + \partial_{[\mu} \lambda_{\nu]}$$

$SU(N)$ Gauge Theory with θ Term

- Action: $$S = -\frac{1}{2g^2}\int \mathrm{tr}(f \wedge \star f) + \frac{\theta}{8\pi^2}\int \mathrm{tr}(f \wedge f)$$

$\mathcal{T}$-symmetry when θ = 0, π

➢ Coupling $\mathbb{Z}_N$ two-form gauge field B as the background gauge field,

$$S = -\frac{1}{2g^2}\int \mathrm{tr}\left[(\mathcal{F} - \mathbb{1}B) \wedge (\mathcal{F} - \mathbb{1}B)\right] + \frac{\theta}{8\pi^2}\int \mathrm{tr}\left[(\mathcal{F} - \mathbb{1}B) \wedge (\mathcal{F} - \mathbb{1}B)\right] + \frac{1}{2\pi}\int u \wedge (\mathrm{tr}\,\mathcal{F} - NB)$$

➢ "$\mathbb{Z}_N$ one-form gauge symmetry"+"$\mathcal{T}$-symmetry when θ = π"

➢ Respecting $\mathbb{Z}_N$ one-form gauge symmetry,

$$Z[B] \xrightarrow{\mathcal{T}} Z[B] \exp\left[i\frac{-1+N+2p}{4\pi N}\int NB \wedge NB\right]$$

$2\pi i \times$(fractional)

➢ Our motivation is to understand in a completely regularized framework (lattice field theory).

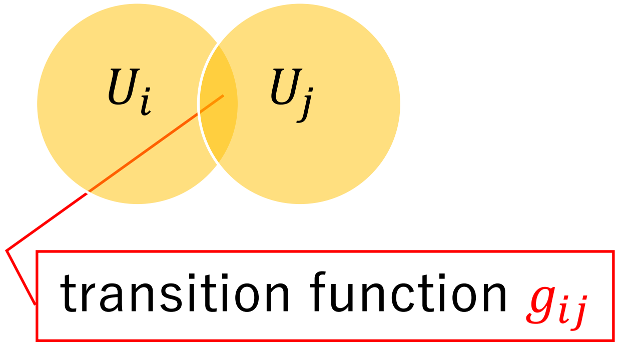
Principal Fiber Bundle

- Covering manifold by patch U_i, each patch has $SU(N)$ gauge field a_i, matter field ϕ_i in an irreducible representation ρ.

- g_{ij} at $U_{ij} = U_i \cap U_j$

$$a_j = g_{ij}^{-1} a_i g_{ij} - i g_{ij}^{-1} \mathrm{d} g_{ij}$$

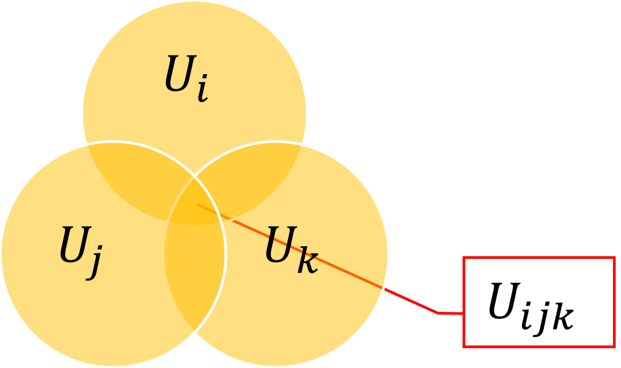
$$\phi_j = \rho(g_{ij}^{-1}) \phi_i$$



transition function g_{ij}

- Cocycle condition at $U_{ijk} = U_i \cap U_j \cap U_k$

$$g_{ij} g_{jk} g_{ki} = 1$$



U_{ijk}

$\mathbb{Z}_N$ One-form Gauge Symmetry and Fiber Bundle

- Especially for the adjoint representation, the cocycle condition can become relaxed.

$$g_{ij}g_{jk}g_{ki} = \exp\left(\frac{2\pi i}{N} n_{ijk}\right)$$

$\in \mathbb{Z}_N$

➢ $\{n_{ijk}\}$ has the gauge redundancy.

➢ The transformation of the transition function is

$$g_{ij} \mapsto \exp\left(\frac{2\pi i}{N}\lambda_{ij}\right) g_{ij}$$

For the invariance of the cocycle condition,

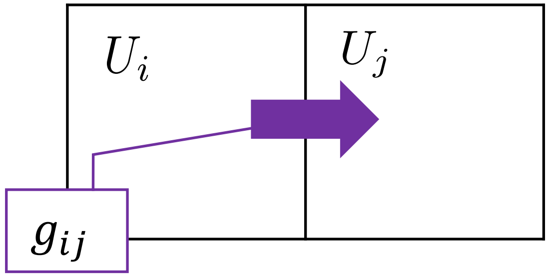
$$n_{ijk} \mapsto n_{ijk} + (\delta\lambda)_{ijk}$$

$$(\delta\lambda)_{ijk} \equiv \lambda_{ij} - \lambda_{ik} + \lambda_{jk}$$

➢ This transformation is $\mathbb{Z}_N$ one-form gauge transformation, $\{n_{ijk}\}$ is $\mathbb{Z}_N$ two-form gauge field.

Fiber Bundle in $SU(N)/\mathbb{Z}_N$ gauge theory

- By the principal fiber bundle in the $SU(N)/\mathbb{Z}_N$ gauge theory, the topological charge can become fractional.
  ('t Hooft, Nucl. Phys. B 153 (1979), van Baal, Commun. Math. Phys. 85 (1982))



$$\text{(new transition function)} \sim \omega_\mu \times \text{(ordinary transition function)}$$

factor for fractionality

※ $Z[B] \xrightarrow{\mathcal{T}} Z[B] \exp\left[i \frac{-1+N+2p}{4\pi N} \int NB \wedge NB \right]$

2πi×(fractional)

Fractional Topological Charge on the Lattice

- Above discussion about mixed ’t Hooft anomaly and the fractional topological charge is in the continuum.

➢ Want to understand in a completely regularized framework (lattice gauge theory)

← our purpose

- Integer topological charge is formulated in the lattice $SU(N)$ gauge theory
  (Lüscher, Commun. Math. Phys. 85 (1982))

➢ We formulated fractional topological charge in the lattice $U(1)$ gauge theory.

← our paper

➢ Apply the simpler formulation of the integer topological charge in the lattice $U(1)$ gauge theory
(Fujiwara, Suzuki, Wu, arXiv:0001029)

Transition Function on the Lattice

Fractionality

$$\text{(new transition function)} \sim \omega_\mu \times \text{(ordinary transition function)}$$

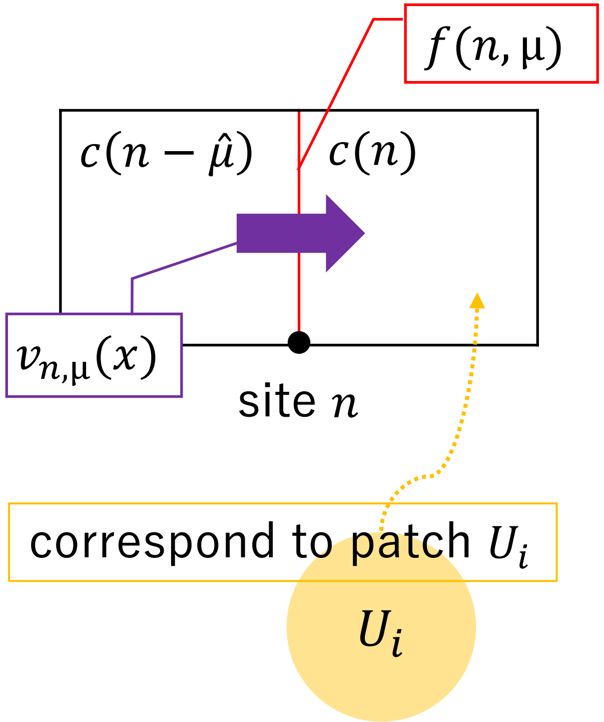
- At $x \in f(n, \mu)$, constructing $U(1)/\mathbb{Z}_q$ transition function,

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x) \quad \text{at } x \in f(n, \mu)$$

➢ ω_μ is the factor for relaxing the cocycle condition,

$$\omega_\mu(x) \equiv \begin{cases} \exp\left(\frac{\pi i}{q} \sum_{\nu \neq \mu} \frac{z_{\mu\nu} x_\nu}{L}\right) & \text{for } x_\mu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

➢ $z_{\mu\nu} \in \mathbb{Z}$ and $z_{\mu\nu} = -z_{\nu\mu}$



Cocycle Condition on the Lattice

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x) \quad \text{at } x \in f(n,\mu)$$

- For the ordinary transition function $\check{v}_{n,\mu}$, the cocycle condition is

$$\check{v}_{n-\hat{\mu},\nu}(x)\check{v}_{n,\mu}(x)\check{v}_{n,\nu}(x)^{-1}\check{v}_{n-\hat{\nu},\mu}(x)^{-1} = 1$$

lattice version of
$g_{ij}g_{jk}g_{ki} = 1$

- For the new transition function $v_{n,\mu}$, owing to ω_μ,

$$v_{n-\hat{\mu},\nu}(x)v_{n,\mu}(x)v_{n,\nu}(x)^{-1}v_{n-\hat{\nu},\mu}(x)^{-1}$$
$$= \begin{cases} \exp\left(\frac{2\pi i}{q} z_{\mu\nu}\right) \in \mathbb{Z}_q & \text{for } x_\mu = x_\nu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

$\mathbb{Z}_q$ “relax”

$\mathbb{Z}_q$ One-form Global Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➢ Link variable

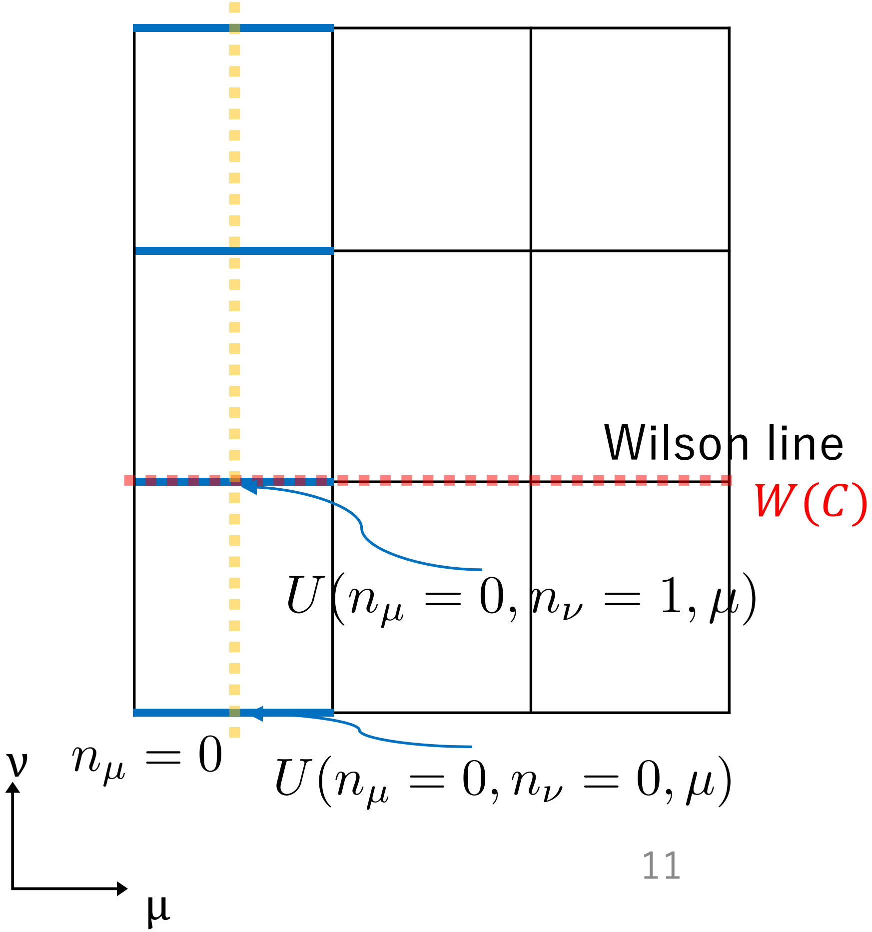
$$U(n,\mu) \to \underbrace{\exp\left(\frac{2\pi i}{q} z_\mu\right)}_{\in \mathbb{Z}_q} U(n,\mu) \qquad n_\mu = 0$$

➢ Transition function

$$\check{v}_{n,\mu}(x) \to \begin{cases} \exp\left(\frac{2\pi i}{q} z_\mu\right) \check{v}_{n,\mu}(x) & \text{for } x_\mu = 1 \\ \check{v}_{n,\mu}(x) & \text{otherwise} \end{cases}$$

➢ Cocycle condition

$$\check{v}_{n-\hat{\nu},\mu}(x)\check{v}_{n,\nu}(x)\check{v}^{-1}_{n,\mu}(x)\check{v}^{-1}_{n-\hat{\mu},\nu}(x) \overset{\text{not } \mathbb{Z}_q \text{ “relax”}}{=} 1$$



$\mathbb{Z}_q$ One-form Gauge Symmetry on the Lattice

- The factor of fractionality ω_μ is related to the $\mathbb{Z}_q$ one-form transform.

➢ Link variable

$$U(n,\mu) \to \exp\left[\frac{2\pi i}{q} z_\mu(n)\right] U(n,\mu)$$

$\in \mathbb{Z}_q$

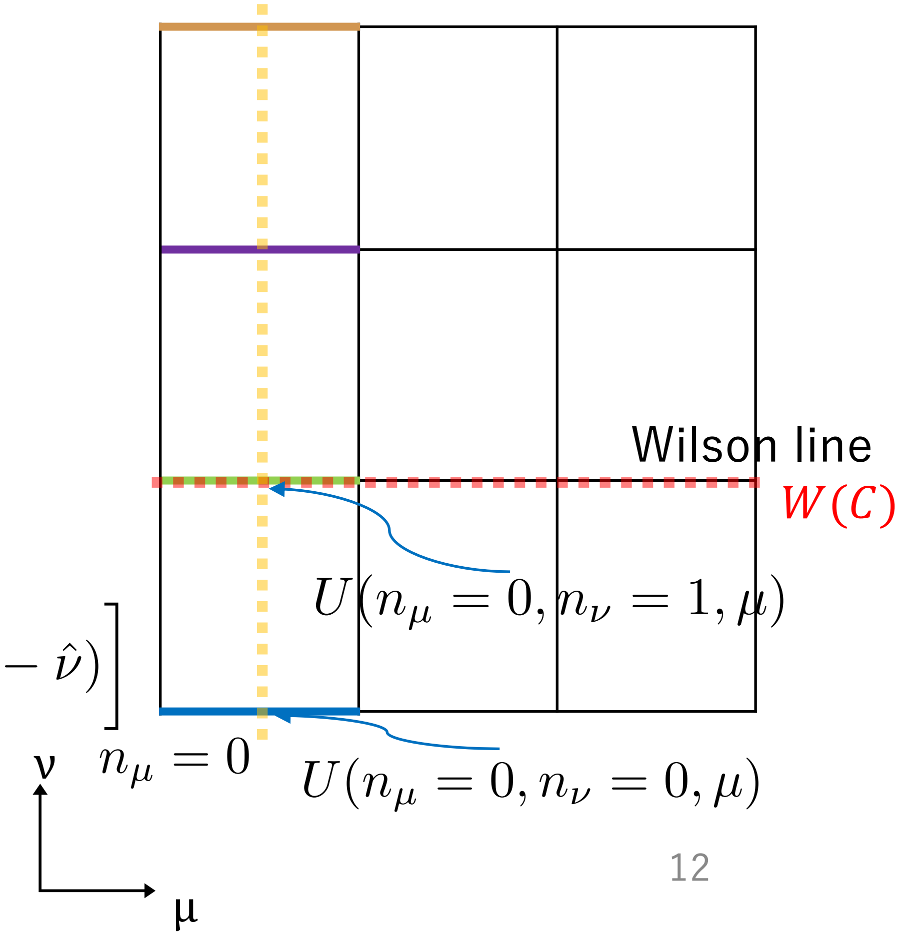
➢ Transition function

$$v_{n,\mu}(x) \to \exp\left[\frac{2\pi i}{q} z_\mu(n-\hat{\mu})\right] v_{n,\mu}(x) \qquad x \in f(n,\mu)$$

➢ Cocycle condition

$\mathbb{Z}_q$ "relax"

$$v_{n-\hat{\nu},\mu}(x) v_{n,\nu}(x) v_{n,\mu}(x)^{-1} v_{n-\hat{\mu},\nu}(x)^{-1} \equiv \exp\left[\frac{2\pi i}{q} z_{\mu\nu}(n-\hat{\mu}-\hat{\nu})\right]$$



Fractional Topological Charge on the Lattice

- In the continuum,

$$Q = \frac{1}{32\pi^2}\int_{T^4} \mathrm{d}^4x\, \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)$$

- Topological charge is calculated by the transition function,

$$Q = -\frac{1}{8\pi^2}\sum_{n\in\Lambda}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\int \mathrm{d}^2x\, [v_{n,\mu}(x)\partial_\rho v_{n,\mu}(x)^{-1}][v_{n-\hat{\mu},\nu}(x)^{-1}\partial_\sigma v_{n-\hat{\mu},\nu}(x)]$$

- For the new transition function $v_{n,\mu}$,

factor of fractionality

$$\omega_\mu(x) \sim \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right)$$

$$Q = \underbrace{\frac{1}{8q^2}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}_{\text{Fractional!!}} + \underbrace{\frac{1}{8\pi q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu}\sum_{n_\mu=0}\check{F}_{\rho\sigma}(n)}_{\text{cross term}}$$
$$+ \underbrace{\frac{1}{32\pi^2}\sum_{n}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\check{F}_{\mu\nu}(n)\check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})}_{\text{integer}}$$

Mixed ’t Hooft Anomaly

- A lattice action is

$$S \equiv \frac{1}{4g_0^2}\sum_{n}\sum_{\mu,\nu}\check{F}_{\mu\nu}(n)\check{F}_{\mu\nu}(n) + S_{\mathrm{matter}} - iq\theta Q$$

By the Witten effect
Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389
Honda, Tanizaki, arXiv:2009.10183

➢ Topological charge is

$$qQ = \frac{1}{8q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma} + \mathbb{Z}$$

✓ invariant under the $\mathbb{Z}_q$ one-form gauge transformation

✓ odd under the lattice $\mathcal{T}$-transformation.

➢ We discussed the mixed ’t Hooft anomaly between the $\mathbb{Z}_q$ one-form gauge symmetry and the $\mathcal{T}$-symmetry.

Mixed ’t Hooft Anomaly

- Including a local counter term, under the $\mathcal{T}$-transformation, e^{iS} is

$$e^{i\pi qQ}e^{-S_{\text{counter}}} \xrightarrow{\mathcal{T}} e^{-i\pi qQ}e^{+S_{\text{counter}}} = e^{-2i\pi qQ}e^{2S_{\text{counter}}}e^{i\pi qQ}e^{-S_{\text{counter}}}$$

$$= \exp\left[-\frac{2\pi i(4k+1)}{8q}\underbrace{\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}\Big]}_{\boxed{0,\pm 8,\pm 16,\cdots}} e^{i\pi qQ}e^{-S_{\text{counter}}}$$

- ➢ The anomaly is canceled for $4k+1=0 \mod q$.
- ➢ This is $\Big[$ impossible for $q \in 2\mathbb{Z}$.
  possible for $q \in 2\mathbb{Z}+1$.
- ➢ This implies the mixed ’t Hooft anomaly between the $\mathbb{Z}_q$ one-form gauge symmetry and the $\mathcal{T}$-symmetry for $q \in 2\mathbb{Z}$.

Conclusion and Future Work

- Conclusion

➢ We formulated the fractional topological charge on the lattice $U(1)$ gauge theory.

➢ Our construction provides the mixed 't Hooft anomaly between the $\mathbb{Z}_q$ one-form gauge symmetry and the $\mathcal{T}$-symmetry for $q \in 2\mathbb{Z}$.

- Future work

➢ The formulation of the fractional topological charge on the lattice $SU(N)$ gauge theory

➢ The formulation of the Witten effect on the lattice