Bond-weighting method for the Grassmann tensor renormalization group

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Based on SA, JHEP11(2022)030

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Advantages of the TRG approach

✔ Tensor renormalization group (TRG) is a typical tensor network method and the application to the 4D systems has recently made remarkable progress

Lagrangian (TRG) approach: Meurice+, Rev. Mod. Phys. 94(2022)025005
Kadoh, PoS(LATTICE2021)633
SA+, PoS(LATTICE2021)530

Hamiltonian (TNS) approach: Bañuls-Cichy, Rep. Prog. Phys. 83(2020)024401

✔ TRG is a deterministic numerical method based on the idea of the real-space renormalization group

- **No sign problem**
- **The computational cost scales logarithmically w. r. t. the system size**
- **Direct evaluation of the Grassmann integrals**
- **Direct evaluation of the path integral**

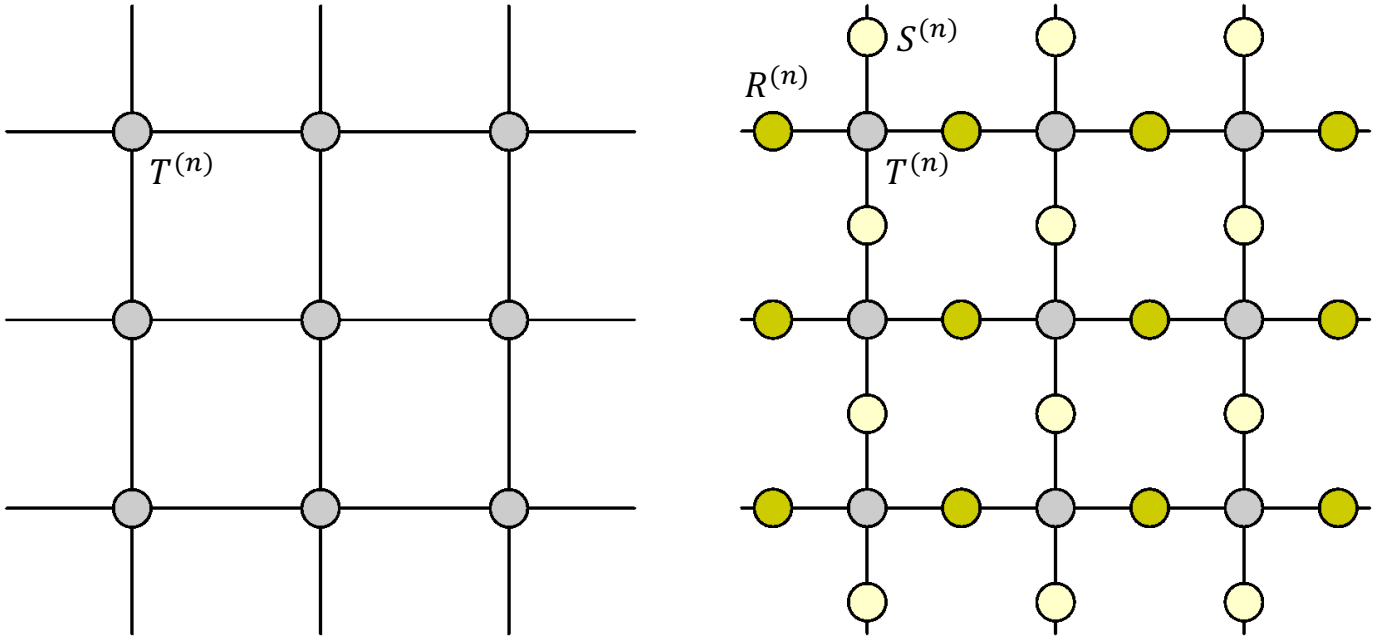
Bond-weighted TRG (BTRG)

Adachi+, PRB105(2022)L060402

✔ **The generalized (improved) TRG algorithm**

One considers the tensor network w/ **bond weights**.

Introducing the bond weights, one could improve the accuracy of the TRG w/o increasing the computational memory and execution time.



A good choice of the hyperparameter

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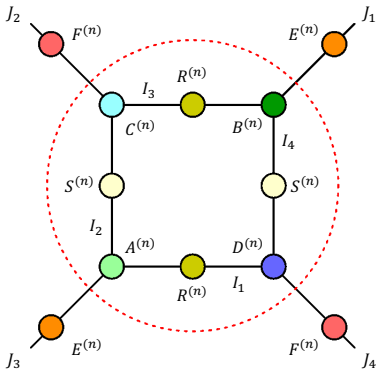
✔ **Introducing the hyperparameter $k \in \mathbb{R}$ in the SVD employed in the TRG**

$$T_{IJ}^{(n)} \approx \sum_{\alpha=1}^{D} U_{I\alpha}^{(n)} \left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\sigma_\alpha^{(n)}\right)^{k} \left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} V_{J\alpha}^{(n)*}$$

n labels the renormalization steps

With $k = 0$, the BTRG is reduced to be the Levin-Nave TRG.



✔ **A good choice of k?**

By the TRG renormalization, $T^{(n+1)} \sim \left[\left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\sigma_\alpha^{(n)}\right)^{k}\right]^4$

By the SVD of $T^{(n+1)}$, $T^{(n+1)} \sim \sigma_\alpha^{(n+1)}$

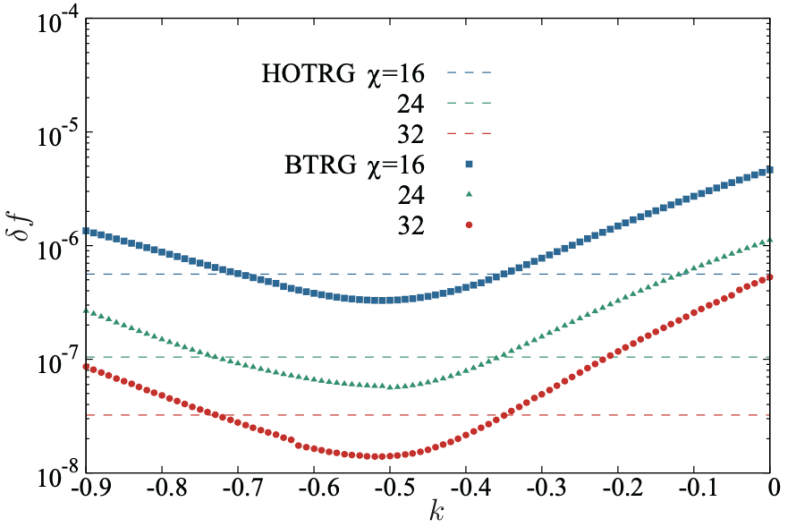
Suppose the singular-value spectrum becomes scale-invariant w/ sufficiently large n, we have

$$\left[\left(\sigma_\alpha^{(n)}\right)^{\frac{1-k}{2}} \left(\sigma_\alpha^{(n)}\right)^{k}\right]^4 = \sigma_\alpha^{(n)} \Rightarrow \boxed{k = -0.5}$$

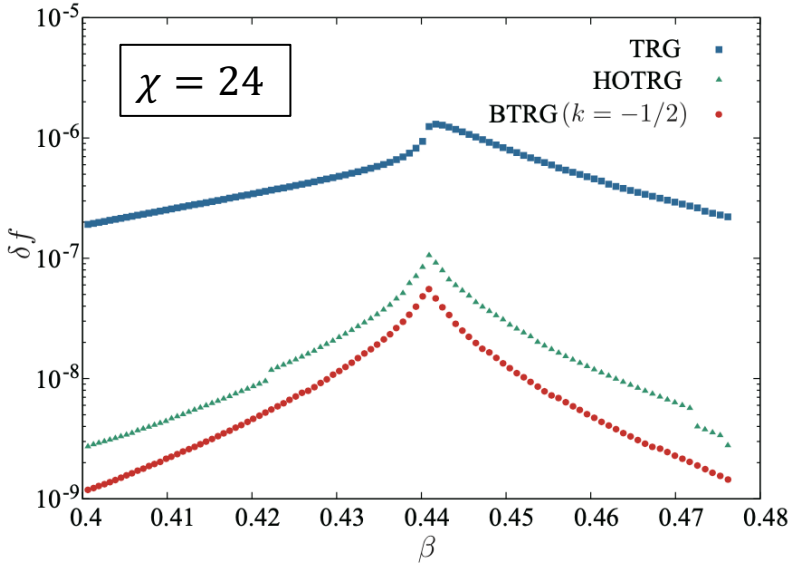
BTRG for the 2D classical Ising model

Adachi+, PRB105(2022)L060402

Relative error vs k



Comparison w/ other TRGs



$k = -0.5$ seems optimal and **the accuracy of the BTRG is higher than the Levin-Nave TRG and the HOTRG with the same bond dimension.**

Introduction of k does not increase the computational cost.
Therefore, the cost of the BTRG is same with the Levin-Nave TRG.

BTRG for the lattice fermion?

✔ $k = -0.5$ seems to be just determined by the geometry of the tensor network.

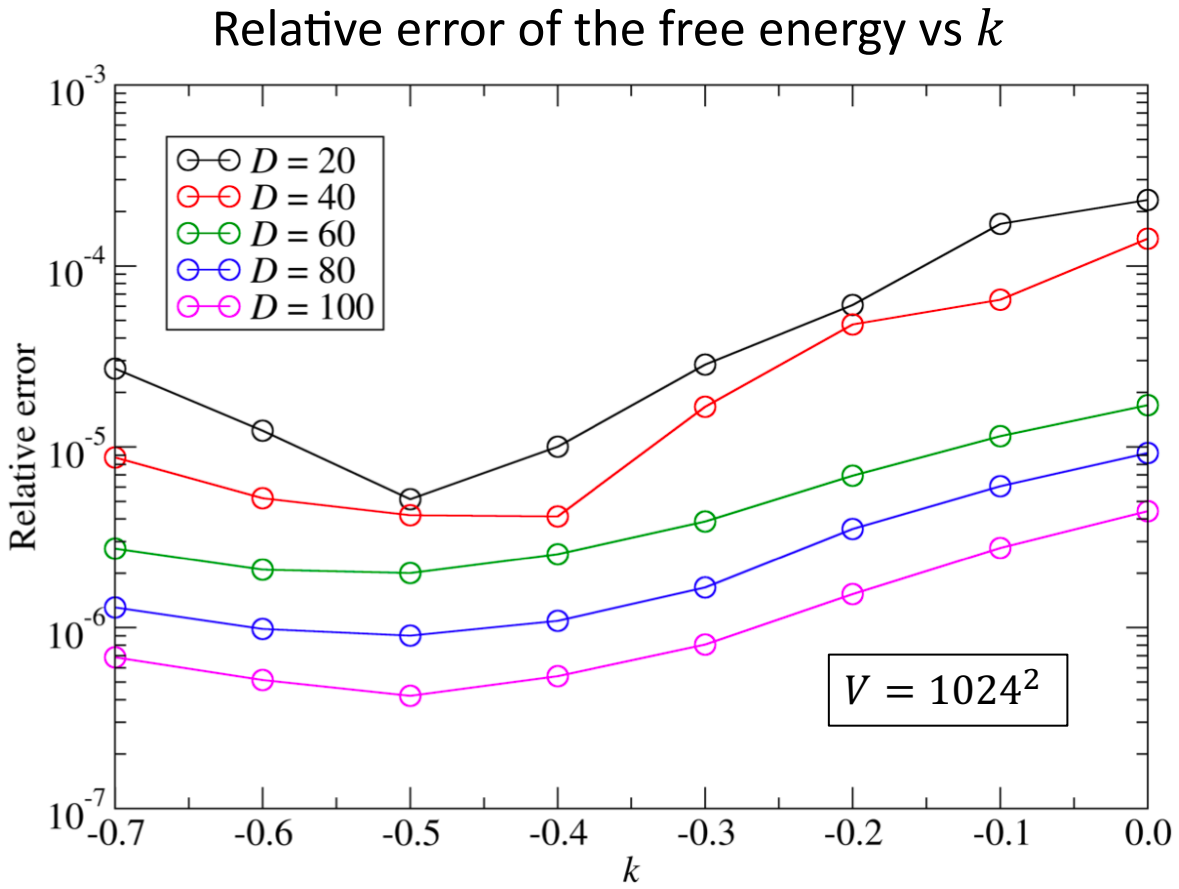
→ **the optimal choice of k does not depend on the details of the model?**

✔ Any TRG algorithm can be extended to evaluate the Grassmann path integral.

→ **BTRG is also efficient for the lattice fermions?**

✔ We develop the Grassmann BTRG algorithm employing the formulation of SA-Kadoh, JHEP10(2021)188, which allows us to extend the given TRG algorithm in a simple way to evaluate the Grassman path integral.

Benchmarking w/ the free massless Wilson fermion in 2D



Again, $k = -0.5$ seems optimal and the bond-weighting method does improve the Grassmann Levin-Nave TRG.

Finite-entanglement scaling

✔ In 1+1D, we have the finite-entanglement scaling based on the Matrix Product State (MPS), which is a tensor network method in the Hamiltonian formalism.

w.r.t. the bond dimension, the correlation length scales with $\xi_D \sim D^{\kappa}$, where $\kappa = \frac{6}{c\left(\sqrt{\frac{12}{c}}+1\right)}$. ($c$: the central charge)

Tagliacozzo+, PRB78(2008)024410,
Pollmann+, PRL102(2009)255701

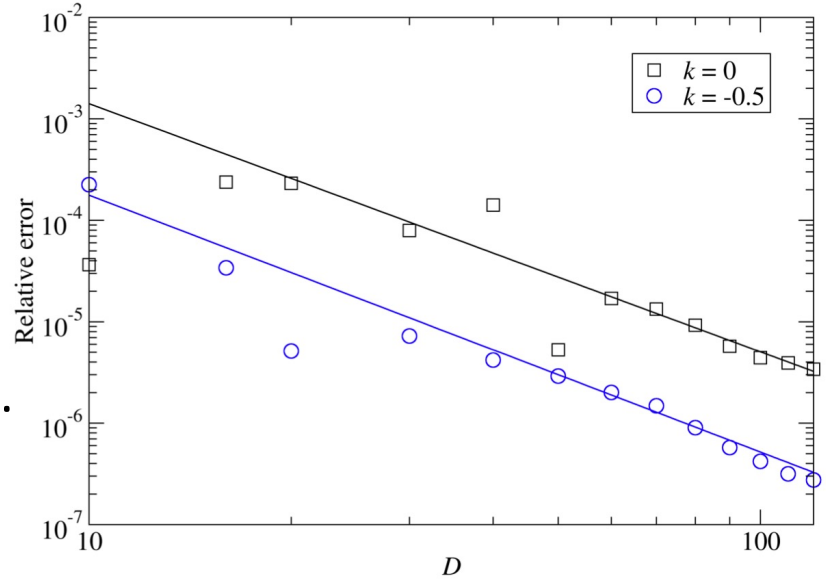
✔ Fitting the relative error of the free energy with $aD^{-2\kappa}$,

$k = -0.5$: $a = 0.06(4)$, $\kappa = 1.26(7)$

$k = 0$: $a = 0.4(3)$, $\kappa = 1.22(8)$

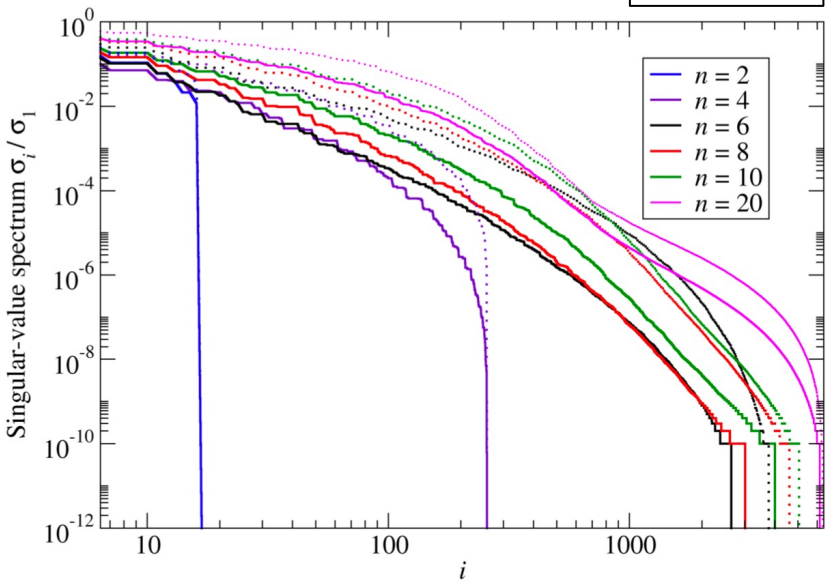
The Grassmann BTRG reproduces $\kappa = 1.344\cdots$, corresponding to $c = 1$.

Relative error of the free energy vs D

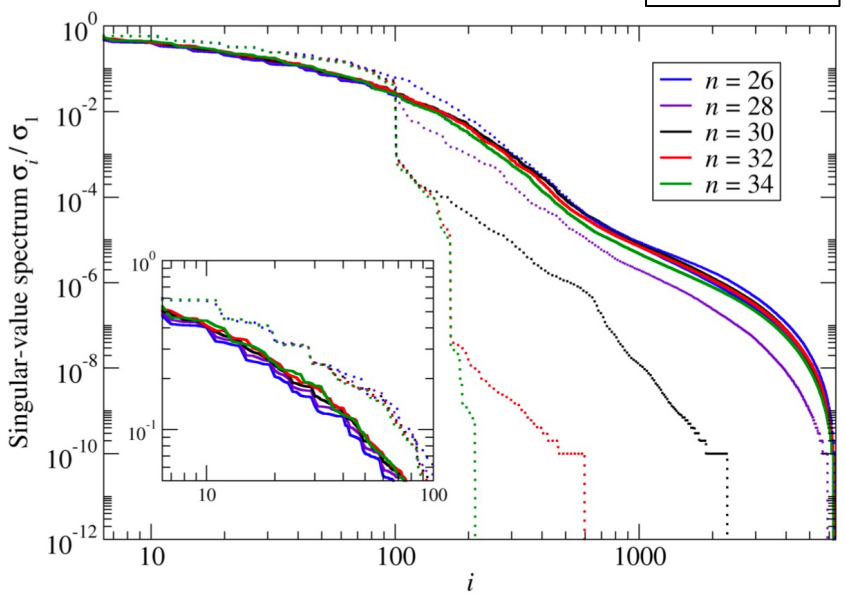


Hierarchy of the singular values

Smaller volume $D = 80$



Larger volume $D = 80$



n labels the renormalization steps or the lattice volume via $V = 2^n$

Grassmann BTRG (solid lines)
Grassmann Levin-Nave TRG (dotted lines)

In smaller volumes, the spectrum obtained by the Grassmann BTRG rapidly decays compared with the Grassmann Levin-Nave TRG.

The Levin-Nave algorithm does not reproduce the scale-invariant structure in the local Grassmann tensor, but the Grassmann BTRG does.

Summary

- Bond-weighting method is a new way to improve the TRG algorithm.

- The method was originally proposed for the spin system.
  We numerically confirmed that the bond-weighting method is also efficient for the lattice fermions.

- Benchmarking with the 2D free Wilson fermions, we have found that the accuracy of the TRG is highly improved. The scale-invariant structure of the local Grassmann tensor is also reproduced.

- **A sample code of the Grassmann BTRG is available on GitHub.**
  2D single-flavor Gross-Neveu-Wilson model at finite density is provided as an example.
  see https://github.com/akiyama-es/Grassmann-BTRG