Exact solution of the finite Grosse-Wulkenhaar model

Naoyuki Kanomata

Tokyo University of Science

December 8, 2022

Joint work with Prof. Akifumi Sako

Nucl. Phys. B 982 (2022) 1-25, arXiv:2205.15798 [hep-th]

KEK Theory Workshop 2022

Introduction 1

- Matrix models were well studied in the 1980s and 1990s in the context of two-dimensional quantum gravity theories.
- Each Feynman diagram in perturbative expansions of the matrix models represents a corresponding simplicial decomposition of a two-dimensional surface. In particular, Feynman diagrams of Φ^3 matrix models can be regarded as triangulations of two-dimensional surfaces. The sum over two-dimensional surfaces corresponds to path integrals of two-dimensional quantum gravity theories.
- Fukuma, Kawai, and Nakayama proved that the Virasoro constraint condition is equivalent to the condition that the solution of the KdV hierarchy satisfies the string equation. Witten showed that the Witten-Kontsevich τ-function satisfies the string equation. Furthermore, Witten conjectured that the Witten-Kontsevich τ-function is the τ function of the KdV hierarchy. Using the Φ^3 matrix model, the proof of this conjecture was done by Kontsevich.

Introduction 2

- Quantum field theories on noncommutative spaces such as Moyal spaces have given a new perspective to matrix models.

Matrix model on noncommutative spaces (Grosse-Wulkenhaar model)

- It corresponds to scalar field theories on noncommutative spaces, which is renormalizable by adding harmonic oscillator potentials to the action.
- Φ^3 matrix model [Grosse-Sako-Wulkenhaar ('17)]
- Φ^4 matrix model [Grosse-Hock-Wulkenhaar ('19)]
- Any n-point function of the Φ^3 matrix model was calculated by solving the Schwinger-Dyson equation exactly by using the Ward-Takahashi identity. The n-point functions of the Φ^3 matrix model in the large N, V limit were calculated in the previous studies by Grosse, Wulkenhaar, and Sako.

Introduction 3

- The difference between the Grosse-Wulkenhaar model (Φ^3 matrix model) and the Kontsevich model is the way the n-point functions are defined. Another difference is that the Grosse-Wulkenhaar model includes terms necessary for renormalization.

In order to mathematically formulate quantum field theories as a toy model, it is necessary to clarify the properties of the matrix model on noncommutative spaces (Grosse-Wulkenhaar model).

To gain insight into the Grosse-Wulkenhaar model, the following studies were conducted.

- We found the exact solutions of the Φ^3 finite matrix model (Grosse-Wulkenhaar model).

Grosse-Wulkenhaar model (Φ^3 matrix model)

Definition(Action of Φ^3 matrix model)

$$S[\Phi] = iV\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3\right)$$

- $\Phi = (\Phi_{ij}), \quad i,j = 1,\cdots,N$: Hermitian matrix
- $E = (E_{k-1}\delta_{km}), \quad k,m = 1,\cdots,N$: diagonal matrix
- $E_{k-1} = \mu^2\left(\frac{1}{2} + e\left(\frac{k-1}{\mu^2 V}\right)\right)$
- $\kappa \in \mathbb{R}$

Definition(Partition function of Φ^3 matrix model)

$$\mathcal{Z}[J] = \int \mathcal{D}\Phi \exp(-S[\Phi] + iV\mathrm{tr}(J\Phi))$$

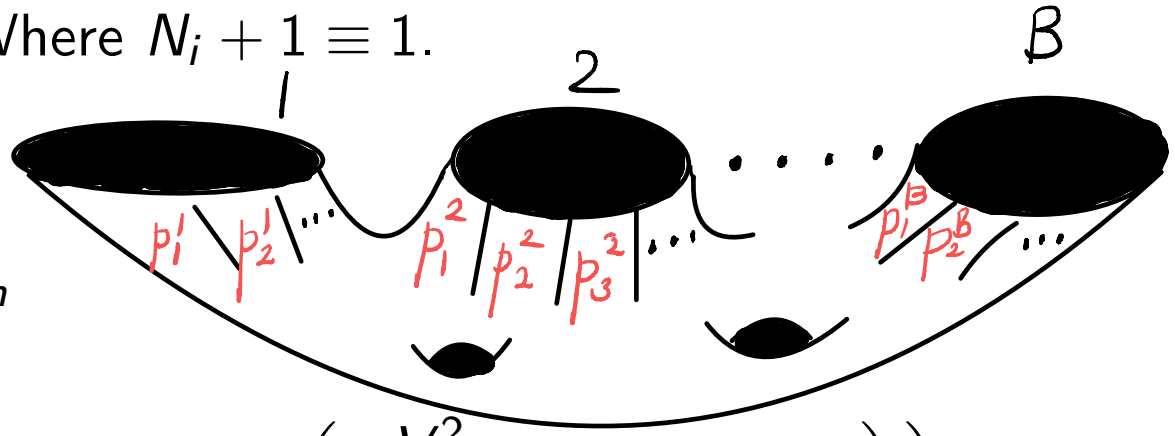
- $J = (J_{mn}), \quad m,n = 1,\cdots,N$: Hermitian matrix
- $\mathcal{D}\Phi = \prod_{i<j} d\Phi^I_{ij} \prod_{i\leq j} d\Phi^R_{ij}$: integral measure

The $n = \sum_{j=1}^{B} N_j$ point functions of the Grosse-Wulkenhaar model are defined as follows:

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$$

$$:= \sum_{B=1}^{\infty} \sum_{1 \le N_1 \le \cdots \le N_B}^{\infty} \sum_{p_1^1, \ldots, p_{N_B}^B = 1}^{N} (iV)^{2-B} \frac{G_{|p_1^1 \ldots p_{N_1}^1| \ldots |p_1^B \ldots p_{N_B}^B|}}{S_{(N_1, \ldots, N_B)}} \prod_{\beta=1}^{B} \frac{\mathbb{J}_{p_1^\beta \ldots p_{N_\beta}^\beta}}{N_\beta}$$

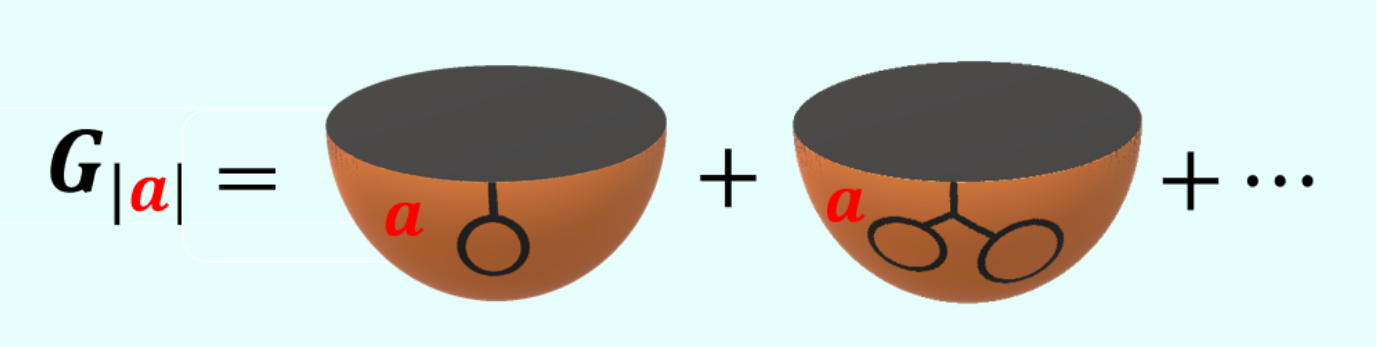
- $N_i \;\; i = 1, \ldots, B \;:\; N_i$-external lines from the i-th boundary
- $\mathbb{J}_{p_1 \ldots p_{N_i}} := \prod_{j=1}^{N_i} J_{p_j p_{j+1}}$ Where $N_i + 1 \equiv 1$.



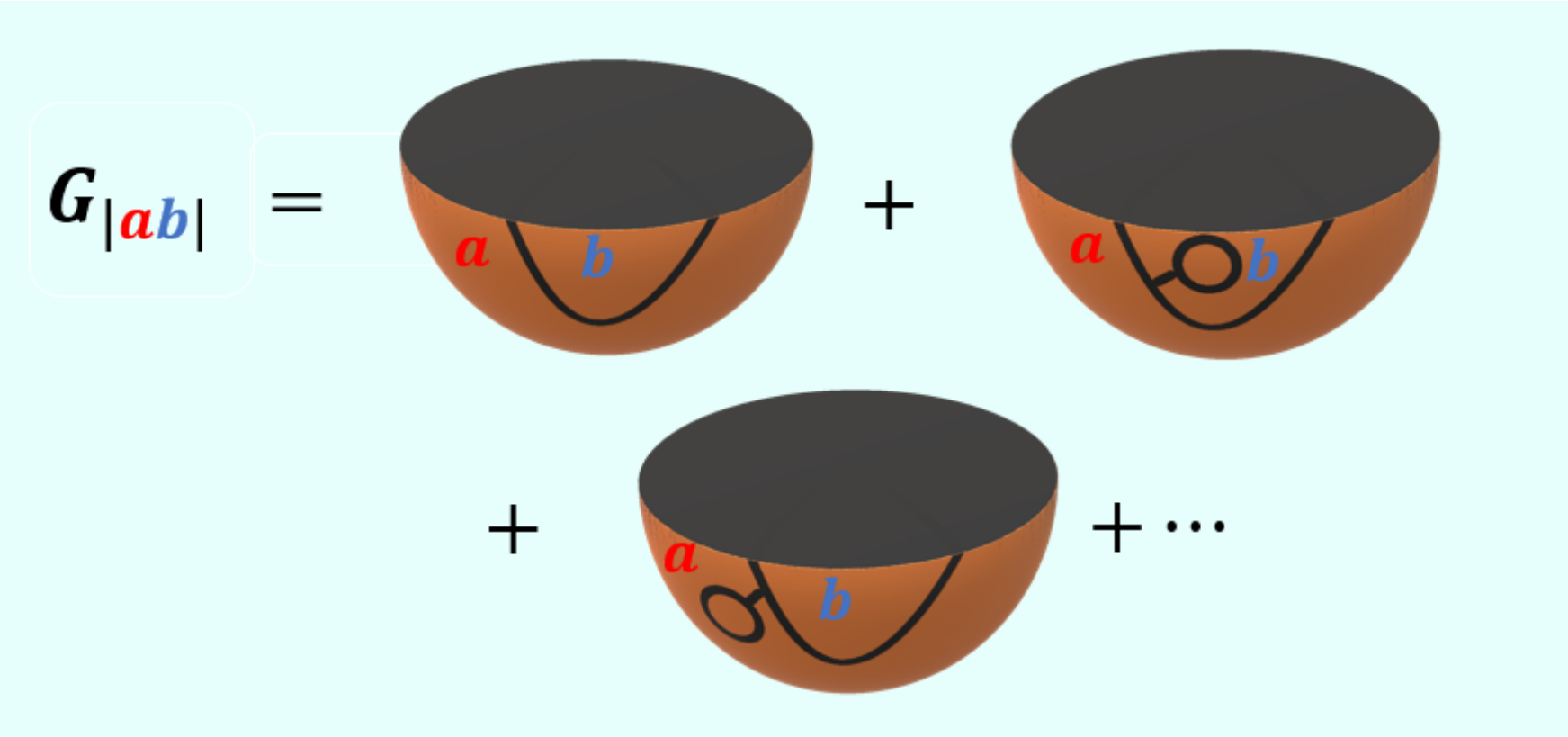
$$\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = 1 + iV \sum_{n=1}^{N} G_{|n|} J_{nn}$$

$$+ \sum_{n,m=1}^{N} \left(\frac{iV}{2} G_{|nm|} J_{nm} J_{mn} + \left(-\frac{V^2}{2} G_{|n|} G_{|m|} J_{nm} J_{mn} \right) \right) + \cdots$$

- Feynman diagrams of 1-point function $G_{|a|}$



- Feynman diagrams of 2-point functions $G_{|ab|}$



Calculation of partition function $\mathcal{Z}[J]$

$$\mathcal{Z}[J] := \int \mathcal{D}\Phi \exp\left(-iV\mathrm{tr}\left(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3\right)\right) \exp\left(iV\mathrm{tr}\left(J\Phi\right)\right)$$

$$\downarrow$$

$$\mathcal{Z}[J] = \exp\left(-i\lambda V\mathrm{tr}\left(\frac{2}{3}(\widetilde{E})^3 - \widetilde{\kappa}\widetilde{E} + \frac{1}{\lambda}J\widetilde{E}\right)\right)$$

$$\int \left(\prod_{i=1}^{N} dx_i \exp\left(-i\frac{\lambda V}{3}x_i^3\right)\right)\left(\prod_{1\leq k<l\leq N}(x_l - x_k)^2\right)$$

$$\int dU \exp\left(i\lambda V\widetilde{\kappa}\mathrm{tr}\{(M - I + K)U\widetilde{X}U^*\}\right)$$

The integration is divided into diagonal elements and off-diagonal elements.

- $\mathcal{D}X = \prod_{i=1}^{N} dx_i \prod_{1\leq k<l\leq N}(\mathrm{x}_l - \mathrm{x}_k)^2 dU$ [3]

Itzykson-Zuber integral [4]

$$\int_{U(N)} \exp\left(t\,\mathrm{tr}\left(AUBU^*\right)\right) dU = c_N \frac{\det\limits_{1\le i,j\le N}\left(\exp\left(t\lambda_i(A)\lambda_j(B)\right)\right)}{t^{\frac{N^2-N}{2}}\Delta(\lambda(A))\Delta(\lambda(B))}$$

- A, B : Hermitian matrix
- $\lambda_i(A), \lambda_i(B)$ $i = 1, \cdots, N$: Eigenvelues of A, B
- $dU = \prod\limits_{1\le i<j\le N} (U^* dU)^R_{ij} (U^* dU)^I_{ij}$
- $t \in \mathbb{C}/\{0\}$
- $\Delta(\lambda(A)) := \prod\limits_{1\le i<j\le N} (\lambda_j(A) - \lambda_i(A))$: Vandermonde determinant
- $c_N := \prod\limits_{i=1}^{N-1} i! \times \pi^{\frac{N(N-1)}{2}}$: normalization constant

The off-diagonal elements are integrated using the Itzykson-Zuber integral.

$$\int dU \exp\left(i\lambda V\widetilde{\kappa}\mathrm{tr}\{(M-I+K)U\widetilde{X}U^*\}\right) = \frac{C}{N!}\frac{\det\limits_{1\le i,j\le N}\exp\left(i\lambda V\widetilde{\kappa}x_i s_j\right)}{\prod\limits_{i<j}(x_j-x_i)\prod\limits_{i<j}(s_j-s_i)}$$

The diagonal elements are integrated using the Airy function.

$$\mathcal{Z}[J] = C'\frac{e^{\frac{-iV}{\lambda}\mathrm{tr}(JE)}A_N(y_1,\cdots,y_N)}{\prod\limits_{1\le t<u\le N}(s_u-s_t)}$$

- s_t : Eigenvelues of $M-I+K$
- $y_j = -\dfrac{V\kappa s_j}{(\lambda V)^{\frac{1}{3}}}$
- $Ai(y)$: Airy function
- $A_N(y_1,\cdots,y_N) = \left(\prod\limits_{1\le i<j\le N}(\partial_{y_i}-\partial_{y_j})\right)Ai(y_1)\cdots Ai(y_N)$

Calculation of 1-Point Function $G_{|a|}$

In the following, J is treated as a diagonal matrix. In the calculation of the 1-point function $G_{|a|}$,

$$
\begin{aligned}
G_{|a|} &= \frac{1}{iV}\frac{\partial \log \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} \\
&= \frac{1}{iV}\frac{1}{\mathcal{Z}[0]}\frac{\partial \mathcal{Z}[J]}{\partial J_{aa}}\bigg|_{J=0} \\
&= \cdots \\
&= -\frac{E_{a-1}}{\lambda} - \frac{\lambda}{iV}\sum_{i=1, i\neq a}^{N}\frac{1}{E_{a-1}^2 - E_{i-1}^2} - \frac{1}{i(\lambda V)^{\frac{1}{3}}}\partial_a \log A_N(z_1, \cdots, z_N)
\end{aligned}
$$

- $z_j = -\dfrac{VE_{j-1}^2}{(\lambda V)^{\frac{1}{3}}\lambda} + \dfrac{V\kappa}{(\lambda V)^{\frac{1}{3}}},\ j = 1, \cdots, N$
- $\partial_a = \dfrac{\partial}{\partial z_a}$

The $n = \sum_{j=1}^{B} N_j$ point functions from the Schwinger-Dyson equation[1][2] are expressed as follows:

$$
\begin{aligned}
&G_{|a_1^1 \dots a_{N_1}^1| \dots |a_1^B \dots a_{N_B}^B|} \\
&= \lambda^{N_1 + \dots + N_B - B} \sum_{k_1=1}^{N_1} \cdots \sum_{k_B=1}^{N_B} G_{|a_{k_1}^1| \dots |a_{k_B}^B|} \\
&\left(\prod_{l_1=1, l_1 \neq k_1}^{N_1} P_{a_{k_1}^1 a_{l_1}^1} \right) \cdots \left(\prod_{l_B=1, l_B \neq k_B}^{N_B} P_{a_{k_B}^B a_{l_B}^B} \right)
\end{aligned}
$$

- $P_{ab} := \dfrac{1}{E_{a-1}^2 - E_{b-1}^2}$

Calculation of n-Point Functions $G_{|a^1|a^2|\cdots|a^n|}$

Theorem n-Point Functions $G_{|a^1|a^2|\cdots|a^n|}$

$$
\begin{aligned}
&G_{|a^1|a^2|\cdots|a^n|} \\
=&(iV)^{n-2}\sum_{\pi}\left\{\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\right\}\prod_{B\in\pi}\frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\bigg|_{J=0}=\cdots \\
=&(iV)^{n-2}\mathcal{C}\sum_{\pi}\left\{\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\bigg|_{x=\mathcal{Z}[0]}\right\}\prod_{B\in\pi}\sum_{S\subset B}\sum_{M\subset\overline{S}}F(S,M,\overline{M})
\end{aligned}
$$

$$
\begin{aligned}
F(S,M,\overline{M}) := &\left(\prod_{i\in S}\left(-iV\frac{E_{a^i}-1}{\lambda}\right)\right)\left(\left\{\prod_{k\in M}\left(-\frac{V}{(\lambda V)^{\frac{1}{3}}}\right)\partial_{a^k}\right\}A_N(z_1,\ldots,z_N)\right) \\
&\left(\left\{\prod_{q\in\overline{M}}\frac{\partial}{\partial t_{a^q}}\right\}\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}\right)
\end{aligned}
$$

- $\sum_{\pi}$ means the sum over all partitions π of the set $\{1, \ldots, n\}$.
- $B = S \sqcup \overline{S}$
- $\overline{S} = M \sqcup \overline{M}$

ex)

① $\{1, 2\}$

② Find Partitions

$\pi_1 = \{\{1,2\}\}$, $\pi_2 = \{\{1\},\{2\}\}$

③ $B = \{1,2\}$

$S = \{1,2\}$, $M = \emptyset$, $\overline{M} = \emptyset$

③' $B = \{1\}$

$S = \{1\}$, $M = \emptyset$, $\overline{M} = \emptyset$

- $\prod_{B \in \pi}$ is the product over all of the parts B of the partition π.
- $\sum_{S \subset B}$ means the sum over all subsets of B.
- $\sum_{M \subset \overline{S}}$ means the sum over all subsets of $\overline{S} = B \backslash S$.

$$F(S, M, \overline{M}) := \left(\prod_{i \in S} \left(-iV \frac{E_{a^i - 1}}{\lambda} \right) \right) \left(\left\{ \prod_{k \in M} \left(-\frac{V}{(\lambda V)^{\frac{1}{3}}} \right) \partial_{a^k} \right\} A_N(z_1, \ldots, z_N) \right)$$
$$\left(\left\{ \prod_{q \in \overline{M}} \frac{\partial}{\partial t_{a^q}} \right\} \frac{1}{\det\limits_{1 \leq l, j \leq N} \left(t_l^{j-1} \right)} \right)$$

Calculation of 2-Point Functions $G_{|a^1|a^2|}$

The 2-point functions $G_{|a^1|a^2|}$ can be calculated by computing the 2-patterns of $\pi = \{\{1,2\}\}$,$\pi = \{\{1\},\{2\}\}$.

First, we calculate the case of $\pi = \{\{1,2\}\}$.

- $|\pi| = 1$
- $B = \{1,2\}$

$$\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\Bigg|_{x=\mathcal{Z}[0]} \prod_{B\in\pi} \frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\Bigg|_{J=0} = \frac{1}{\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{a^1 a^1}\partial J_{a^2 a^2}}\Bigg|_{J=0}$$

$$F(\{1,2\},\emptyset,\emptyset) = \left(-iV\frac{E_{a^1-1}}{\lambda}\right)\left(-iV\frac{E_{a^2-1}}{\lambda}\right)A_N(z_1,\ldots,z_N)\frac{1}{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}$$

$$\vdots$$

From this, the calculation result for $\pi = \{\{1,2\}\}$ is as follows:

$$
\begin{aligned}
&\frac{1}{\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{a^1 a^1} \partial J_{a^2 a^2}} \bigg|_{J=0} \\
&= \left(\frac{\det\limits_{1 \le l, j \le N} \left(t_l^{j-1} \right)}{A_N(z_1, \ldots, z_N)} \right) \Bigg\{ F(\{1,2\}, \emptyset, \emptyset) + F(\emptyset, \{1,2\}, \emptyset) + F(\emptyset, \emptyset, \{1,2\}) \\
&\quad + \sum_{l,n=1, l \neq n}^{2} \Big(F(\{l\}, \{n\}, \emptyset) + F(\{l\}, \emptyset, \{n\}) + F(\emptyset, \{l\}, \{n\}) \Big) \Bigg\}
\end{aligned}
$$

Next we calculate the case of $\pi = \{\{1\},\{2\}\}$.

- $|\pi| = 2$
- $B = \{1\}, or \{2\}$

$$\left(\frac{d}{dx}\right)^{|\pi|}(\log x)\Bigg|_{x=\mathcal{Z}[0]} \prod_{B\in\pi} \frac{\partial^{|B|}\mathcal{Z}[J]}{\prod_{j\in B}\partial J_{a^j a^j}}\Bigg|_{J=0} = -\frac{1}{\mathcal{Z}[0]^2}\frac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\Bigg|_{J=0}\frac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\Bigg|_{J=0}$$

From this, the calculation result for $\pi = \{\{1\},\{2\}\}$ is as follows:

$$\begin{aligned}
&-\frac{1}{\mathcal{Z}[0]^2}\frac{\partial\mathcal{Z}[J]}{\partial J_{a^1a^1}}\Bigg|_{J=0}\frac{\partial\mathcal{Z}[J]}{\partial J_{a^2a^2}}\Bigg|_{J=0} \\
&= -\left(\frac{\det\limits_{1\le l,j\le N}\left(t_l^{j-1}\right)}{A_N(z_1,\ldots,z_N)}\right)^2 \prod_{l=1}^{2}\Big(F(\{l\},\emptyset,\emptyset)+F(\emptyset,\{l\},\emptyset)+F(\emptyset,\emptyset,\{l\})\Big)
\end{aligned}$$

Summarizing the results of the calculations, we obtain

$$
\begin{aligned}
G_{|a^1|a^2|} &= \frac{1}{\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{a^1 a^1} \partial J_{a^2 a^2}}\bigg|_{J=0} - \frac{1}{\mathcal{Z}[0]^2} \frac{\partial \mathcal{Z}[J]}{\partial J_{a^1 a^1}}\bigg|_{J=0} \frac{\partial \mathcal{Z}[J]}{\partial J_{a^2 a^2}}\bigg|_{J=0} = \cdots \\
&= \left(\frac{\det\limits_{1 \le l, j \le N} \left(t_l^{j-1} \right)}{A_N(z_1, \ldots, z_N)} \right) \\
&\quad \Bigg\{ F(\{1,2\}, \emptyset, \emptyset) + F(\emptyset, \{1,2\}, \emptyset) + F(\emptyset, \emptyset, \{1,2\}) \\
&\quad + \sum_{l,n=1, l \neq n}^{2} \Big(F(\{l\}, \{n\}, \emptyset) + F(\{l\}, \emptyset, \{n\}) + F(\emptyset, \{l\}, \{n\}) \Big) \Bigg\} \\
&\quad - \left(\frac{\det\limits_{1 \le l, j \le N} \left(t_l^{j-1} \right)}{A_N(z_1, \ldots, z_N)} \right)^2 \\
&\quad \prod_{l=1}^{2} \Big(F(\{l\}, \emptyset, \emptyset) + F(\emptyset, \{l\}, \emptyset) + F(\emptyset, \emptyset, \{l\}) \Big)
\end{aligned}
$$

Overall Summary and Future Prospect

The Φ^3 matrix model (Grosse-Wulkenhaar model) was constructed based on the scalar Φ^3 theory on noncommutative spaces. Therefore, we can mathematically formulate the scalar Φ^3 theory on noncommutative spaces by clarifying the properties of the Φ^3 matrix model (Grosse-Wulkenhaar model).

- In this study, we obtained the exact solutions of the n-point functions of the finite Grosse-Wulkenhaar model (Φ^3 matrix model).
- It is known that any $G_{|a_1^1 \ldots a_{N_1}^1| \ldots |a_1^B \ldots a_{N_B}^B|}$ can be expressed using $G_{|a^1| \ldots |a^n|}$ type n-point functions. Thus we focus on rigorous calculations of $G_{|a^1| \ldots |a^n|}$. The formula for $G_{|a^1| \ldots |a^n|}$ is obtained, and it is achieved by using the partition function $\mathcal{Z}[J]$ calculated by the Harish-Chandra-Itzykson-Zuber integral.
- In the future, we would like to clarify the properties of the Φ^4 matrix model (Grosse-Wulkenhaar model) in order to mathematically formulate the scalar Φ^4 theory on noncommutative spaces.

References

[1] H. Grosse, A. Sako and R. Wulkenhaar, Exact solution of matricial Φ^3_2 quantum field theory, Nucl. Phys. B **925**, 319-347 (2017) doi:10.1016/j.nuclphysb.2017.10.010 [arXiv:1610.00526 [math-ph]].

[2] A. Hock, Matrix field theory, Ph.D. Thesis, WWU Münster, 2020, arXiv:2005.07525.

[3] T. Kimura, Mathematical Physics of Random Matrices, Morikita Press, 2021, ISBN:9784627063013.

[4] T. Tao, http://terrytao.wordpress.com/2013/02/08/the-harish-chandra-itzykson-zuber-integral-formula/.

[5] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. **147**, 1-23 (1992) doi:10.1007/BF02099526