Reality from maximizing overlap in the periodic complex action theory

Keiichi Nagao

Ibaraki Univ.

Dec. 8, 2022 @ KEK

Based on the work with Holger Bech Nielsen
Prog. Theor. Exp. Phys. **2022** (9) 091B01(arXiv:2203.07795 [quant-ph])

Introduction

H.B.Nielsen, M.Ninomiya, Proc. Bled 2006, p.87

Complex action theory (CAT) $\Leftarrow$ Extension of quantum theory

- coupling parameters are complex
- corresponding Hamiltonian is non-normal.
  * PT symmetric Hamiltonian : Bender and Boettcher etc.

Less constraints, more fundamental.

It would be nice if we could remove a constraint of an action being real. $\Leftarrow$ One of the main two benefits

Expected to give falsifiable predictions
$\Rightarrow$ Intensively studied by H. B. Nielsen and M. Ninomiya

Properties of the future-included theory

H.B.Nielsen, M.Ninomiya, Proc. Bled 2006, p.87

We consider a "future-included" theory, in which not only a past state $|A(T_A)\rangle$ at the initial time T_A, but also a future state $|B(T_B)\rangle$ at the final time T_B is given:

$$i\hbar \frac{d}{dt}|A(t)\rangle = \hat{H}|A(t)\rangle, \quad -i\hbar \frac{d}{dt}\langle B(t)| = \langle B(t)|\hat{H},$$

where $\hat{H}$ is non-normal.

The normalized matrix element $\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{O}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$ is expected to work as an "expectation value" of $\hat{O}$.

* $\langle \hat{O} \rangle^{BA}$ is called the weak value in the real action theory (RAT).

Y. Aharonov, D. Z. Albert, L. Vaidman, Phys. Rev. Lett. 60 (1988) pp.1351-1354

KN, H.B.Nielsen, PTEP **2013** 023B04

If we regard $\langle\hat{O}\rangle^{BA}$ as an expectation value in the future-included theory, then, utilizing $\frac{d}{dt}\langle\mathcal{O}\rangle^{BA} = \langle\frac{i}{\hbar}[\hat{H},\mathcal{O}]\rangle^{BA}$, we obtain

- Heisenberg equation
- Ehrenfest's theorem:

$$\frac{d}{dt}\langle\hat{q}_{new}\rangle^{BA} = \frac{1}{m}\langle\hat{p}_{new}\rangle^{BA},$$
$$\frac{d}{dt}\langle\hat{p}_{new}\rangle^{BA} = -\langle V'(\hat{q}_{new})\rangle^{BA}.$$

* momentum relation $p = m\dot{q}$

KN, H.B.Nielsen, IJMP **A27**(2012) 1250076;Erratum-ibid, **A32**(2017) 1792003

* complex coordinate and momentum formalism

KN, H.B.Nielsen, PTP126 (2011)102

- Conserved probability current density

Thus $\langle\hat{O}\rangle^{BA}$ seems to play the role of an expectation value in the future-included theory. However, $\langle\hat{O}\rangle^{BA}$ is complex in general.

Difference of the philosophies

$$\langle \hat{O} \rangle^{BA} \equiv \frac{\langle B(t)|\hat{O}|A(t)\rangle}{\langle B(t)|A(t)\rangle}$$

	Theory of Aharonov et.al.	Our theory
mainly interested in	experiments in laboratories	whole universe
look at the path s.t. $\vert\langle B(t)\vert A(t)\rangle\vert$ is	small	large
because of	amplification of detection	less conditions, natural initial states

Our philosophy:
our universe could be realized by a path (including initial and final conditions) selected from a superposition of many possible paths of our universe that are given randomly.

Four types of quantum theory

Quantum theory can be classified into four types, according to whether its action is real or not, and whether the future is included or not.

Table: Four types of quantum theory.

	Real action	Complex action
Future not included	FNI RAT	FNI CAT
Future included	FI RAT	FI CAT

* Complex action suggests future-included theory

KN and H.B.Nielsen, PTEP **2017** 111B01

Modified inner product for $\hat{H}$

KN and H.B.Nielsen, PTP **125**(2011)633

$$\hat{H}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$$

$|\lambda_i\rangle$: eigenstates of $\hat{H}$, but not orthogonal in the usual inner product
$I(|\lambda_i\rangle, |\lambda_j\rangle) \equiv \langle\lambda_i|\lambda_j\rangle \neq \delta_{ij}$.
$\lambda_i (i = 1, \dots)$: complex

$$\hat{H} = PDP^{-1},$$

where $P = (|\lambda_1\rangle, |\lambda_2\rangle, \dots)$, $D =$ diag$(\lambda_1, \lambda_2, \dots)$

Let us considr a transition from $|\lambda_i\rangle$ to $|\lambda_j\rangle$ $(i \neq j)$ fast in time Δt

$$|I(|\lambda_j\rangle, \exp\left(-\frac{i}{\hbar}\hat{H}\Delta t\right)|\lambda_i\rangle)|^2 \neq 0,$$

since $\langle\lambda_i|\lambda_j\rangle \neq 0$, even though $\hat{H}$ cannot bring the system from $|\lambda_i\rangle$ to $|\lambda_j\rangle$ $(i \neq j)$.
$\Rightarrow$ Such a transition should be prohibited.

We define $I_Q(|f\rangle, |i\rangle) \equiv \langle f|_Q\, i\rangle \equiv \langle f|Q|i\rangle$ s.t. $I_Q(|\lambda_i\rangle, |\lambda_j\rangle) = \delta_{ij}$, and impose $\langle\psi_1(t)|_Q\psi_2(t)\rangle = \langle\psi_2(t)|_Q\psi_1(t)\rangle^* \to Q^\dagger = Q$

Also, we define $\dagger_Q$ for

- any operator A: $\langle\psi_2|_Q A|\psi_1\rangle^* = \langle\psi_1|_Q A^{\dagger_Q}|\psi_2\rangle \to A^{\dagger_Q} = Q^{-1}A^\dagger Q$
- kets and bras: $|\psi_1\rangle^{\dagger_Q} \equiv \langle\psi_1|_Q$, $(\langle\psi_2|_Q)^{\dagger_Q} \equiv |\psi_2\rangle$

When A satisfies $A^{\dagger_Q} = A$, we call A Q-Hermitian.
We choose Q as $Q = (P^\dagger)^{-1}P^{-1}$.

* $\hat{H}$ is Q-normal: $[\hat{H}, \hat{H}^{\dagger_Q}] = [\hat{H}_{Qh}, \hat{H}_{Qa}] = 0$

$$\hat{H} = \hat{H}_{Qh} + \hat{H}_{Qa}, \quad \hat{H}_{Qh} \equiv \frac{\hat{H} + \hat{H}^{\dagger_Q}}{2}, \quad \hat{H}_{Qa} \equiv \frac{\hat{H} - \hat{H}^{\dagger_Q}}{2}.$$

* A similar inner product is studied also in F. G. Scholtz, H. B. Geyer and F. J. W. Hahne, Ann. Phys. 213 (1992) 74,
A. Mostafazadeh, J.Math.Phys.43, 3944 (2002).

From now on, let us adopt the modified inner product I_Q for all quantities. So, we slightly change the final state $\langle B(T_B)|$ as $\langle B(T_B)| \to \langle B(T_B)|_Q$.

$$-i\hbar\frac{d}{dt}\langle B(t)|_Q = \langle B(t)|_Q\hat{H} \quad \Leftrightarrow \quad i\hbar\frac{d}{dt}|B(t)\rangle = \hat{H}^{\dagger^Q}|B(t)\rangle.$$

Then $\langle\hat{O}\rangle^{BA}$ is replaced with

$$\langle\hat{O}\rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q\hat{O}|A(t)\rangle}{\langle B(t)|_Q A(t)\rangle.}$$

We investigate $\langle\hat{O}\rangle_Q^{BA}$ with the Q-normalized $|A(T_A)\rangle$ and $\langle B(T_B)|$: $\langle A(T_A)|_Q A(T_A)\rangle = 1$, $\langle B(T_B)|_Q B(T_B)\rangle = 1$.

Theorem on the normalized matrix element $\langle\hat{O}\rangle_Q^{BA}$

KN, H.B.Nielsen, PTEP **2015** 051B01; PTEP **2017** 081B01

Theorem 1 *Assume that* $\hat{H}$ *is non-normal but diagonalizable and that the imaginary part of its eigenvalues are bounded from above, and define a modified inner product* I_Q*. Let* $|A(t)\rangle$ *and* $|B(t)\rangle$ *time-develop according to* $|A(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}(t-T_A)}|A(T_A)\rangle$, $|B(t)\rangle = e^{-\frac{i}{\hbar}\hat{H}^{\dagger^Q}(t-T_B)}|B(T_B)\rangle$*, and be normalized by* $\langle A(T_A)|_Q A(T_A)\rangle = 1$, $\langle B(T_B)|_Q B(T_B)\rangle = 1$.
Next determine $|A(T_A)\rangle$ *and* $|B(T_B)\rangle$ *so as to maximize* $|\langle B(t)|_Q A(t)\rangle|$.
Then, provided that $\hat{O}^{\dagger^Q} = \hat{O}$, $\langle\hat{O}\rangle_Q^{BA} \equiv \frac{\langle B(t)|_Q\hat{O}|A(t)\rangle}{\langle B(t)|_Q A(t)\rangle}$ *becomes* real *and time-develops under a* Q-Hermitian *Hamiltonian.*

We call this way of thinking the maximization principle.

* $\mathrm{Im}\lambda_i$ are bounded from above to avoid the Feynman path integral $\int e^{\frac{i}{\hbar}S}\mathcal{D}$path being divergently meaningless.

$\Rightarrow$ Some $\mathrm{Im}\lambda_i$ take the maximal value B, and we denote the corresponding subset of $\{i\}$ as A.

Among the four types of quantum theory, only in the future-included CAT, initial (and final) conditions are determined in the Feynman path integral. $\Leftarrow$ The other benefit of the CAT.

Periodic complex action theory

KN, H.B.Nielsen, PTEP **2022** 091B01

In the future-included CAT, taking $T_B = T_A + t_p$, we impose the following periodic condition on $|A(T_A)\rangle$ and $|B(T_B)\rangle$:

$$|B(T_B)\rangle = |A(T_A)\rangle. \tag{1}$$

$\Rightarrow$ The remaining quantity to be adjusted becomes the period t_p.

It is very interesting that even the period t_p is regarded as a parameter that is adjusted via the maximization principle.

* Another periodic condition: $|A(T_A)\rangle = |A(T_A + t_p)\rangle = e^{-\frac{i}{\hbar}\hat{H}t_p}|A(T_A)\rangle$
$\Rightarrow$ $e^{-\frac{i}{\hbar}\hat{H}t_p}$ has to have $|A(T_A)\rangle$ as the eigenstate for its eigenvalue 1, so we do not adopt this condition.

Then, $\langle\hat{O}\rangle_Q^{BA}$ is written as

$$\begin{aligned}
\langle\hat{O}\rangle_Q^{BA} &= \frac{\langle A(T_A)|_Q e^{\frac{i}{\hbar}\hat{H}(t-T_B)}\hat{O}e^{-\frac{i}{\hbar}\hat{H}(t-T_A)}|A(T_A)\rangle}{\langle A(T_A)|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}|A(T_A)\rangle} \equiv \langle\hat{O}\rangle_Q^{A} \\
&\simeq \frac{\sum_n \langle\hat{O}\rangle_Q^{A_n}\ \langle A_n|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}|A_n\rangle}{\sum_n \langle A_n|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}|A_n\rangle} = \frac{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\hat{O}\right)}{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)},
\end{aligned}$$

where we have taken a basis $|A_n\rangle$ s.t. the state $|A(T_A)\rangle$ maximizing $|\langle A(T_A)|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}|A(T_A)\rangle|$ is included.

Weighting various $\langle\hat{O}\rangle_Q^{A_n}$ by $\langle A_n|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}|A_n\rangle$ replaces maximizing $|\langle A(T_A)|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}|A(T_A)\rangle|$ crudely in a quantitative way.

In the periodic CAT we propose our "expectation value" for $\hat{O}$ by

$$\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\hat{O}\right)}{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)}. \tag{2}$$

This quantity is

- generically complex
- independent of the time $t \Rightarrow$ the situation is like that in general relativity and quantum cosmology

Since whether $\langle \hat{O} \rangle_{\text{periodic time}}$ could be real is most important, we focus on this question.

Seeking a condition for $\langle \hat{O} \rangle_{\text{periodic time}}$ to be real provided that $\hat{O}$ is $\mathcal{Q}$-Hemitian, we propose two theorems:

Theorem 3. *As a prerequisite, assume that a given Hamiltonian* $\hat{H}$ *is non-normal but diagonalizable, that the maximal value* B *of the imaginary parts of the eigenvalues of* $\hat{H}$ *is equal to or smaller than zero, and that* $|B|$ *is much smaller than the distances between any two real parts of the eigenvalues of* $\hat{H}$*, and define a modified inner product* I_Q*. Then, provided that an operator* $\hat{O}$ *is* Q*-Hermitian,* $\hat{O}^{\dagger^Q} = \hat{O}$*,* $\langle\hat{O}\rangle_{\text{periodic time}} \equiv \frac{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\hat{O}\right)}{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)}$ *becomes real for selected periods* t_p *such that* $|\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)|$ *is maximized.*

Outline of proof of Theorem 3

The numerator of Eq.(2) is expressed as

$$\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\hat{O}\right) = \sum_n \langle\lambda_n|_Q e^{-\frac{i}{\hbar}\hat{H}t_p}\hat{O}|\lambda_n\rangle \simeq e^{\frac{B}{\hbar}t_p}\sum_{n\in A}\langle\lambda_n|_Q\hat{O}|\lambda_n\rangle e^{-i\theta_n}.$$

- $|\lambda_n\rangle$ is an eigenstate of $\hat{H}$:
  $\hat{H}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle$, $\langle\lambda_n|_Q\lambda_m\rangle = \delta_{nm}$, $\sum_m |\lambda_m\rangle\langle\lambda_m|_Q = 1$.
- introduced $\theta_n \equiv \frac{1}{\hbar}\mathrm{Re}\lambda_n t_p$
- supposed that t_p is sufficiently large
  $\Rightarrow$ the terms coming from the subset A dominate most significantly

The denominator is written as $\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right) \simeq e^{\frac{B}{\hbar}t_p}\sum_{n\in A} e^{-i\theta_n}$.

$$\langle\hat{O}\rangle_{\text{periodic time}} \simeq \frac{\sum_{n\in A}\langle\lambda_n|_Q\hat{O}|\lambda_n\rangle e^{-i\theta_n}}{\sum_{n\in A} e^{-i\theta_n}}. \quad (3)$$

We introduce the function $f(t_p) \equiv |\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)|^2$:

$f(t_p) \simeq e^{\frac{2}{\hbar}Bt_p} \sum_{n,m\in A} \cos\left\{\frac{1}{\hbar}(\mathrm{Re}\lambda_m - \mathrm{Re}\lambda_n)t_p\right\}$,

$\frac{df(t_p)}{dt_p} \simeq \frac{1}{\hbar} \sum_{n,m\in A} \Big[2B\cos\left\{\frac{1}{\hbar}(\mathrm{Re}\lambda_m - \mathrm{Re}\lambda_n)t_p\right\}$

$-\sin\left\{\frac{1}{\hbar}(\mathrm{Re}\lambda_m - \mathrm{Re}\lambda_n)t_p\right\}(\mathrm{Re}\lambda_m - \mathrm{Re}\lambda_n)\Big] e^{\frac{2B}{\hbar}t_p}$.

We are assuming that $B \leq 0$ and $|B|$ is much smaller than the distances between any two real parts of the eigenvalues of $\hat{H}$.
$\Rightarrow$ The second term in the square brackets contributes significantly in $\frac{df(t_p)}{dt_p}$.

* $B \leq 0$ has to be supposed so that $|\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)| = e^{\frac{B}{\hbar}t_p}|\sum_{n\in A} e^{-i\theta_n}|$ does not diverge when we seek t_p s.t. $|\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)|$ is maximized.

So, for θ_i s.t.

$$\theta_i = \theta_c \pmod{2\pi} \text{ for } \forall i \in A \Leftrightarrow \mathrm{Re}\lambda_i t_p = \hbar\theta_c \equiv C \pmod{2\pi\hbar} \text{ for } \forall i \in A, \tag{4}$$

$\frac{d^2 f(t_p)}{d{t_p}^2} < 0$ and $f(t_p)$ is maximized.

If t_p satisfying Eq.(4) exist, the phase factor $e^{-i\theta_n}$ becomes the same for $\forall n \in A$ in Eq.(3), so $\langle\hat{O}\rangle_{\text{periodic time}}$ is reduced to

$$\langle\hat{O}\rangle_{\text{periodic time}} \simeq \frac{\sum_{n\in A}\langle\lambda_n|_Q\hat{O}|\lambda_n\rangle}{\sum_{n\in A} 1}. \tag{5}$$

This is real for Q-Hermitian $\hat{O}$. Thus Theorem 3 will be proven.

We prove the existence of such t_p, via a number-theoretical argument, according to n_A, the order of the Hilbert space $\mathcal{H}_A$ that is labeled by the subset A.

Summary

In the periodic CAT, extending the weak value of $\hat{O}$ to $\langle \hat{O} \rangle_{\text{periodic time}} \equiv \frac{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\hat{O}\right)}{\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)}$, we presented a Theorem 3 stating that $\langle \hat{O} \rangle_{\text{periodic time}}$ becomes real provided that $\hat{O}$ is Q-Hermitian, for the period t_p selected s.t. $|\mathrm{Tr}\left(e^{-\frac{i}{\hbar}\hat{H}t_p}\right)|$ is maximized, in the case where $B \leq 0$ and $|B| << |\mathrm{Re}\lambda_m - \mathrm{Re}\lambda_n|$ for $\forall m, n \ (m \neq n)$.

Theorem 3 suggests that, if our universe is periodic, then even the period could be an adjustment parameter to be determined in the Feynman path integral.

This is a variant of the maximization principle that we previously proposed.

Outlook

We have supposed that the whole universe is a closed time-like curve (CTC) and considered very special cases for simplicity.

It would be interesting to

- investigate more complicated universes in general relativity and quantum cosmology.
- propose a model that results in an exactly periodic universe and provides for t_p an order of magnitude identifiable with the age of our universe via any kind of maximization principle.
- provide $\langle \hat{O} \rangle_{\text{periodic time}}$ with the time t dependence by introducing a clock operator $\hat{T}_{\text{clock}}(t)$.
- study the harmonic oscillator model we previously proposed more in detail, and the extention of the Q-Hilbert space

KN and H.B.Nielsen, PTEP **2019** 073B01

Recently we introduced a couple of density matrices to deal with mixed states both in the future-included and future-not-included CAT.

KN and H.B.Nielsen, arXiv:2209.11619 [hep-th]

Utilizing the density matrices, it would be intriguing to study

- the von Neumann entropy
- classical dynamics via Wigner function.
- master equation by interpreting our theory as a subsystem in a larger system