Renormalization group approach to cMERA

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based on arXiv:2211.05534
Dec. 8, KEK Theory Workshop 2022
Shizuoka University

Abstract

Motivation

To construct the bulk from the boundary in the context of AdS/CFT,
build the cMERA non-perturbatively for interacting theories

Result

- Derive a non-perturbative flow equation(ERG eq) for wave functionals
- Check the validity of the flow equation to the first-order perturbation in ϕ^4 theory

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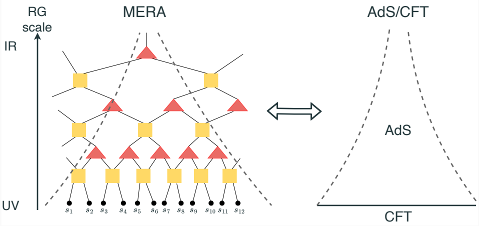
MERA and AdS/CFT

AdS/CFT correspondence

Gravity on AdS ⇔ CFT on the boundary[Maldacena, 1999]

Classical gravity ⇔ Large N strong coupling gauge theory

How to construct the bulk theory
from the boundary theory?



Correspondence between MERA & AdS/CFT

RG scale corresponds to the bulk direction

MERA and AdS/CFT

AdS/CFT correspondence

Gravity on AdS $\Leftrightarrow$ CFT on the boundary[Maldacena, 1999]

Classical gravity $\Leftrightarrow$ Large N strong coupling gauge theory

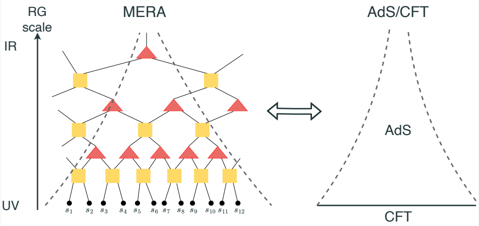
How to construct the bulk theory
from the boundary theory?

Hint: MERA[Vidal, 2007] & AdS/CFT correspondence
[Swingle, 2012]

MERA:
A numerical method for computing the wave function of the ground state

- Multi-layering of the system by the scale of the RG
- Treating the entanglement properly by tensors called disentangler
- Applying variational method to obtain the wave function of the ground state

Each layer $\leftrightarrow$ RG scale



Correspondence between MERA & AdS/CFT
RG scale corresponds to the bulk direction

MERA and AdS/CFT

MERA network ⇔ discrete AdS [Swingle, 2012]

⇓

MERA enables the explicitly construction
of the bulk geometry

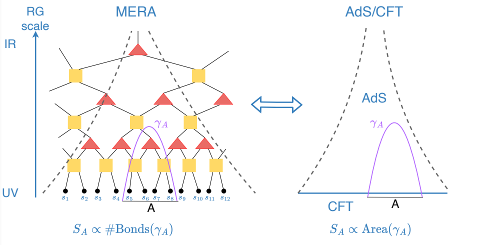
Continuous geometry from continuous fields

⇒ continuous MERA(cMERA)[Haegeman et al., 2013]

for free fields: [Nozaki et al., 2012] (using the variational method)

Nontrivial to construct trial functions
of the variational method for interacting theories

for interacting fields: [Fliss et al., 2017, Cotler et al., 2019]
[Fernandez-Melgarejo and Molina-Vilaplana, 2022]



Correspondence between MERA & AdS/CFT
γ_A : minimal surface of A

Exact renormalization group

- Continuous geometry $\rightarrow$ cMERA
- Strong coupling theory to extract classical geometry $\rightarrow$ Non-perturbation theory

$\Rightarrow$ Should consider the non-perturbative cMERA

Exact renormalization group

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$\Rightarrow$ Should consider the non-perturbative cMERA

How to construct the non-perturbative cMERA?

Constructing the cMERA:

$\Updownarrow$ equivalent

Obtaining the scale dependence of wave functionals

(c.f. RG scale corresponds to the bulk direction)

Non-perturbative cMERA

$\Rightarrow$ Derive a functional differential equation for wave functionals
from the exact renormalization group(ERG)

Exact renormalization group(or Functional RG)

Gives a functional differential equation(ERG equation)
that describes non-perturbatively the scale-dependence of the effective action

We consider the ERG equation for wave functional in scalar fields theories

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Exact renormalization group(ERG)

Requirement

The partition function is unchanged
under the infinitesimal change of the effective cutoff Λ

Exact renormalization group(ERG)

Requirement

The partition function is unchanged
under the infinitesimal change of the effective cutoff Λ

$$0 = -\Lambda \frac{\partial}{\partial \Lambda} \int \mathcal{D}\phi e^{-S_\Lambda[\phi]}$$

$$\Rightarrow -\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta \phi(p)} \left[G_\Lambda[\phi](p) e^{-S_\Lambda[\phi]} \right]$$

$\Lambda:$ the effective cutoff, $S_\Lambda:$ the effective action

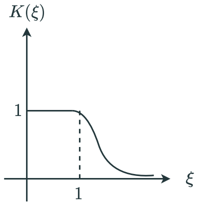
$G_\Lambda[\phi](p):$ the UV regularization,
corresponds to a continuum blocking(coarse-graining) procedure

$$G_\Lambda[\phi](p) = \frac{1}{2} \dot{C}_\Lambda(p) \frac{\delta}{\delta \phi(-p)} (S_\Lambda - 2\hat{S})$$

$\hat{S}:$ the seed action

$\dot{C}_\Lambda \equiv -\Lambda \partial_\Lambda C_\Lambda:$ an ERG integration kernel

typically, $C_\Lambda(p) = K(p^2/\Lambda^2)/(p^2 + m^2)$



The Polchinski equation [Polchinski, 1984]

Take the seed action $\hat{S}$ to the free part S_0

$$\hat{S} = S_0 = \int_p \frac{1}{2}\phi(p) C_\Lambda^{-1}(p)\phi(-p)$$

$$\begin{cases} -\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} & = \int_p \frac{\delta}{\delta \phi(p)} \left[G_\Lambda[\phi](p) e^{-S_\Lambda[\phi]} \right] \\ G_\Lambda[\phi](p) & = \frac{1}{2} \dot{C}_\Lambda(p) \frac{\delta}{\delta \phi(-p)} (S_\Lambda - 2\hat{S}) \end{cases}$$

$$\Rightarrow -\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta \phi(p)} \left[\frac{1}{2} \dot{C}_\Lambda(p) \left\{ \frac{\delta}{\delta \phi(-p)} (S_\Lambda - 2S_0) \right\} e^{-S_\Lambda[\phi]} \right]$$

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$$\Rightarrow -\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda[\phi]} = \int_p \frac{\delta}{\delta \phi(p)} \left[\frac{1}{2}\dot{C}_\Lambda(p) \left\{ \frac{\delta}{\delta\phi(-p)}(S_\Lambda - 2S_0) \right\} e^{-S_\Lambda[\phi]} \right]$$

Put $S_\Lambda = S_0 + S_{\text{int}}$

The Polchinski equation for S_{int}

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_{\text{int}}} = -\frac{1}{2} \int_p \dot{C}_\Lambda(p) \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} e^{-S_{\text{int}}}$$

The functional differential equation for the interacting part of S_Λ

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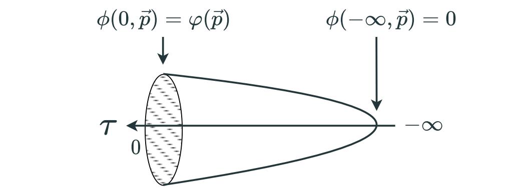
Path integral representation of wave functionals

Derive an ERG equation for the wave functional of the ground state



$$\Psi_\Lambda[\varphi(\vec{p})] = \langle \varphi(\vec{p}) | \Psi_\Lambda \rangle$$

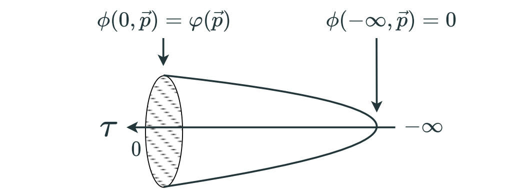
$$= \int_{\phi(0,\vec{p})=\varphi(\vec{p})} D\phi \; e^{-\int_{-\infty}^{0} d\tau L_\Lambda[\phi]}$$

$$\left(= \int_{\phi(0,\vec{p})=\varphi(\vec{p})} D\phi \; e^{-\underline{\int_0^{+\infty}} d\tau L_\Lambda[\phi]} \right)$$

(assume $L_\Lambda \in \mathbb{R} \to \Psi_\Lambda \in \mathbb{R}$)

Path integral representation of wave functionals

Derive an ERG equation for the wave functional of the ground state



$$\Psi_\Lambda[\varphi(\vec{p})] = \langle \varphi(\vec{p}) | \Psi_\Lambda \rangle$$
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(assume $L_\Lambda \in \mathbb{R} \to \Psi_\Lambda \in \mathbb{R}$)

$$\Psi_\Lambda^2[\varphi] = \int \mathcal{D}\phi \prod_{\vec{p}} \delta[\phi(0,\vec{p}) - \varphi(\vec{p})] e^{-S_\Lambda[\phi]}$$

ERG equation for Ψ_Λ

Act $-\Lambda\frac{\partial}{\partial\Lambda}$ on the both sides, we obtain

$$-\Lambda\frac{\partial}{\partial\Lambda}\Psi_\Lambda^2[\varphi] = \int \mathcal{D}\phi \prod_{\vec{k}} \delta[\phi(0,\vec{k}) - \varphi(\vec{k})] \int_{\tau,\tau',\vec{p}} \frac{\delta}{\delta\phi(\tau,\vec{p})}\left[\frac{1}{2}\dot{C}_\Lambda(\tau-\tau',\vec{p})\left\{\frac{\delta}{\delta\phi(\tau',-\vec{p})}(S_\Lambda - 2S_0)\right\}e^{-S_\Lambda}\right]$$

Substitute $\begin{cases} S_0 & = \int_p \frac{1}{2}\phi(p)C_\Lambda^{-1}(p)\phi(-p), \\ -\Lambda\frac{\partial}{\partial\Lambda}e^{-S_\Lambda[\phi]} & = \int_p \frac{\delta}{\delta\phi(p)}\left[\frac{1}{2}\dot{C}_\Lambda(p)\left\{\frac{\delta}{\delta\phi(-p)}(S_\Lambda - 2S_0)\right\}e^{-S_\Lambda[\phi]}\right] \end{cases}$

$$\Rightarrow \int \mathcal{D}\phi \prod_{\vec{k}} \delta[\phi(0,\vec{k}) - \varphi(\vec{k})] \int_{\tau,\tau',\vec{p}} \left[-\frac{1}{2}\dot{C}_\Lambda(\tau-\tau',\vec{p})\frac{\delta^2}{\delta\phi(\tau,\vec{p})\delta\phi(\tau',-\vec{p})}e^{-S_\Lambda[\phi]}\right]$$

$$- \int \mathcal{D}\phi \prod_{\vec{k}} \delta[\phi(0,\vec{k}) - \varphi(\vec{k})] \int_{\tau,\tau',\tau'',\vec{p}} \frac{\delta}{\delta\phi(\tau,\vec{p})}\left[\dot{C}_\Lambda(\tau-\tau',\vec{p})C_\Lambda^{-1}(\tau'-\tau'',\vec{p})\phi(\tau'',\vec{p})e^{-S_\Lambda[\phi]}\right]$$

$\uparrow$ Total derivative except $\tau = \tau' = 0$(first line) & $\tau = 0$(second line)

Typically, $\dot{C}_\Lambda(0,\vec{p}) = \dot{K}(\vec{p})/(2\sqrt{\vec{p}^2 + m^2})$, where $K(\vec{p})$ damps rapidly for $\vec{p}^2 > \Lambda^2$

ERG equation for the wave functional of the ground state

$$-\Lambda\frac{\partial}{\partial\Lambda}\Psi_\Lambda = -\frac{1}{2}\int_{\vec{p}} \dot{C}_\Lambda(0,\vec{p})\left\{\frac{\delta^2\Psi_\Lambda}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} + \frac{1}{\Psi_\Lambda}\frac{\delta\Psi_\Lambda}{\delta\varphi(\vec{p})}\frac{\delta\Psi_\Lambda}{\delta\varphi(-\vec{p})}\right\}$$

$$- \int_{\vec{p}} \frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})}\varphi(\vec{p})\frac{\delta\Psi_\Lambda}{\delta\varphi(\vec{p})} - \frac{V}{2}\Psi_\Lambda \int_{\vec{p}} \frac{\dot{C}_\Lambda(0,\vec{p})}{C_\Lambda(0,\vec{p})}$$

Wave functional for a free theory

We can check that Ψ_0, the ground-state wave functional of a free theory, satisfies the ERG equation
The wave functional

$$\Psi^{(0)}_\Lambda[\varphi(\vec{p})] = \int_{\phi(0,\vec{p})=\varphi(\vec{p})} D\phi \; e^{-\int_{-\infty}^0 d\tau L_0}$$

$$L_0 = \int_{\vec{p}} \frac{1}{2} K_{\vec{p}}^{-1} \left[\partial_\tau \phi(\tau,\vec{p}) \partial_\tau \phi(\tau,-\vec{p}) + (\vec{p}^2 + m^2)\phi(\tau,\vec{p})\phi(\tau,-\vec{p})\right]$$

$$\Psi^{(0)}_\Lambda = \exp\left[-\int_{\vec{p}} \frac{1}{2} K_{\vec{p}}^{-1} \omega_{\vec{p}} \, \varphi(\vec{p})\varphi(-\vec{p}) + \frac{V}{4}\int_{\vec{p}} \log\left(2K_{\vec{p}}^{-1}\right)\omega_{\vec{p}}\right]$$

$\rightarrow$ satisfies the ERG equation for the wave functional

ERG eq for the interaction part of wave functionals

$$\Psi_\Lambda[\varphi] = \int_{\phi(0,\vec{p})=\varphi(\vec{p})} D\phi \; e^{-\int_{-\infty}^{0} d\tau (L_0 + L_{\rm int})}$$

Parametrize

$$\Psi_\Lambda[\varphi] = e^{I[\varphi]}\Psi_\Lambda^{(0)} \,, \qquad \Psi_\Lambda^{(0)}[\varphi]: \text{the free part of the wave functional}$$

$$I[\varphi]: \text{the “interaction part” of } \Psi_\Lambda$$

ERG eq for the interaction part of Ψ_Λ

$$-\Lambda \frac{\partial}{\partial \Lambda} I = -\frac{1}{2} \int_{\vec{p}} \dot{C}(0,\vec{p}) \left[\frac{\delta^2 I}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} + \frac{\delta I}{\delta\varphi(\vec{p})} \frac{\delta I}{\delta\varphi(-\vec{p})} \right]$$

Counterpart of the Polchinski equation

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1st-order perturbation of ϕ^4 theory

$$L_0 = \int_{\vec{p}} \frac{1}{2} K_{\vec{p}}^{-1} \left[\partial_\tau \phi(\tau, \vec{p}) \partial_\tau \phi(\tau, -\vec{p}) + (\vec{p}^2 + m^2) \phi(\tau, \vec{p}) \phi(\tau, -\vec{p}) \right]$$

$$L_{\text{int}} = \frac{\delta m^2}{2} \int_{\vec{p}} \phi(\tau, \vec{p}) \phi(\tau, -\vec{p}) + \frac{\lambda}{4!} \int_{\vec{p}_1 \ldots \vec{p}_4} \phi(\tau, \vec{p}_1) \phi(\tau, \vec{p}_2) \phi(\tau, \vec{p}_3) \phi(\tau, \vec{p}_4) \tilde{\delta}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$

$\Psi_\Lambda \equiv e^I \Psi_\Lambda^{(0)}, \quad \Psi_\Lambda^{(0)}$: the free part of the wave functional

I : 1PI diagrams

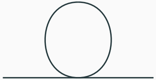
1st-order perturbation of ϕ^4 theory

The ERG equation for I :

$$-\Lambda\frac{\partial}{\partial\Lambda}I = -\frac{1}{2}\int_{\vec{p}}\dot{C}(0,\vec{p})\left[\frac{\delta^2 I}{\delta\varphi(\vec{p})\delta\varphi(-\vec{p})} + \underbrace{\frac{\delta I}{\delta\varphi(\vec{p})}\frac{\delta I}{\delta\varphi(-\vec{p})}}_{=0}\right]$$

1PI diagrams for 1st-order perturbation of I :

$$= -\frac{\delta m^2}{2}\int_{\vec{p}}\varphi(\vec{p})\varphi(-\vec{p})\frac{1}{2\omega_{\vec{p}}}$$



$$= -\frac{\lambda}{4!}\int_{\vec{p}_1\vec{p}_2}\varphi(\vec{p}_1)\varphi(-\vec{p}_1)\ \frac{3K_2}{2\omega_1(\omega_1+\omega_2)}$$

$$= -\frac{\lambda}{4!}\int_{\vec{p}_1\cdots\vec{p}_4}\varphi_1\cdots\varphi_4\ \frac{\tilde{\delta}(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)}{\omega_1+\omega_2+\omega_3+\omega_4}\ ,$$

1st-order perturbation of ϕ^4 theory

Using the flow equation for δm^2 which is obtained by the Polchinski eq

$$-\Lambda \frac{\partial}{\partial \Lambda} \delta m^2 = \frac{\lambda}{2} \int_{\vec{p}} \frac{\dot{K}_\Lambda(\vec{p})}{2\omega_{\vec{p}}}$$

$\Rightarrow$ I satisfies the ERG equation

The ERG equation is valid to the 1st order perturbation of ϕ^4 theory

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Summary

- Derive the ERG equation that determines non-perturbatively the scale dependence of wave functionals in scalar field theories
- Check the validity of the ERG equation to the 1st-order perturbation

Future Work

- Determine the entangler of cMERA non-perturbatively
- Calculate the entanglement entropy
- Extract the geometry from the information metric

  cf. For the case of the massless scalar field,

  pure AdS is extracted from the information metric[Nozaki et al., 2012]

Backup slides

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Notation

$(d+1)$ dimensional Euclidean spacetime

$$\int_p \equiv \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \, , \quad \int_{\vec{p}} \equiv \int \frac{d^d p}{(2\pi)^d} \, , \quad \int_\tau \equiv \int d\tau \, ,$$

$$\tilde{\delta}(\vec{p}) = (2\pi)^d \delta(\vec{p})$$

$$V = \tilde{\delta}(0)$$

We frequently use $\phi(p)$ and $\phi(\tau, \vec{p})$, which are transformed each other by

$$\phi(p) = \phi(E, \vec{p}) = \int d\tau \phi(\tau, \vec{p}) e^{-iE\tau} \, .$$