Path integrals of perturbative strings on all the curved backgrounds from string geometry theory and a potential energy of the backgrounds

Matsuo Sato (Hirosaki U.)

M.S.	Int.J.Mod.Phys.A34 (2019)1950126
M.S. , Y. Sugimoto (POSTECH)	Eur.Phys.J.C80 (2020) 789
M.S. , Y. Sugimoto	Nucl.Phys.B956 (2020) 115019
M.S., T. Tohshima (Hirosaki U.)	in preparation
M. Honda (Waseda U.), M.S.	Int.J.Mod.Phys.A 35 (2020) 2050176
M. Honda, M.S., T. Tohshima	Adv.High Energy Phys. (2021) 9993903
M.S., Y. Sugimoto, K. Uzawa (Kwansei Gakuin U.)	Phys.Rev.D 106 (2022) 8, 086006
M.S., K. Uzawa	arXiv:2211.16959

Motivation

Backgrounds in string theory

Perturbative string theory

One theory is formulated for each background. ex. The string theory on $AdS_5 \times S^5$

Non-perturbative formulation of string theory

Perturbative string theories on **all** the backgrounds should be derived from a **single** theory.

Review

String geometry theory

- String geometry theory is one of the candidates of non-perturbative formulation of string theory.

- We can derive the path-integral of the **IIA, IIB, SO(32) type I, and SO(32) and E8xE8 heterotic** superstring theories in the flat background from the **single** theory by considering <u>fluctuations around fixed flat background</u> in the corresponding charts, respectively.

- All the **five** 10-dimensional supergravities (IIA, IIB, SO(32) type I, and SO(32) and E8xE8 heterotic) are obtained by consistent truncations of the fields to **five** string background configurations in the **single** string geometry theory. This implies that all **the string backgrounds and their dynamics are included in string geometry theory**.

- The theory unifies particles and the space-time. Macroscopically, space-time = string manifold, particle = a fluctuation of string manifold

Formalism

For presentations, bosonic closed only.
In general, supersymmetric open and closed.

- $Z = \int \mathcal{D}\mathbf{G}\mathcal{D}\Phi\mathcal{D}\mathbf{B}e^{iS}$

$$S = \int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X(\bar{\tau})\sqrt{-\mathbf{G}}e^{-2\Phi}\left[\mathbf{R} + 4\nabla_I\Phi\nabla^I\Phi - \frac{1}{2}|\mathbf{H}|^2\right]$$

- coordinates on a string manifold
  - infinite dimensional manifold
  - h : metric on a worldsheet Σ
  - $\bar{\tau}$: global time on Σ
  - $X(\bar{\tau}) : \Sigma|_{\bar{\tau}} \to \mathbf{R}^d$

* h is a discrete variable

$\Downarrow$

h is constant because there is no derivative w. r. t. h

- $\mathbf{R}$: scalar curvature of a metric $\mathbf{G}_{IJ}$ on a string manifold
- Φ : scalar field on a string manifold
- $\mathbf{H}$: 3-form field strength of a 2-form $\mathbf{B}$ on a string manifold

$$\nabla_I\Phi\nabla^I\Phi = \nabla_d\Phi\nabla^d\Phi + \int d\bar{\sigma}\bar{e}\nabla_{(\mu\bar{\sigma})}\Phi\nabla^{(\mu\bar{\sigma})}\Phi$$

Derive the path-integrals of perturbative strings in all the curved string backgrounds from String Geometry

M.S. , Y. Sugimoto (POSTECH) , K. Uzawa (Kwansei Gakuin U.)

Identify physical D.O.F.

- Consider only static configurations

$$
\begin{aligned}
\partial_d G_{MN} &= 0 \\
\partial_d B_{MN} &= 0 \\
\partial_d \Phi &= 0
\end{aligned}
$$

- Consider fluctuations around a flat background for simplicity. (up to the 1st order in classical fluctuations and up to the 2nd order in quantum fluctuations)

$$
G_{MN} = \underbrace{\widehat{G}_{MN}}_{\text{flat}} + \underbrace{h_{MN}}_{\text{fluctuations}}
$$

- Fix the general covariance to the harmonic gauge

$$
\widehat{\nabla}^M \psi_{MN} = 0
$$

where $\psi_{MN} := h_{MN} - \frac{1}{2}\widehat{G}^{IJ} h_{IJ} \widehat{G}_{MN} + 2\widehat{G}_{MN}\phi$

Identify physical D.O.F. (cont.)

- D.O.F. of strings

$$\psi_{dd} = \bar{\psi}_{dd} + \tilde{\psi}_{dd}$$

classical quantum

where $\bar{\psi}_{dd} = \int \mathcal{D}X' \underbrace{G(X;X')}_{\text{the Green function on the flat string manifold}} \int d\bar{\sigma}\sqrt{\bar{h}}[\alpha' R_{\bar{h}}\Phi(X'(\bar{\sigma})) + \frac{1}{\bar{e}^2}G_{\mu\nu}(X'(\bar{\sigma}))\partial_{\bar{\sigma}}X'^{\mu}\partial_{\bar{\sigma}}X'^{\nu}]$

- D.O.F. of string backgronds (string background configuration)

$$\boxed{\begin{aligned} \bar{G}_{(\mu\bar{\sigma})(\nu\bar{\sigma}')} &= G_{\mu\nu}(X(\bar{\sigma}))\frac{\bar{e}^3}{\sqrt{\bar{h}}}\delta_{\bar{\sigma}\bar{\sigma}'} \\ \bar{B}_{(\mu\bar{\sigma})(\nu\bar{\sigma}')} &= B_{\mu\nu}(X(\bar{\sigma}))\frac{\bar{e}^3}{\sqrt{\bar{h}}}\delta_{\bar{\sigma}\bar{\sigma}'} \\ \bar{\phi} &= \int d\bar{\sigma}\bar{e}\Phi(X(\bar{\sigma})) \end{aligned}}$$

where $\left[\begin{aligned} G_{\mu\nu}(x) &= \delta_{\mu\nu} + h_{\mu\nu}(x) \\ B_{\mu\nu}(x) &= b_{\mu\nu}(x) \\ \Phi(x) &= \phi(x) \end{aligned}\right]$ are string backgrounds

Propagator

- Normalize the leading part of the kinetic term by rescaling $\tilde{\psi}_{dd}$
- Delete the first order term by shifting $\tilde{\psi}_{dd}$
- Then, the action plus the gauge fixing term becomes

$$S + S_{fix} = \int \mathcal{D}\bar{\tau}\mathcal{D}\bar{h}\mathcal{D}X(\bar{\tau})\tilde{\psi}_{dd} H(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\tilde{\psi}_{dd}$$

where

$$\begin{aligned} H(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X},\ X,\ \bar{h}) = & \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}(\eta_{\mu\nu} - h_{\mu\nu}(X(\bar{\sigma})))(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^{\mu}})(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^{\nu}}) \\ & -\frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\frac{1}{\bar{e}}\frac{\partial}{\partial X^{\mu}}\frac{1}{\bar{e}}\frac{\partial}{\partial X_{\mu}}\bar{\psi}_{dd} \\ & + \int d\bar{\sigma}\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}}X^{\mu}\bar{e}(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^{\mu}}) + \int d\bar{\sigma}\, i\frac{\sqrt{\bar{h}}}{\bar{e}^2}\partial_{\bar{\sigma}}X^{\nu}B_{\nu}{}^{\mu}(X(\bar{\sigma}))\bar{e}(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X^{\mu}}) \end{aligned}$$

ADM decomposition $\quad \bar{h}_{mn} = \begin{pmatrix} \bar{n}^2 + \bar{n}_{\bar{\sigma}}\bar{n}^{\bar{\sigma}} & \bar{n}_{\bar{\sigma}} \\ \bar{n}_{\bar{\sigma}} & \bar{e}^2 \end{pmatrix}$

- Differential equation for the propagator $\quad \Delta_F(\bar{h}, X(\bar{\tau});\ \bar{h},' X'(\bar{\tau}')) = < \tilde{\psi}_{dd}(\bar{h}, X(\bar{\tau}))\tilde{\psi}_{dd}(\bar{h},' X'(\bar{\tau}')) >$

$$H(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\Delta_F(\bar{h}, X;\ \bar{h}', X') = \delta(\bar{h} - \bar{h}')\delta(X - X')$$

2-point correlation function

- In order to compare with perturbative strings,
  take the Schwinger representation of the propagator by using the first quantization formalism.

  operators $(\hat{\bar{h}}, \hat{X})$ conjugate momenta $(\hat{p}_{\bar{h}}, \hat{p}_X)$ eigen states $|\bar{h}, X>$ $|p_{\bar{h}}, p_X>$

- $H(-i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\Delta_F(\bar{h}, X;\ \bar{h}', X') = \delta(\bar{h}-\bar{h}')\delta(X-X')$

$$\updownarrow$$

$$\begin{aligned}\Delta_F(\bar{h}, X;\ \bar{h}', X') &= <\bar{h}, X|\hat{H}^{-1}(\hat{p}_X, \hat{X}, \hat{\bar{h}})|\bar{h}', X'> \\ &= i\int_0^\infty dT <\bar{h}, X|e^{-iT\hat{H}}|\bar{h}', X'>\end{aligned}$$

- 2-point correlation function of diffeo inv. states (We integrate X in the end.)

$$\Delta_F(X;\ X') := i\int_0^\infty dT < X|_{out} e^{-iT\hat{H}}|X'>_{in}$$

where

$$< X|_{out} := \int \mathcal{D}h < \bar{h}, X(\bar{\tau}=\infty)|$$

$$|X'>_{in} := \int \mathcal{D}h'|\bar{h}', X'(\bar{\tau}=-\infty)>$$

Path integral representation

- path integral representation

$$\Delta_F(X;\ X')$$
$$= \ i\int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}X \int \mathcal{D}p_T\mathcal{D}p_X \exp\left(i\int_0^1 dt\left(p_T(t)\frac{d}{dt}T(t) + p_X(t)\cdot\frac{d}{dt}X(t) - T(t)H(p_X(t), X(t), \bar{h})\right)\right)$$

- Move onto Lagrange formalism from the canonical formalism by integrating out p_X .

Change a gauge

- $\Delta_F(X;\ X')$

$$
\begin{aligned}
= \ & \int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}X\mathcal{D}p_T \\
& \exp\Bigg(i\int_0^1 dt \Big(p_T(t)\frac{d}{dt}T(t) + \frac{1}{2}\int d\bar{\sigma}\sqrt{\bar{h}}\,T(t)\,\alpha'\,R_{\bar{h}}\Phi(X(\bar{\tau},t)) \\
& + \int d\bar{\sigma}\Big(\sqrt{\bar{h}}G_{\mu\nu}(X(\bar{\sigma},\bar{\tau},t))(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_t X^\nu(\bar{\sigma},\bar{\tau},t) + \bar{h}^{01}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X^\nu(\bar{\sigma},\bar{\tau},t) \\
& + \frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}} X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X^\nu(\bar{\sigma},\bar{\tau},t)) + iB_{\mu\nu}(X^\mu(\bar{\sigma},\bar{\tau},t))\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X^\nu(\bar{\sigma},\bar{\tau},t)\Big)\Big)\Bigg)
\end{aligned}
$$

- This path integral is obtained

  if $F_1(t) := \frac{d}{dt}T(t) = 0$ gauge is chosen in a covariant form w.r.t. t diffeo:

  * $T(t)$ is transformed as an einbein.

  * F.P. ghosts are free and contribute to only the overall factor.

- $T(t)$ disappears under $\frac{d\bar{\tau}}{d\bar{\tau}'} = T(t)$

- Take $\bar{\tau} = t$ gauge.

Conclusion

Path-integrals of perturbative strings in curved string backgrounds

- We obtain

$$\Delta_F(X;\ X') = Z \int_{X'}^{X} \mathcal{D}h \mathcal{D}X e^{iS_s}$$

$$S_s = \int_{-\infty}^{\infty} d\tau \int d\sigma \sqrt{h(\sigma,\tau)} \frac{1}{2}\Bigg(\Big(h^{mn}(\sigma,\tau) G_{\mu\nu}(X(\sigma,\tau)) + i\epsilon^{mn}(\sigma,\tau) B_{\mu\nu}(X(\sigma,\tau)) \Big) \partial_m X^{\mu}(\sigma,\tau) \partial_n X^{\nu}(\sigma,\tau)$$
$$+\alpha' R_{\bar{h}} \Phi(X(\sigma,\tau)) \Bigg)$$

- We have derived the path-integrals of all order perturbative strings in general backgrounds.

Backups

Non-perturbative Formulation of Superstring Theory Based on String Geometry

Matsuo Sato (Hirosaki U.)

Contents

0. Introduction

1. String geometry

2. Non-perturbative formulation of string theory

3. String geometry solution that represents a perturbative vacuum of string theory

4. Derive all order scattering amplitudes of perturbative string

5. General supersymmetric case that includes open strings

6. String geometry and a new type of supersymmetric matrix models

7. Unification of particles and the space-time

8. Future directions

0. Introduction

Elementary particle physics

Fundamental problems in elementary particle physics

- Many parameters that cannot be determined by the standard model
- Hierarchy problems
- How to describe the very early universe

⋮

Theory of gravity at Plank scale = **quantum gravity** is necessary

String theory is one of the most strong candidates of **quantum gravity**

String theory

- Is understood well perturbatively.
- Has many perturbatively stable vacua.

Advantages of perturbative string theory In **bottom up** point of view,

Can reproduce almost physics in the experimental region
by choosing a perturbative vacuum (geometry) appropriately.

cf. Calabi-Yau compactification, flux compactification

We can find a new phenomenological mechanism by understanding physics in terms of geometry.

(Chosen by hand)

Disadvantages of perturbative string theory In **top down** point of view,

We might say ``This is just replacing the problem of choosing parameters of physics to choosing geometry."

Cannot determine a vacuum. That is, **NO prediction**.

↓

Non-perturbative formulation of string theory is necessary.

(Chosen automatically)

Motivation

<u>T-duality</u>

- IIA string on a background $\longleftrightarrow$ IIB string on another background

  Observed values by strings coincide.

  $\downarrow$

  "Space observed by strings are the same."

  ||

  Expected to be "geometric principle of string theory."

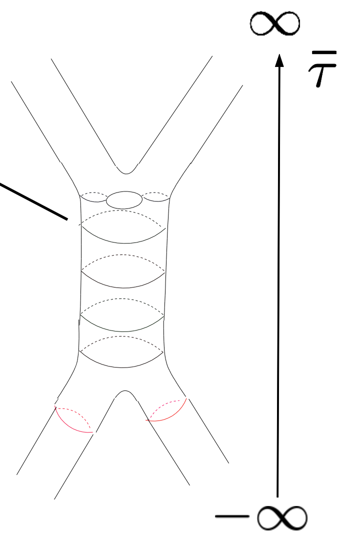
Q. Is there **spaces T-dual to each other**? $\partial_a X'^9 = i\epsilon_{ab}\partial_b X^9$

A. Yes: **moduli spaces of Riemann surfaces embedded on-shell in the backgrounds**.

$$X : \Sigma \to M \longleftrightarrow X' : \Sigma \to M'$$

Extend off-shell

Extend non-perturbatively. (**Consider collections of** $\Sigma|_{\bar{\tau}=const.}$)



- $\boxed{\text{Spaces where } \textit{curves parametrized by } \bar{\tau}(-\infty < \bar{\tau} < \infty) \textit{ reproduce the right moduli space of the Riemann surfaces in a target manifold.}}$

  ||

  string geometry

  criterion to define string topology

- Construct string theory by regarding string geometry as geometric principle.

1. String geometry

1. 1 String model space

Global time $\bar{\tau}$

- On Σ , there exists an unique Abelian differential dp that has simple poles with residues f_i at Punctures Pi where $\Sigma_i f_i = 0$, if it is normalized to have purely imaginary periods with respect to all contours.

- Global time $\bar{\tau}$ is defined by $\bar{w} = \bar{\tau} + i\bar{\sigma} := \int^P dp$ (Krichever, Novikov 1987)

  $\bar{\tau} = +\infty \ (-\infty)$ on Pi with $f_i > 0 \ (f_i < 0)$

- Determine f_i

  0. $\Sigma_i f_i = 0$: f_i conservation law (if we choose the outgoing direction as positive.)

  1. Divide Pi s to arbitrary incoming and outgoing sets.

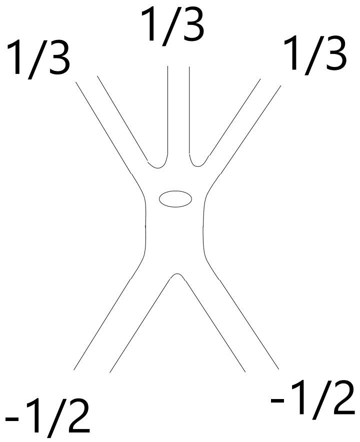
  2. Divide -1 to incoming $f^i \equiv \frac{-1}{N_{in}}$ and 1 to outgoing $f^i \equiv \frac{1}{N_{out}}$

  ↓

  - f_i are determined uniquely on Σ

  ↓

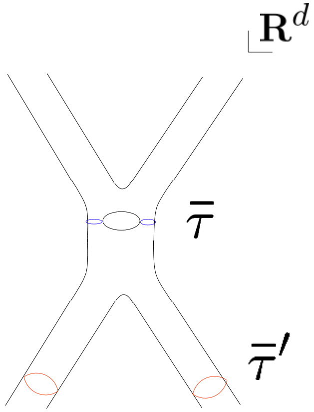
  - $\bar{\tau}$ is uniquely determined.



String model space E

Collection of string states $[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]$

- $\Sigma|_{\bar{\tau}} \cong S^1 \cup \cdots \cup S^1$ many body states of strings



- $X_{\hat{D}}(\bar{\tau}) : \Sigma|_{\bar{\tau}} \to \mathbf{R}^d$

  $\hat{D}$: backgrounds (B, dilaton)

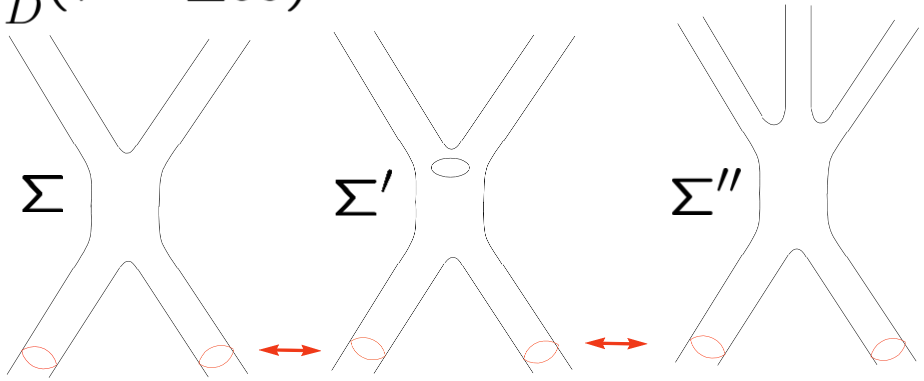
- $[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]$: equivalence class

  at $\bar{\tau} \cong \pm\infty$ $\quad \Sigma \cong C^2 \cup \cdots \cup C^2$

  Here, $\Sigma|_{\bar{\tau}\cong\pm\infty} = \Sigma'|_{\bar{\tau}\cong\pm\infty}, \; X_{\hat{D}}(\bar{\tau} \cong \pm\infty) = X'_{\hat{D}}(\bar{\tau} \cong \pm\infty)$

  $$\downarrow$$

  $$(\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong \pm\infty) \sim (\Sigma', X'_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong \pm\infty)$$



$(\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong -\infty) \sim (\Sigma', X'_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong -\infty) \sim (\Sigma'', X''_{\hat{D}}(\bar{\tau}), \bar{\tau} \cong -\infty)$

- $E := \bigcup_{\hat{D}} \{[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]\}$

1. 2 String toplology

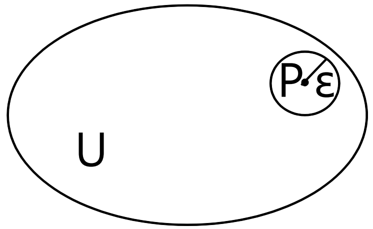
String topology

- ϵ open neighborhood

$$U([\Sigma, X_{0\hat{D}}(\bar{\tau}_0), \bar{\tau}_0], \epsilon) := \left\{ [\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}] \,\middle|\, \sqrt{|\bar{\tau} - \bar{\tau}_0|^2 + \|X_{\hat{D}}(\bar{\tau}) - X_{0\hat{D}}(\bar{\tau}_0)\|^2} < \epsilon \right\}$$

s.t. $$\|X_{\hat{D}}(\bar{\tau}) - X_{\hat{D}0}(\bar{\tau}_0)\|^2 = \int_0^{2\pi} d\bar{\sigma} (X^{\mu}_{\hat{D}}(\bar{\tau}, \bar{\sigma}) - X^{\mu}_{\hat{D}0}(\bar{\tau}_0, \bar{\sigma}))^2$$

- U is defined to be an open set if there exists ϵ such that an ϵ open neighborhood $\subset$ U for an arbitrary point P $\in$ U.



- The open sets satisfies the axiom of topology.

$$(i) \quad \emptyset, E \in \mathcal{U}$$

$$(ii) \quad U_1, U_2 \in \mathcal{U} \Rightarrow U_1 \cap U_2 \in \mathcal{U}$$

$$(iii) \quad U_\lambda \in \mathcal{U} \Rightarrow \cup_{\lambda \in \Lambda} U_\lambda \in \mathcal{U}$$

1. 2 String manifold

General coordinate transformation

- Σ does not transform to $\bar{\tau},\ X_{\hat{D}}$ and vice versa, because Σ is a discrete variable, whereas $\bar{\tau},\ X_{\hat{D}}$ are continuous variables by definition of the neighbourhoods.

- $\bar{\tau}$ and $\bar{\sigma}$ do not transform to each other because the string states are defined by $\bar{\tau}$ constant lines.

- Under these restrictions, the most general coordinate transformation is given by

$$[\bar{h}_{mn}(\bar{\sigma},\bar{\tau}),\bar{\tau},X^{\mu}_{\hat{D}}(\bar{\tau})] \mapsto [\bar{h}'_{mn}(\bar{\sigma}'(\bar{\sigma}),\bar{\tau}'(\bar{\tau},X_{\hat{D}}(\bar{\tau}))),\bar{\tau}'(\bar{\tau},X_{\hat{D}}(\bar{\tau})),X'^{\mu}_{\hat{D}}(\bar{\tau}')(\bar{\tau},X_{\hat{D}}(\bar{\tau}))]$$

$\Sigma \longleftrightarrow \bar{h}_{mn}(\bar{\sigma},\bar{\tau})$ up to diff × Weyl

- String manifolds $\mathfrak{M}$ are constructed by patching open sets of E by general coordinate transformations.

Example of string manifolds $\mathcal{M}_D$

- $\mathcal{M}_D := \{[\Sigma, x_D(\bar{\tau}), \bar{\tau}]\}$

  where $x_D(\bar{\tau}) : \Sigma|_{\bar{\tau}} \to M$ which has target mertic $ds^2 = dx_D^\mu(\bar{\tau}, \bar{\sigma}) dx_D^\nu(\bar{\tau}, \bar{\sigma}) G_{\mu\nu}(x_D(\bar{\tau}, \bar{\sigma}))$

- D: backgronds including the target metric . D is fixed on a string manifold.

- Open sets of $\mathcal{M}_D \longleftrightarrow$ Open sets of E

  homeomorphic

  diffeo: $[\Sigma, x_D(\bar{\tau}), \bar{\tau}] \mapsto [\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]$

  $X_{\hat{D}}(\bar{\tau})(x_D(\bar{\tau}))$ Is induced by the diffeomorphism transformation of the target space

  $x \mapsto X = X(x)$

  $\downarrow$

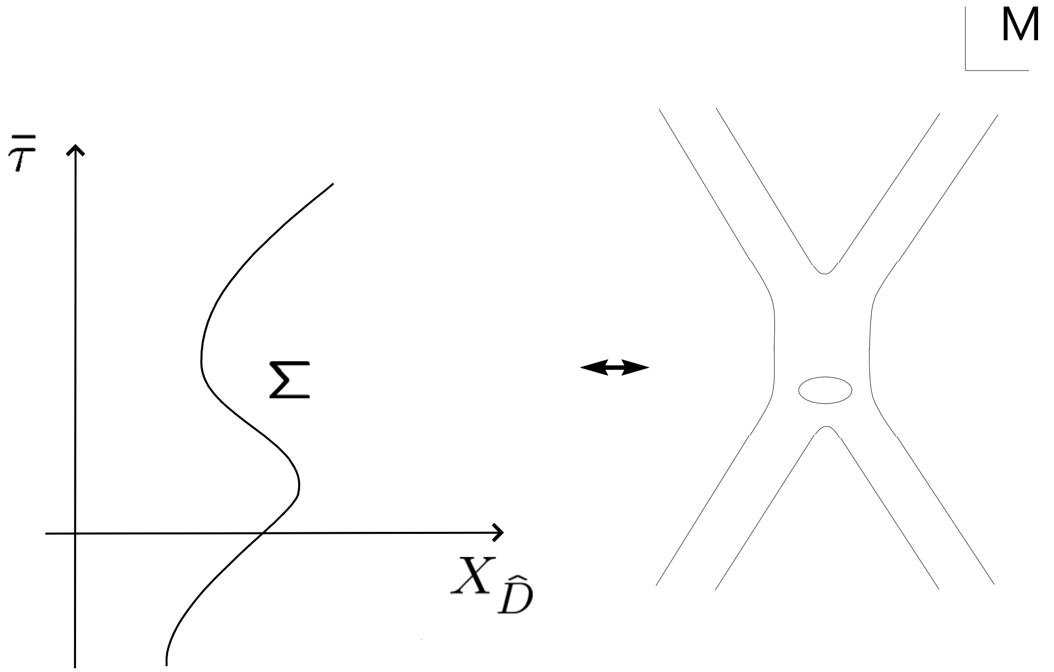
  $X_{\hat{D}}(\bar{\tau}, \bar{\sigma}) = X(x_D(\bar{\tau}, \bar{\sigma}))$

Example of string manifolds $\mathcal{M}_D$ (cont'd)

- Trajectories in asymptotic processes on $\mathcal{M}_D$ represents 2-dim. Riemann surfaces in the target manifold.

↓

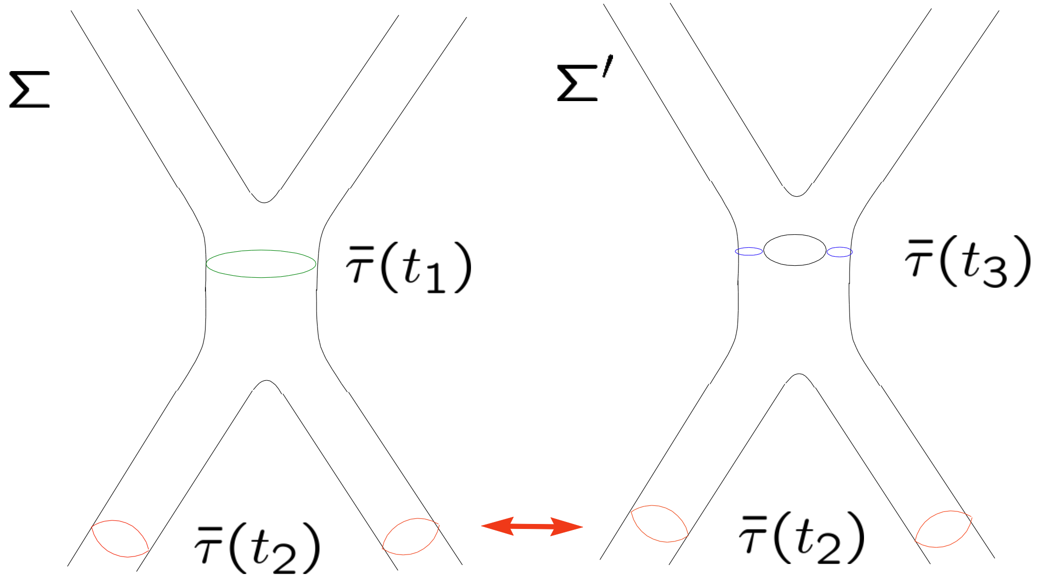
reproduce the right moduli space.



Example of string manifolds $\mathcal{M}_D$ (cont'd)

- By a general trajectory,
  string states on different two-dimensional Riemann surfaces that have **different genus numbers** can be connected continuously.

v.s. the moduli space



1. 4 Riemannian string manifold

Riemannian string manifold

- cotangent vectors

cotangent space of manifolds are spanned by continuous variables:

$$dX^{\mu}_{\hat{D}}(\bar{\sigma}, \bar{\tau}) \quad \text{Treat } (\mu\bar{\sigma}) \text{ as indices.} \qquad d\bar{\tau}$$

$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$

$$dX^{(\mu\bar{\sigma})}_{\hat{D}} \qquad\qquad\qquad\qquad dX^{d}_{\hat{D}} \quad \xrightarrow[\text{summarize}]{} \quad dX^{I}_{\hat{D}} \; (I = d, (\mu\bar{\sigma}))$$

Take summation by $\int d\bar{\sigma}\bar{e}(\bar{\sigma}, \bar{\tau})$ $\qquad (\bar{e} := \sqrt{\bar{h}_{\bar{\sigma}\bar{\sigma}}})$

Invariant under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$

Transformed as scalar under $\bar{\tau} \mapsto \bar{\tau}'(\bar{\tau}, X)$

- metric

$$ds^2(\bar{h}, \bar{\tau}, X_{\hat{D}}) = G_{IJ}(\bar{h}, \bar{\tau}, X_{\hat{D}}) dX^{I}_{\hat{D}} dX^{J}_{\hat{D}}$$

2 Non-perturbative formulation of string theory

Non-perturbative formulation of superstring theory

- $$Z = \int \mathcal{D}G\mathcal{D}A e^{-S}$$

$$S = \frac{1}{G_N} \int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X_{\hat{D}} \sqrt{G}(-R + \frac{1}{4} G_N G^{I_1 I_2} G^{J_1 J_2} F_{I_1 J_1} F_{I_2 J_2})$$

$\mathcal{D}h$: the invariant measure of h_{mn} on Σ .

$h_{mn} \longleftrightarrow \bar{h}_{mn}$

diff × Weyl

$F_{\mathbf{IJ}}$: field strength of an u(1) gauge field $A_{\mathbf{I}}$

- **The theory is background independent.**

diffeomorphism invariance

- Under $(\bar{\tau}, X) \mapsto (\bar{\tau}'(\bar{\tau}, X), X'(\bar{\tau}, X))$

  $G_{IJ}(\bar{h}, \bar{\tau}, X)$: symmetric tensor $\quad A_I(\bar{h}, \bar{\tau}, X)$: vector

  Action is manifestly invariant

- Under $\bar{h}_{mn} \to \bar{h}'_{mn}$, $G_{IJ}(\bar{h}, \bar{\tau}, X)$ and $A_I(\bar{h}, \bar{\tau}, X)$ are defined as scalars

  Under $\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})$,
  - the fields that have index $\bar{\sigma}$ transform as scalars.
  - $\int d\bar{\sigma}\bar{e}(\bar{\sigma}, \bar{\tau})$ is invariant.

  $\downarrow$

  The action is invariant under $\boxed{\bar{\sigma} \mapsto \bar{\sigma}'(\bar{\sigma})}$

* In a supersymmetric case, the action is invariant under $(\bar{\sigma}, \bar{\theta}^{\alpha}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'^{\alpha}(\bar{\sigma}, \bar{\theta}))$

3 String geometry solution that represents a perturbative vacuum of string theory

Perturbative vacuum solution (Extension of Majumdar-Papapetrou solution (1947,1948))

- $$\bar{ds}^2 = 2\lambda\bar{\rho}(\bar{h})N^2(X)(dX^d)^2 + \int d\bar{\sigma}\bar{e}\int d\bar{\sigma}'\bar{e}' N^{\frac{2}{2-D}}(X)\frac{\bar{e}^3(\bar{\sigma},\bar{\tau})}{\sqrt{\bar{h}(\bar{\sigma},\bar{\tau})}}\delta_{(\mu\bar{\sigma})(\mu'\bar{\sigma}')}dX^{(\mu\bar{\sigma})}dX^{(\mu'\bar{\sigma}')}$$

$$\bar{A}_d = i\sqrt{\frac{2-2D}{2-D}}\frac{\sqrt{2\lambda\bar{\rho}(\bar{h})}}{\sqrt{G_N}}N(X), \qquad \bar{A}_{(\mu\bar{\sigma})} = 0$$

is a solution to the equations of motion. ($\bar{h}_{mn}(\bar{\sigma},\bar{\tau})$, $\bar{\tau}$, $X^\mu(\bar{\sigma},\bar{\tau})$ are all independent.)

where $\bar{\rho}(\bar{h}) := \frac{1}{4\pi}\int d\bar{\sigma}\sqrt{\bar{h}}\bar{R}_{\bar{h}}$ ($\bar{R}_{\bar{h}}$ is the scalar curvature of $\bar{h}_{mn}$)

$$D := \int d\bar{\sigma}\bar{e}\delta_{(\mu\bar{\sigma})(\mu\bar{\sigma})} = d2\pi\delta(0) \quad \text{(index volume)}$$

$$N(X) = \frac{1}{1+v(X)} \quad \left(v(X) = \frac{\alpha}{\sqrt{d-1}}\int d\bar{\sigma}\epsilon_{\mu\nu}X^\mu\partial_{\bar{\sigma}}X^\nu\right)$$

- We derive all the perturbative string amplitudes on flat spacetime from the fluctuations around this solution.
- The solution is defined on $\mathcal{M}_D$ where the target metric is fixed to be flat .
- The equations of motion are differential equations with respect to $\bar{\tau}$, $X^\mu(\bar{\sigma},\bar{\tau})$

$\downarrow$

The functions of $\bar{h}_{mn}(\bar{\sigma},\bar{\tau})$ are constants in the solution **(determined by the consistency of the fluctuations.)**

4 Derive all order scattering amplitudes of perturbative string

Propagators around the perturbative vacuum

1. Expand the action around the perturbtive vacuum up to 2nd order:
$$G_{IJ} = \bar{G}_{IJ} + \tilde{G}_{IJ}$$
$$A_I = \bar{A}_I + \tilde{A}_I$$

2. Take $G_N \to 0$. Then, the fluctuations of the gaguge field are suppressed.

3. Take the harmonic gauge to fix diffeo. Then, the gauge fixing term is added.
$$S_{fix} = \frac{1}{G_N} \int \mathcal{D}h \mathcal{D}\bar{\tau} \mathcal{D}X \sqrt{\bar{G}} \frac{1}{2} \Big(\bar{\nabla}^J (\tilde{G}_{IJ} - \frac{1}{2} \bar{G}_{IJ} \tilde{G}) \Big)^2$$

4. Take slowly varying field limit:

derivative expansion $\begin{cases} \tilde{G}_{IJ} \to \frac{1}{\alpha} \tilde{G}_{IJ} \\ \partial_K \tilde{G}_{IJ} \to \partial_K \tilde{G}_{IJ} \\ \partial_K \partial_L \tilde{G}_{IJ} \to \alpha \partial_K \partial_L \tilde{G}_{IJ} \end{cases}$ and $\alpha \to 0$

5. Normalize to obtain canonical kinetic term: $\tilde{H}_{IJ} := Z_{IJ} \tilde{G}_{IJ}$

6. Take $D \to \infty$

Propagators around the perturbative vacuum (cont'd)

- $S + S_{fix} = \int \mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X \frac{1}{4}\tilde{H} H(-i\frac{\partial}{\partial\bar{\tau}}, -i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\tilde{H}$ +(terms do not mix with $\tilde{H}$)

  $\tilde{H}$ is one of the modes of $\tilde{H}_{d\,(\mu\bar{\sigma})}$

  $$H(p_{\bar{\tau}}, p_X, X, h) = \frac{1}{2}\frac{1}{2\lambda\bar{\rho}} p_{\bar{\tau}}^2 + \int_0^{2\pi} d\bar{\sigma} \left(\sqrt{\bar{h}}\left(\frac{1}{2}(p_X^\mu)^2 + \frac{1}{2}\bar{e}^{-2}(\partial_{\bar{\sigma}} X^\mu)^2\right) + i\bar{e}\bar{n}^{\bar{\sigma}}\partial_{\bar{\sigma}} X_\mu p_X^\mu\right)$$

  ADM decomposition $\bar{h}_{mn} = \begin{pmatrix} \bar{n}^2 + \bar{n}_{\bar{\sigma}}\bar{n}^{\bar{\sigma}} & \bar{n}_{\bar{\sigma}} \\ \bar{n}_{\bar{\sigma}} & \bar{e}^2 \end{pmatrix}$

- Differential equation for the propagator $\Delta_F(\bar{h}, \bar{\tau}, X;\ \bar{h}', \bar{\tau}', X')$

  $$H(-i\frac{\partial}{\partial\bar{\tau}}, -i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\Delta_F(\bar{h}, \bar{\tau}, X;\ \bar{h}', \bar{\tau}', X') = \delta(\bar{h} - \bar{h}')\delta(\bar{\tau} - \bar{\tau}')\delta(X - X')$$

Schwinger representation of the propagator = path integral of the perturbative strings

- In order to compare with perturbative strings,
  Take the Schwinger representation of the propagator by using the first quantization formalism.

  operators $(\hat{\bar{h}}, \hat{\bar{\tau}}, \hat{X})$ conjugate momenta $(\hat{p}_{\bar{h}}, \hat{p}_{\bar{\tau}}, \hat{p}_X)$ eigen states $|\bar{h}, \bar{\tau}, X>$

- $$H(-i\frac{\partial}{\partial\bar{\tau}}, -i\frac{1}{\bar{e}}\frac{\partial}{\partial X}, X, \bar{h})\Delta_F(\bar{h}, \bar{\tau}, X;\ \bar{h},' \bar{\tau},' X') = \delta(\bar{h}-\bar{h}')\delta(\bar{\tau}-\bar{\tau}')\delta(X-X')$$

$\Updownarrow$

$$\begin{aligned}\Delta_F(\bar{h}, \bar{\tau}, X;\ \bar{h},' \bar{\tau},' X') &= <\bar{h}, \bar{\tau}, X|\hat{H}^{-1}(\hat{p}_{\bar{\tau}}, \hat{p}_X, \hat{X}, \hat{\bar{h}})|\bar{h},' \bar{\tau},' X'> \\ &= \int_0^\infty dT <\bar{h}, \bar{\tau}, X|e^{-T\hat{H}}|\bar{h},' \bar{\tau},' X'>\end{aligned}$$

- $$\Delta_F(X;\ X') := \int_0^\infty dT < X|_{out} e^{-T\hat{H}}|X'>_{in}$$

$$< X|_{out} := \int \mathcal{D}h < \bar{h}, \bar{\tau} = \infty, X|$$

$$|X'>_{in} := \int \mathcal{D}h' |\bar{h}', \bar{\tau} = -\infty, X'>$$

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- path integral representation

$$
\begin{aligned}
&\Delta_F(X;\ X') \\
=\ &\int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X \int \mathcal{D}p_T\mathcal{D}p_{\bar{\tau}}\mathcal{D}p_X \\
&\exp\Biggl(-\int_0^1 dt\Bigl(-ip_T(t)\frac{d}{dt}T(t) - ip_{\bar{\tau}}(t)\frac{d}{dt}\bar{\tau}(t) - ip_X(t)\cdot\frac{d}{dt}X(t) \\
&+T(t)H(p_{\bar{\tau}}(t), p_X(t), X(t), \bar{h})\Bigr)\Biggr)
\end{aligned}
$$

* By introducing $p_T(t)$, constant T → field T(t)

- move onto Lagrange formalism from the canonical formalism by integrating out $p_{\bar{\tau}},\ p_X$.

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- $$\Delta_F(X;\ X')$$

$$= \int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X\mathcal{D}p_T$$

$$\exp\Biggl(-\int_0^1 dt\Bigl(-ip_T(t)\frac{d}{dt}T(t) + \lambda\bar{\rho}\frac{1}{T(t)}(\frac{d\bar{\tau}(t)}{dt})^2$$

$$+\int d\bar{\sigma}\sqrt{\bar{h}}(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_t X_\mu(\bar{\sigma},\bar{\tau},t) + \bar{h}^{01}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_\mu(\bar{\sigma},\bar{\tau},t)$$

$$+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}} X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_\mu(\bar{\sigma},\bar{\tau},t))\Bigr)\Biggr)$$

- This path integral is obtained

  if $F_1(t) := \frac{d}{dt}T(t) = 0$ gauge is chosen in the next covariant form w.r.t. t diffeo:

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- Covariant form w.r.t. t diffeo

$$
\begin{aligned}
&\Delta_F(X;\ X') \\
= \ & Z_1 \int_{X'}^{X} \mathcal{D}T\mathcal{D}h\mathcal{D}\bar{\tau}\mathcal{D}X \ \mathrm{exp}\Bigg(-\int_0^1 dt\Big(+\lambda\bar{\rho}\frac{1}{T(t)}(\frac{d\bar{\tau}(t)}{dt})^2 \\
&+\int d\bar{\sigma}\sqrt{\bar{h}}(\frac{1}{2}\bar{h}^{00}\frac{1}{T(t)}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_t X_\mu(\bar{\sigma},\bar{\tau},t) + \bar{h}^{01}\partial_t X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_\mu(\bar{\sigma},\bar{\tau},t) \\
&+\frac{1}{2}\bar{h}^{11}T(t)\partial_{\bar{\sigma}} X^\mu(\bar{\sigma},\bar{\tau},t)\partial_{\bar{\sigma}} X_\mu(\bar{\sigma},\bar{\tau},t))\Big)\Bigg)
\end{aligned}
$$

* $T(t)$ is transformed as an einbein.

- $T(t)$ disappears under $\dfrac{d\bar{\tau}}{d\bar{\tau}'} = T(t)$:

$$
\begin{aligned}
\bar{h}^{00} &= T^2\bar{h}'^{00} & \sqrt{\bar{h}} &= \frac{1}{T}\sqrt{\bar{h}'} & (\frac{d\bar{\tau}(t)}{dt})^2 &= T^2(\frac{d\bar{\tau}'(t)}{dt})^2 \\
\bar{h}^{01} &= T\bar{h}'^{01} & & & & \\
\bar{h}^{11} &= \bar{h}'^{11} & \bar{\rho} &= \frac{1}{T}\bar{\rho}' & &
\end{aligned}
$$

* This action is still invariant under the diffeomorphism with respect to t if $\bar{\tau}$ transforms in the same way as t.

- Take $\bar{\tau} = t$ gauge.

Schwinger representation of the propagator = path integral of the perturbative strings (cont'd)

- $$\Delta_F(X;\ X')$$

$$= Z\int_{X'}^{X}\mathcal{D}h\mathcal{D}X\exp\Bigg(-\int d\bar{\tau}\int d\bar{\sigma}\sqrt{\bar{h}}(\frac{\lambda}{4\pi}\bar{R}(\bar{\sigma},\bar{\tau})$$

$$+\frac{1}{2}\bar{h}^{00}\partial_{\bar{\tau}}X^{\mu}(\bar{\sigma},\bar{\tau})\partial_{\bar{\tau}}X_{\mu}(\bar{\sigma},\bar{\tau})+\bar{h}^{01}\partial_{\bar{\tau}}X^{\mu}(\bar{\sigma},\bar{\tau})\partial_{\bar{\sigma}}X_{\mu}(\bar{\sigma},\bar{\tau})+\frac{1}{2}\bar{h}^{11}\partial_{\bar{\sigma}}X^{\mu}(\bar{\sigma},\bar{\tau})\partial_{\bar{\sigma}}X_{\mu}(\bar{\sigma},\bar{\tau}))\Bigg)$$

- Diff $\times$ Weyl transformation gives

$$\Delta_F(X;\ X') = Z\int_{X'}^{X}\mathcal{D}h\mathcal{D}X e^{-\lambda\chi}e^{-S_s}$$

$$S_s = \int_{-\infty}^{\infty}d\tau\int d\sigma\sqrt{h(\sigma,\tau)}\left(\frac{1}{2}h^{mn}(\sigma,\tau)\partial_m X^{\mu}(\sigma,\tau)\partial_n X_{\mu}(\sigma,\tau)\right)$$

χ: Euler number

- We obtain the all-order perturbative scattering amplitudes that possess the moduli in the string theory, by inserting asymptotic states.

- The consistency of the fluctuations around the backgrounds $\rightarrow$ **the critical dimension d=26**.
(**d=10** in the supersymmetric cases)

5 General supersymmetric case that includes open strings

Supersymmetric generalization including open strings

So far	General
Riemann surface Σ	super Riemann surface $\mathbf{\Sigma}$ with or without boundaries
$X_{\hat{D}} : \Sigma\|_{\bar{\tau}} \to \mathbf{R}^d$	$\mathbf{X}_{\hat{D}} : \mathbf{\Sigma}\|_{\bar{\tau}} \to \mathbf{R}^d$ Boundaries have CP factors and map to D-branes
$\hat{D}$: background (B, dilaton)	$\hat{D}$: background (B, dilaton , RR, submanifolds of M that represent D-branes and O-planes gauge fields on D-branes)
model space $E := \bigcup_{\hat{D}} \{[\Sigma, X_{\hat{D}}(\bar{\tau}), \bar{\tau}]\}$	$\mathbf{E} := \bigcup_{\hat{D}_T} \{[\mathbf{\Sigma}, \mathbf{X}_{\hat{D}_T}(\bar{\tau}), \bar{\tau}]\}$ (T= IIA, IIB, I) • For T=I, Ω projected • For T=IIA (T=IIB, I), IIA (IIB) GSO projection is attached on asymptotic states * We can define GSO projection because functions over the model space are functions of ψ^μ_α $\mathbf{X}^\mu_{\hat{D}_T} = X^\mu + \bar{\theta}^\alpha \psi^\mu_\alpha + \frac{1}{2}\bar{\theta}^2 F^\mu$
index $(\mu\bar{\sigma})$	$(\mu\bar{\sigma}\bar{\theta})$

Non-perturbative formulation of superstring theory

- $$Z = \int \mathcal{D}G \mathcal{D}A e^{-S}$$

  $$S = \int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}_{\hat{D}}\sqrt{G}(-R + \frac{1}{4}G_N G^{\mathbf{I}_1\mathbf{I}_2}G^{\mathbf{J}_1\mathbf{J}_2}F_{\mathbf{I}_1\mathbf{J}_1}F_{\mathbf{I}_2\mathbf{J}_2})$$

- **The theory is background independent.**

Supersymmetry is a part of the diffeomorphisms symmetry

$$(\bar{\sigma}, \bar{\theta}^{\alpha}) \mapsto (\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\theta}'^{\alpha}(\bar{\sigma}, \bar{\theta}))$$

$$\updownarrow$$

$$[\mathbf{E}_M{}^A(\bar{\sigma}, \bar{\tau}, \bar{\theta}^{\alpha}), \mathbf{X}^{\mu}_{\hat{D}_T}(\bar{\tau}), \bar{\tau}] \mapsto [\mathbf{E}'_M{}^A(\bar{\sigma}'(\bar{\sigma}, \bar{\theta}), \bar{\tau}, \bar{\theta}'^{\alpha}(\bar{\sigma}, \bar{\theta})), \mathbf{X}'^{\mu}_{\hat{D}_T}(\bar{\tau})(\mathbf{X}_{\hat{D}_T})), \bar{\tau}]$$

- These are dimensional reductions in $\bar{\tau}$ direction of the two-dimensional $\mathcal{N} = (1, 1)$ local susy trans.
- supercharges $\xi^{\alpha} Q_{\alpha} = \xi^{\alpha} (\frac{\partial}{\partial \bar{\theta}^{\alpha}} + i \gamma^1_{\alpha\beta} \bar{\theta}^{\beta} \frac{\partial}{\partial \bar{\sigma}})$
- The number of supercharges is the same as of the two-dimensional ones.
- The supersymmetry algebra closes in a field-independent sense as in ordinary supergravities.

Derive the all order perturbative superstring scattering amplitudes

- We obtain the all-order scattering amplitudes that possess the supermoduli in the perturbative type IIA, IIB and SO(32) type I superstring,
  if we consider the fluctuations after fixing IIA, IIB and SO(32) type I charts, respectively.

- **These amplitudes are derived from the single theory**.

- The consistency of the fluctuations around the backgrounds $\longrightarrow$ **d=10**

- We obtain amplitudes of the superstrings with Dirichlet and Neumann boundary conditions in the normal and tangential directions to the D-submanifolds, respectively.

$\downarrow$

D-submanifolds represent D-brane backgrounds where back reactions from the D-branes are ignored.

6 String geometry and a new type of supersymmetric matrix models

String geometry and a new type of supersymmetric matrix models

Gravity and a matrix moldel (Hanada-Kawai-Kimura 2006)

Equations of motion of $S_e = \frac{1}{G_N} \int d^{10}x \sqrt{g}(-R + \frac{1}{4} G_N F_{\mu\nu} F^{\mu\nu})$

↕ equivalent

Equations of motion of $S_m = tr(-[A_\mu, A_\nu][A^\mu, A^\nu])$ where we replace $A_\mu \equiv \nabla_\mu$

String geometry and a matrix model

Equations of motion of $S = \int \mathcal{D}\mathbf{E}\mathcal{D}\bar{\tau}\mathcal{D}\mathbf{X}\sqrt{G}(-R + \frac{1}{4} G_N G^{\mathbf{I}_1\mathbf{I}_2} G^{\mathbf{J}_1\mathbf{J}_2} F_{\mathbf{I}_1\mathbf{J}_1} F_{\mathbf{I}_2\mathbf{J}_2})$

↕ equivalent

Equations of motion of $S_M = \int \mathcal{D}\mathbf{E} tr(-[A_\mathbf{I}(\mathbf{E}), A_\mathbf{J}(\mathbf{E})][A^\mathbf{I}(\mathbf{E}), A^\mathbf{J}(\mathbf{E})])$ where we replace $A_\mathbf{I} \equiv \nabla_\mathbf{I}$

↑ (extended) large N reduction ?

More simple

$S_{M_0} = tr(-[A_\mathbf{I}, A_\mathbf{J}][A^\mathbf{I}, A^\mathbf{J}])$ (a supersymmetric matrix model that has ∞ indices $\mathbf{I} = (d, (\mu\bar{\sigma}\bar{\theta}))$)

is interesting.

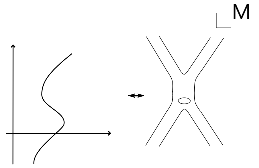
Worldsheets can be derived in general by perturbations of matrix models

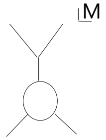
7 Unification of particles and the space-time

Unification of space-time and particles

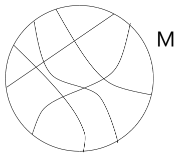
- space-time and string geometry

asymptotic trajectory on $\mathfrak{M}_D$ with target M = string world-sheet in M $\xrightarrow[\text{macro}]{}$ trajectory of a particle in M





Space-time M is identified by: observing all trajectories of a particle in M.



$\therefore$ $\mathfrak{M}_D$ is observed as M macroscopically.

Conversely, we see a string, if we microscopically observe a point of the space-time.

- particle and string geometry

A fluctuation of $\mathfrak{M}_D$ = string $\xrightarrow[\text{macro}]{}$ particle

Conversely, we see a string, if we microscopically observe a particle.

- **unification of space-time and particle**

Macroscopically, space-time = string manifold particle = a fluctuation of string manifold

Appendix

$$
\begin{aligned}
\|X_{\hat{D}}(\bar{\tau}) - X_{\hat{D}0}(\bar{\tau}_0)\|^2 &= \int_0^{2\pi} d\bar{\sigma}(X^\mu(\bar{\tau},\bar{\sigma}) - X_0^\mu(\bar{\tau}_0,\bar{\sigma}))^2 \\
&+ \int_0^{2\pi} d\bar{\sigma}(\bar{\psi}^\mu(\bar{\tau},\bar{\sigma}) - \bar{\psi}_0^\mu(\bar{\tau}_0,\bar{\sigma}))(\psi^\mu(\bar{\tau},\bar{\sigma}) - \psi_0^\mu(\bar{\tau}_0,\bar{\sigma})) \\
&+ \int_0^{2\pi} d\bar{\sigma}(F(\bar{\tau},\bar{\sigma}) - F_0(\bar{\tau}_0,\bar{\sigma}))^2
\end{aligned}
$$