Symmetries in QFTs and applications

Yuya Tanizaki (Yukawa institute, Kyoto)

Solving Quantum Field Theories

UV ← microscopic description
- Quantum Chromodynamics
- (Lattice) quantum many-body systems
- Conformal field theories (CFT_{UV})

Renormalization Group (RG) flow

IR ← macroscopic description
- Nonlinear σ-model (chiral Lagrangian)
- Fermi liquid, spin liquid (TQFT), ...
- CFT_{IR}.

This is a very tough problem!

Power of Symmetry

Sometimes, we can know about low-energy dynamics using symmetry **without solving microscopic Hamiltonian.**

<u>e.g.</u> In '60s, people didn't know about <u>Quantum Chromodynamics (QCD)</u>.
fundamental theory of strong interaction.

current algebra (chiral effective Lagrangian)

$\Longrightarrow$ successful description of low-energy properties of strong interaction

Why this was possible?

$\boxed{\text{Universality due to SSB of chiral symmetry.}}$

Continuous Symmetry in QFT

Noether : If classical action $S[\phi]$ is invariant under continuous transformation,

then there is a conserved current J^μ :

$$\partial_\mu J^\mu = 0.$$

$$\Downarrow$$

In QFT, this becomes Ward-Takahashi identity :

$$\langle \partial_\mu J^\mu(x) \; \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \sum_i \delta(x-x_i) \langle \mathcal{O}_1(x_1) \cdots \delta\mathcal{O}_i(x_i) \cdots \mathcal{O}_n(x_n) \rangle.$$

$$\Downarrow$$

Various theorems related to symmetry.

(such as Nambu-Goldstone theorem, 't Hooft anomaly matching, etc.)

WT identity provides a powerful tool to analyze QFTs.

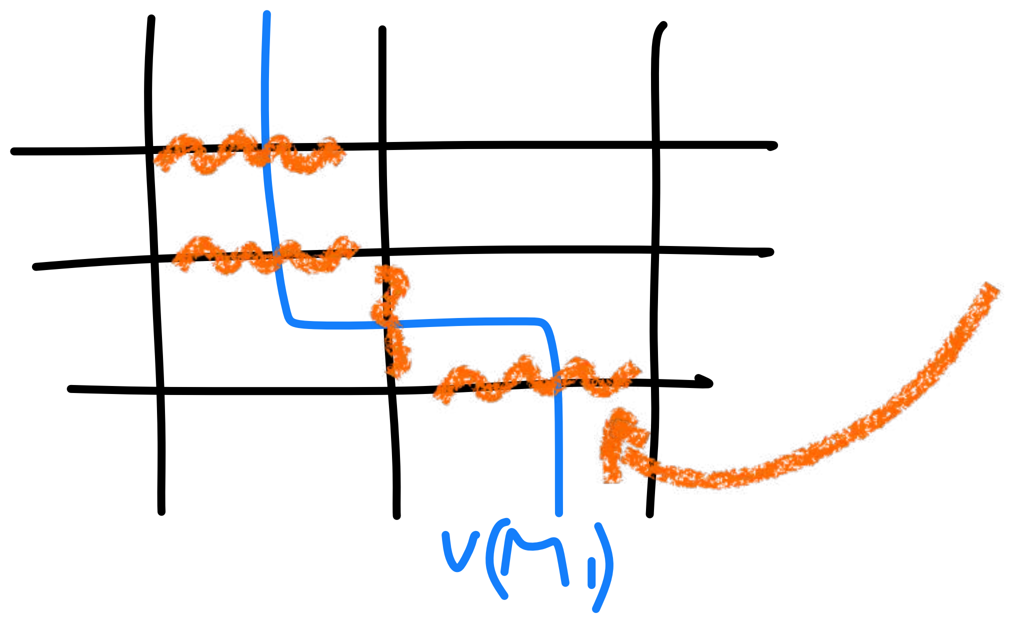
$\Rightarrow$ We'd like to extend its scope!

$$\boxed{\begin{array}{c}\text{Generalization of Symmetry}\\ \parallel \\ \text{Generalization of Ward-Takahashi identity}\end{array}}$$

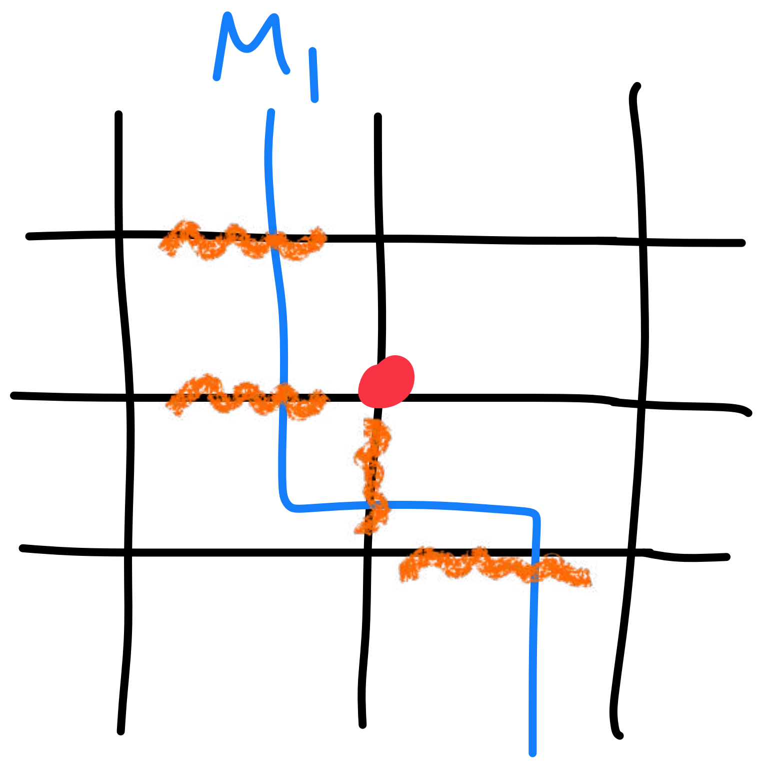
Example WT-type identity for $\mathbb{Z}_2$ symmetry of Ising model

$$H_{\text{Ising}} = J \sum_{\langle x, x' \rangle : \text{nearest neighbors}} s(x)\, s(x') \qquad (s(x) = \pm 1)$$

$V(M_1)$: defect operator



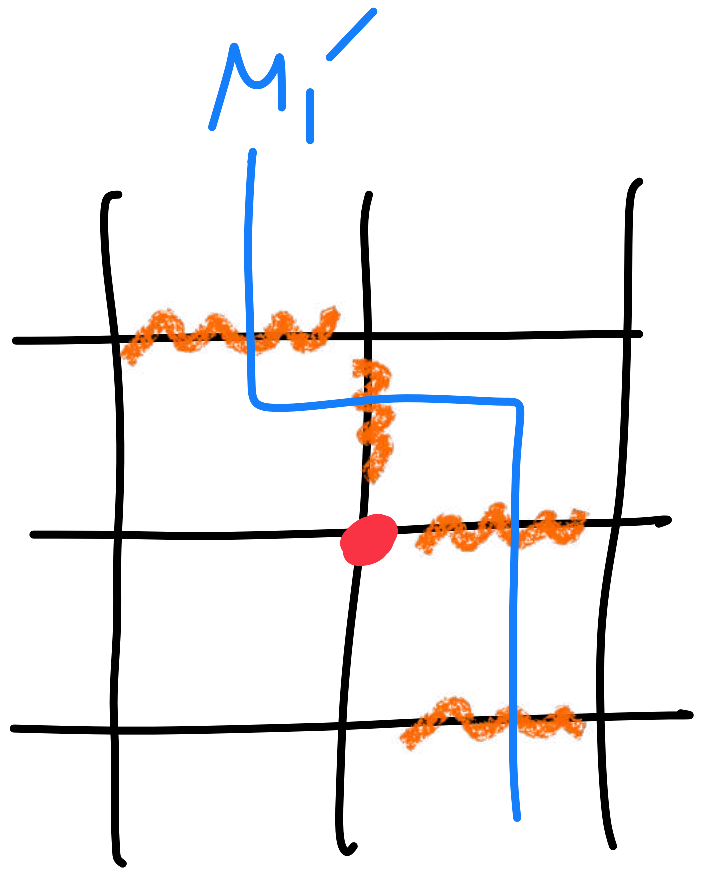
For these $\langle x, x' \rangle$,
change $J \to -J$ in the Hamiltonian.



$\mathbb{Z}_2$ operation for $S(x)$ at ● :

$s \to -s$

$\Longrightarrow$



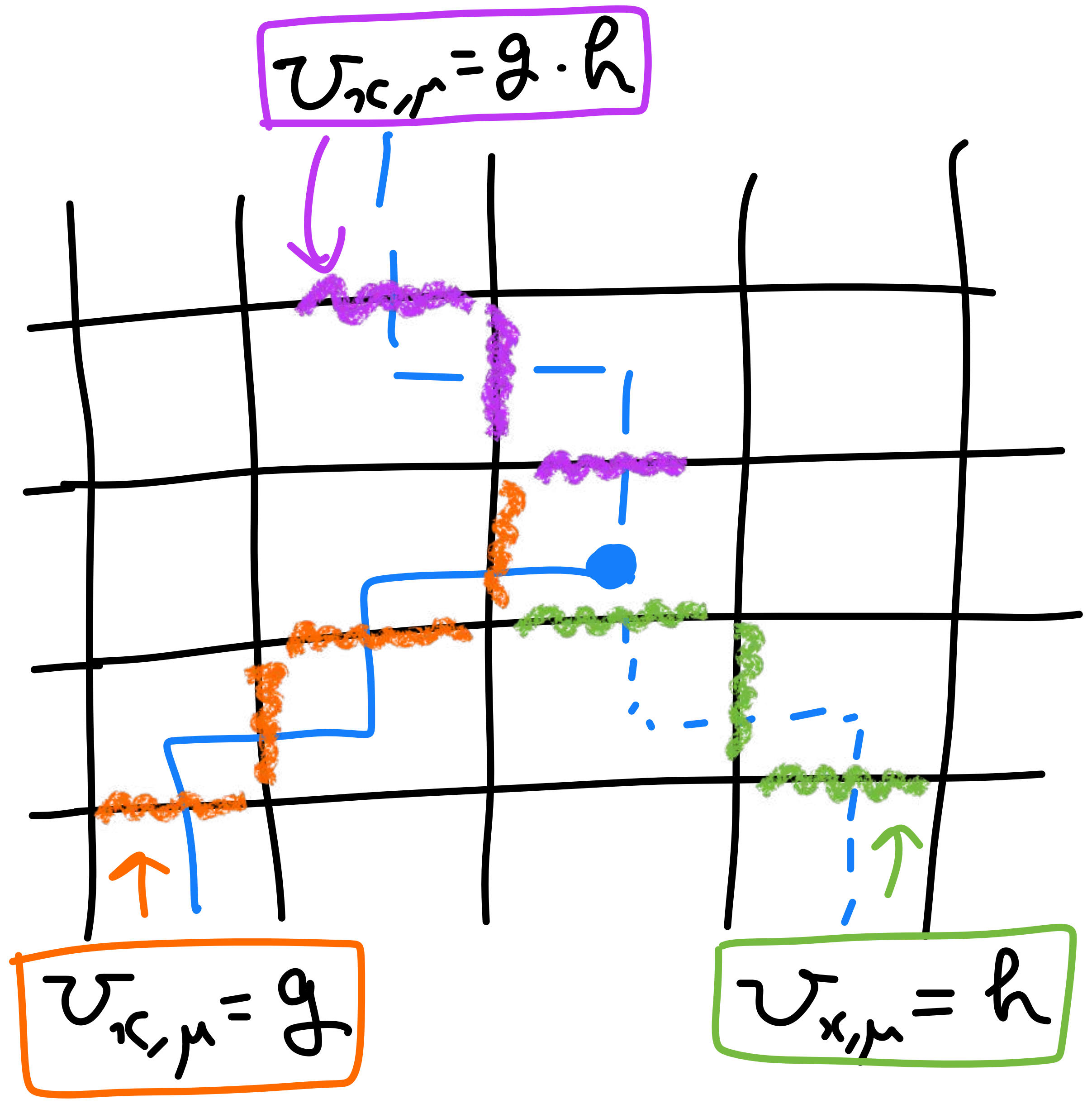
WT-type identity

$$\langle V(M_1) \cdots \rangle$$
$$= \langle V(M_1') \cdots \rangle.$$

Network of topological defects

<u>Model</u> $$S[\phi] = \kappa \sum_{x,\mu} \left(\phi^\dagger_{x+\mu} \cdot \phi_x + \phi^\dagger_x \cdot \phi_{x+\mu} \right) + V[\phi] \qquad (\phi_x \in \mathbb{C}^N)$$

Assume that G is a symmetry (For $g \in G$, $\phi_x \mapsto R(g)\cdot\phi_x$ w/ $R(g) \in U(N)$).



(Network of topological defects (consistent with fusion rule))

$\Longleftrightarrow$

$$S_{\text{twist}} = \kappa \sum_{x,\mu} \left(\phi^\dagger_{x+\mu} R(U_{x,\mu}) \phi_x + \phi^\dagger_x \cdot R(U_{x,\mu})^\dagger \cdot \phi_{x+\mu} \right) + V[\phi]$$

with the twisted hopping (i.e. "link variable")

$U_{x,\mu} \in G$.

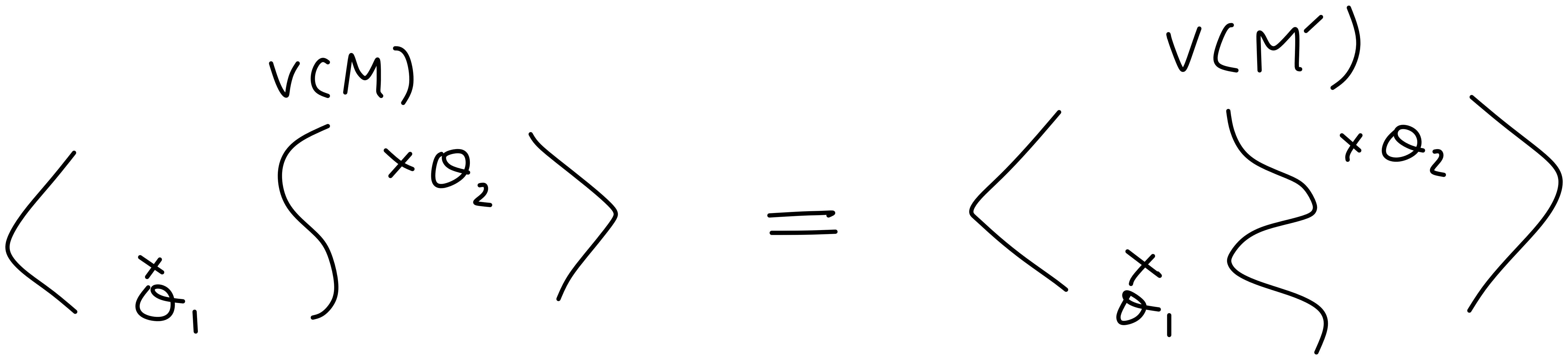
(Background G gauge field (with flatness condition))

Modern Definition of (Generalized) Symmetry

Take-home message

$$\boxed{\text{Symmetry} = \text{Topological defect operators}}$$

<u>Topological</u>



$\Longleftrightarrow$ Conservation Law

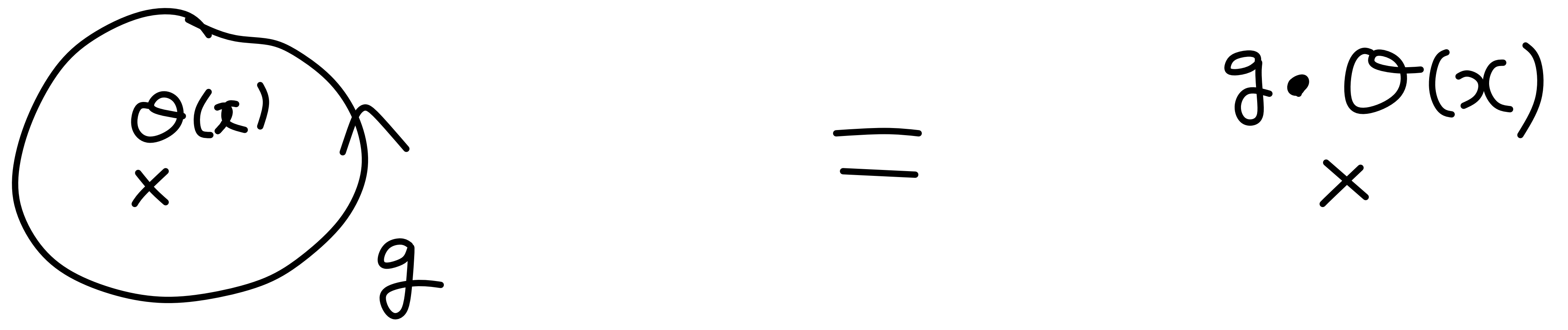
Ordinary Symmetry in Modern Viewpoints

d-dim. QFT has a global symmetry G.

$\overset{\text{def}}{\Longleftrightarrow}$

- $\exists\, V_g(M_{d-1})$: topological codim-1 defect operator for each $g \in G$

- 

- 

(Valid for both continuous and discrete symmetries)

Various generalizations

- **p-form symmetry** (Gaiotto, Kapustin, Seiberg, Willet '14)

  Topological defects have codim-$(p+1)$: $V_g(M_{d-p-1})$.

  (* Esp., 1-form symmetry generalizes the center sym. in gauge theories.)

- **n-group symmetry** (Sharpe '15, Cordova, Dumitrescu, Intriligator '17, YT, Ünsal '19, Hidaka, Nitta, Yokokura '20, …)

  $\approx$ Mixture of 0-, 1-, …, (n-1)-form symmetries.

- **non-invertible symmetry** (Bhardwaj, Tachikawa '17, … in 2d QFTs.
  2d ~ higher dim Nguyen, YT, Ünsal, Koide, Nagoya, Yamaguchi, Choi, Cordova, Hsin, Lam, Shao, Kaidi, Ohmori, Zheng, …)

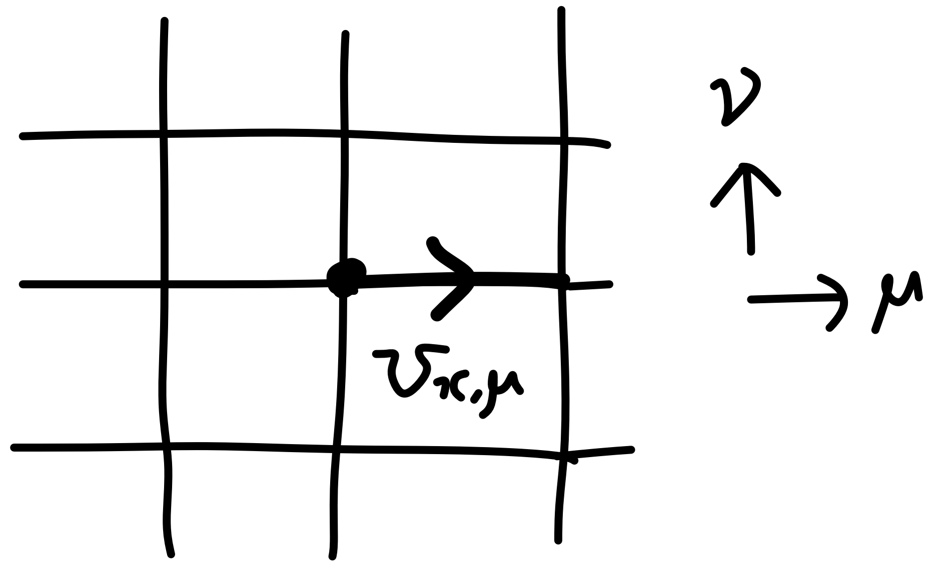
  Transformation rule does not form a group

  (We'll discuss more later.)

1-form symmetry in gauge theories

$\mathbb{Z}_N$ 1-form symmetry of $SU(N)$ Yang-Mills theory

↖ Sophisticated version of "center symmetry"



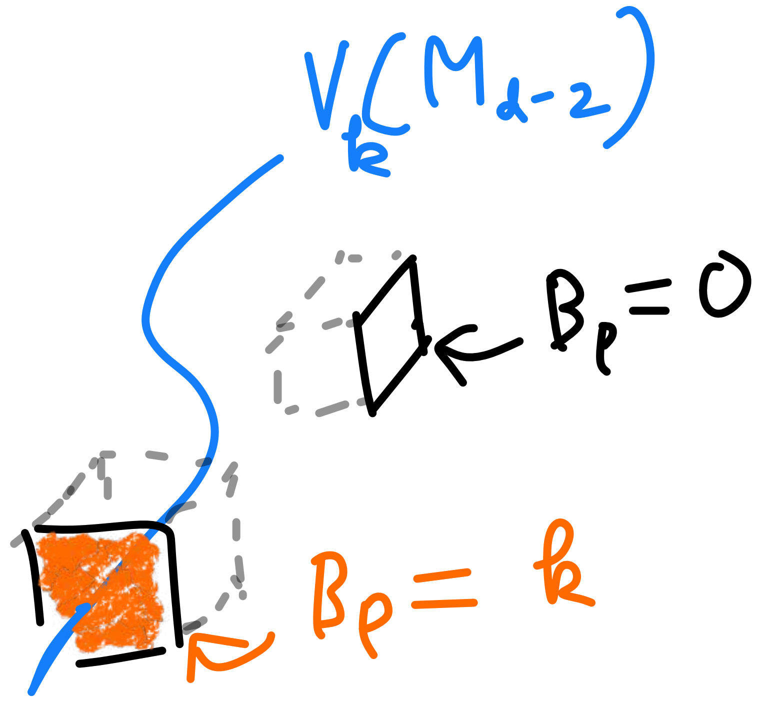
Wilson's lattice formulation

$$\begin{cases} U_{x,\mu} \in SU(N) & \text{link variable} \\ U_p = \mathcal{P} \prod_{(x,\mu)\in p} U_{x,\mu} & \text{plaquette} \end{cases}$$

$$S[U] = -K \sum_p \left(\mathrm{tr}(U_p) + \mathrm{tr}(U_p^\dagger) \right).$$

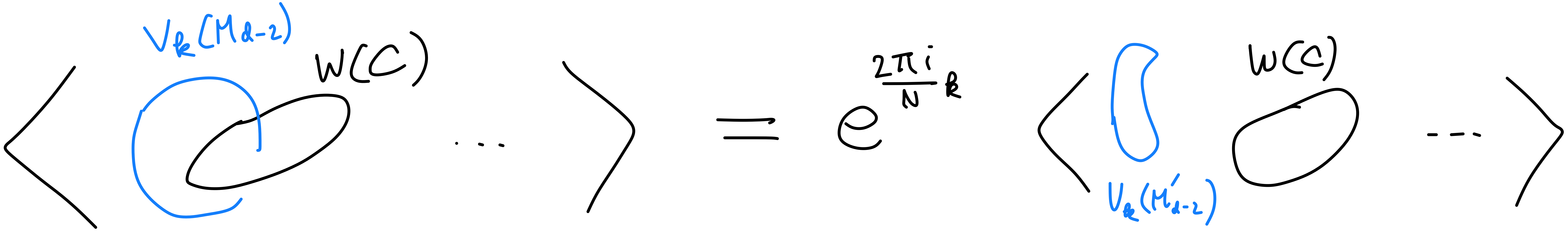
This theory has codim-2 $\mathbb{Z}_N$ topological defect : $\mathbb{Z}_N^{[1]}$ symmetry!

$$S_{twist}[U, B] = -K \sum_p \left(e^{\frac{2\pi i}{N} B_p} \mathrm{tr}(U_p) + e^{-\frac{2\pi i}{N} B_p} \mathrm{tr}(U_p^\dagger) \right).$$



(By performing the $\mathbb{Z}_N$ transformation, $U_{x,\mu} \to e^{\frac{2\pi i k}{N}} U_{x,\mu}$, $V_k(M_{d-2})$ can be deformed continuously.)

$\mathbb{Z}_N^{[1]}$ detects the N-ality charge of Wilson loops:



<u>SSB</u>
(Wilson's criterion)

$$\begin{cases} \langle W(C) \rangle \sim e^{-\sigma \cdot \mathrm{Area}(C)} & \Longleftrightarrow \quad \mathbb{Z}_N^{[1]} \text{ is unbroken.} \\ \langle W(C) \rangle \sim e^{-\mu \cdot \mathrm{Length}(C)} & \Longleftrightarrow \quad \mathbb{Z}_N^{[1]} \text{ is spontaneously broken.} \end{cases}$$

(This generalizes the off-diagonal long range order (ODLRO).)

<u>Gauging</u> Background gauge field $\equiv$ Network of codim-2 $\mathbb{Z}_N$ defects.

(This is called a $\mathbb{Z}_N$ 2-form gauge field.)

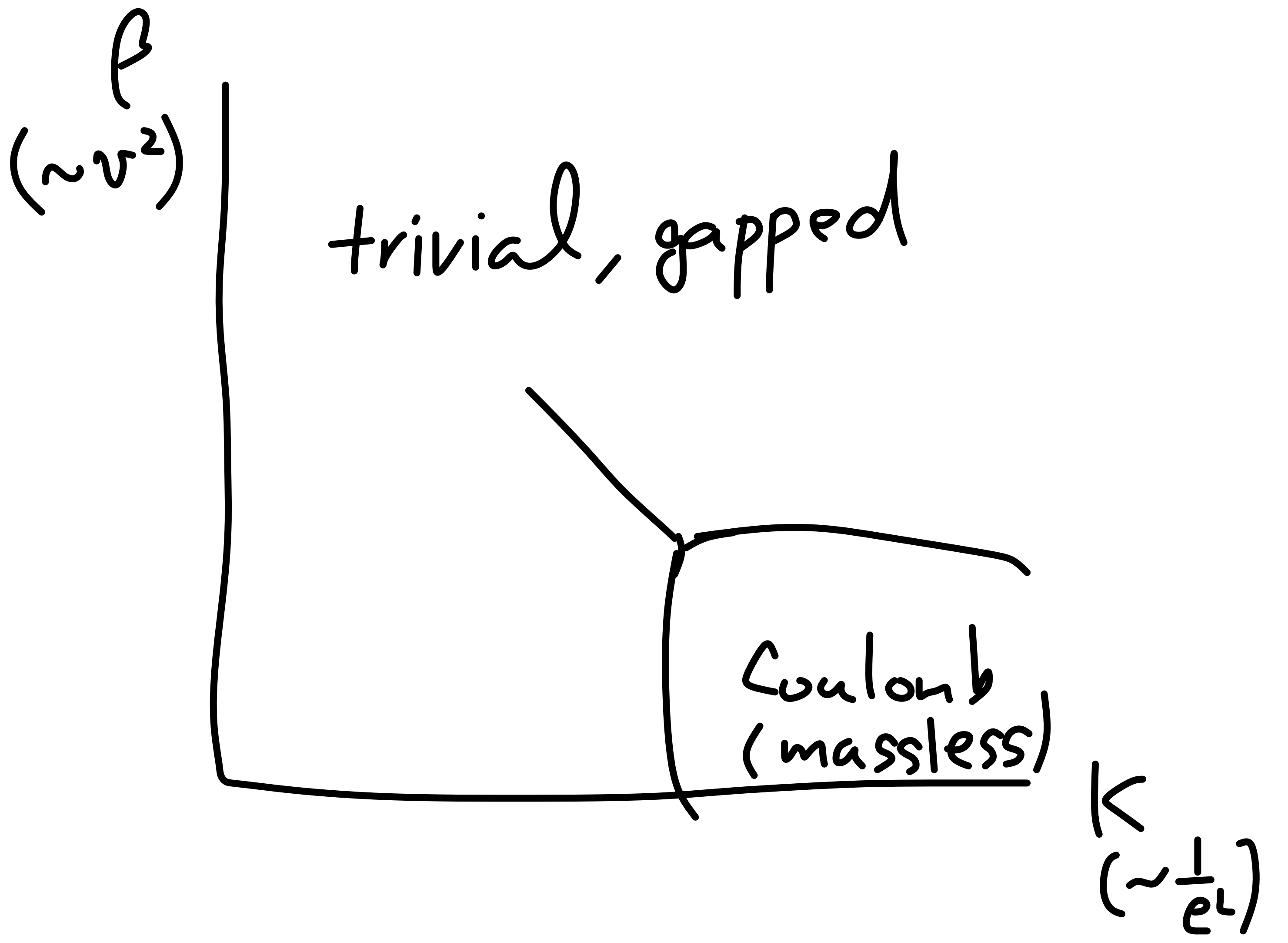
Application 1 — Fradkin–Shenker revisited (Application of 1-form symmetry)

They consider charge-N $U(1)$-Higgs model on a lattice

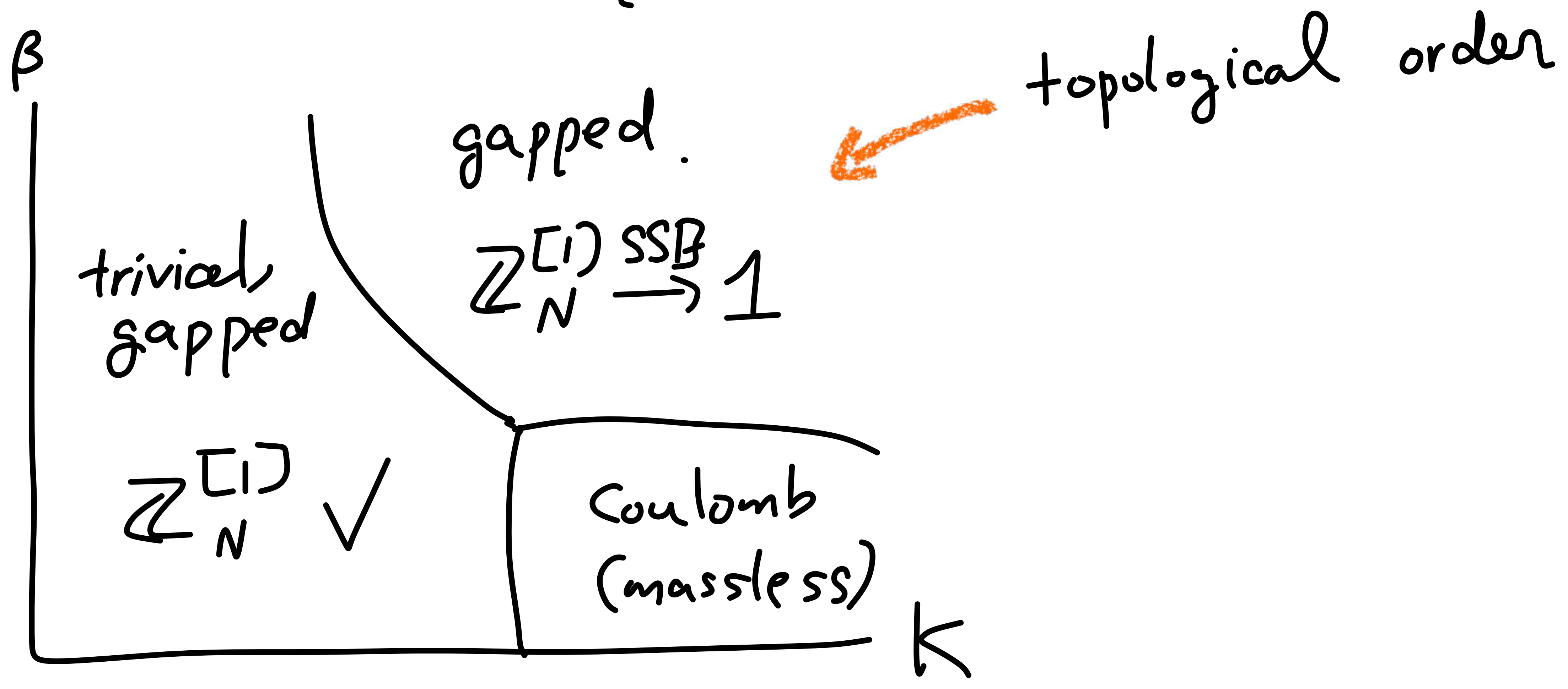
$$S = \beta \sum_{i,\mu} \cos(\partial_\mu \theta + N a_\mu) + K \sum_{\square} \cos(f_{\mu\nu})$$

$$\left(\leftrightarrow \quad S = \frac{1}{2e^2}\int |da|^2 + \underbrace{\int \left\{ |(\partial_\mu + iNa_\mu)\phi|^2 + \right\}}_{U(1)_E^{[1]} \xrightarrow{\text{explicit}} \mathbb{Z}_N^{[1]}} + \underbrace{\text{monopoles}}_{U(1)_M^{[1]} \xrightarrow{\text{explicit}} \times} \right).$$

$\underline{N = 1}$ (No symmetry)



$\underline{N \geq 2}$ ($\mathbb{Z}_N^{[1]}$ symmetry)



Application 2 Spontaneous CP breaking for YM @ $\theta=\pi$

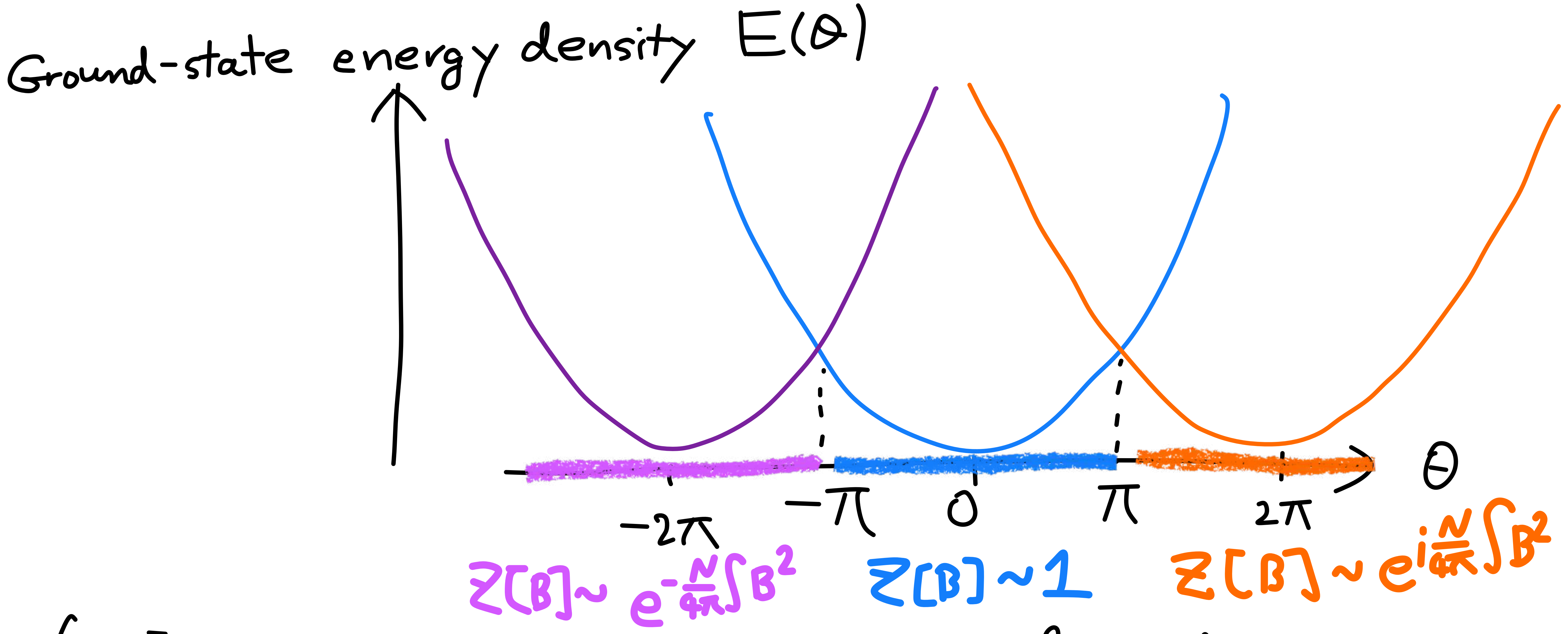
(Gaiotto, Kapustin, Komargodski, Seiberg '17)

$\boxed{\text{YM confining } \underline{\textbf{vacua}} \text{ have a rich structure as SPT states with } \mathbb{Z}_N^{[1]}.}$

B : $\mathbb{Z}_N$ 2-form gauge field (= Background gauge field for $\mathbb{Z}_N^{[1]}$)

$$Z_{\theta+2\pi}[B] = \underbrace{e^{i\frac{N}{4\pi}\int B\wedge B}} \times Z_\theta[B].$$

2π-periodicity of θ is violated by a local counterterm of B.



(→ For extension to the case with fundamental quarks, see Kikuchi, YT '17, Shimizu, Yonekura '17, etc.)

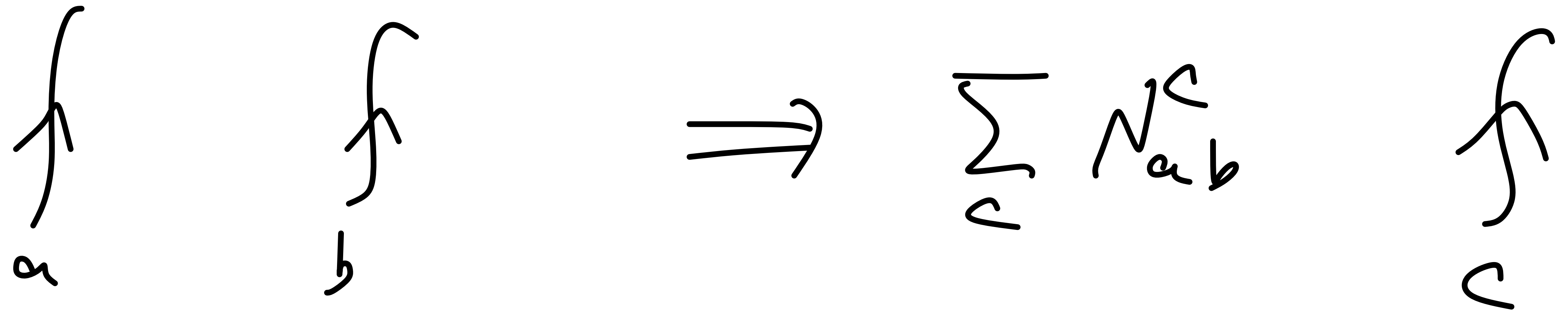
Non-invertible symmetries

Noninvertible Symmetry

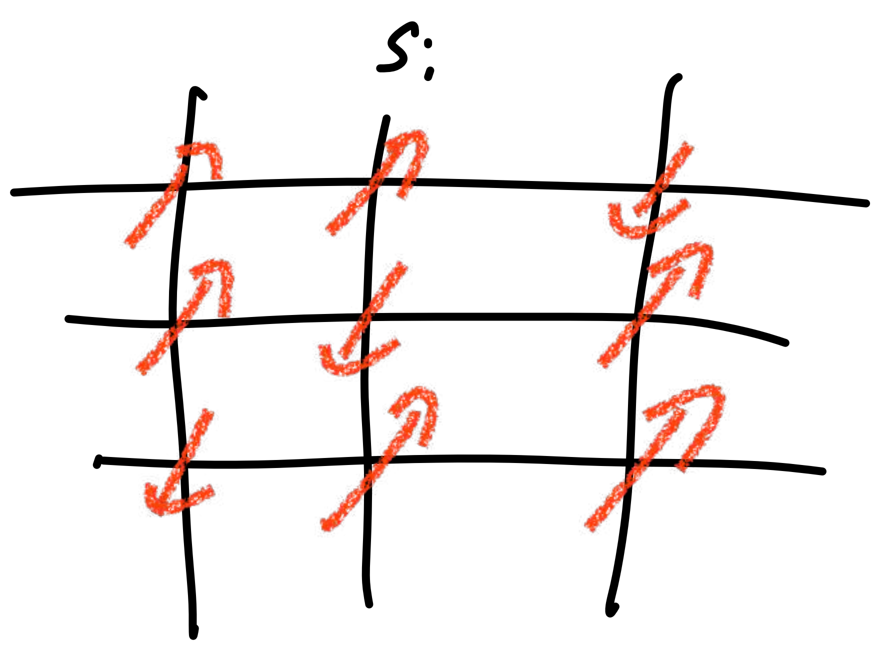
Usually, symmetry forms a group:

Is this necessary?

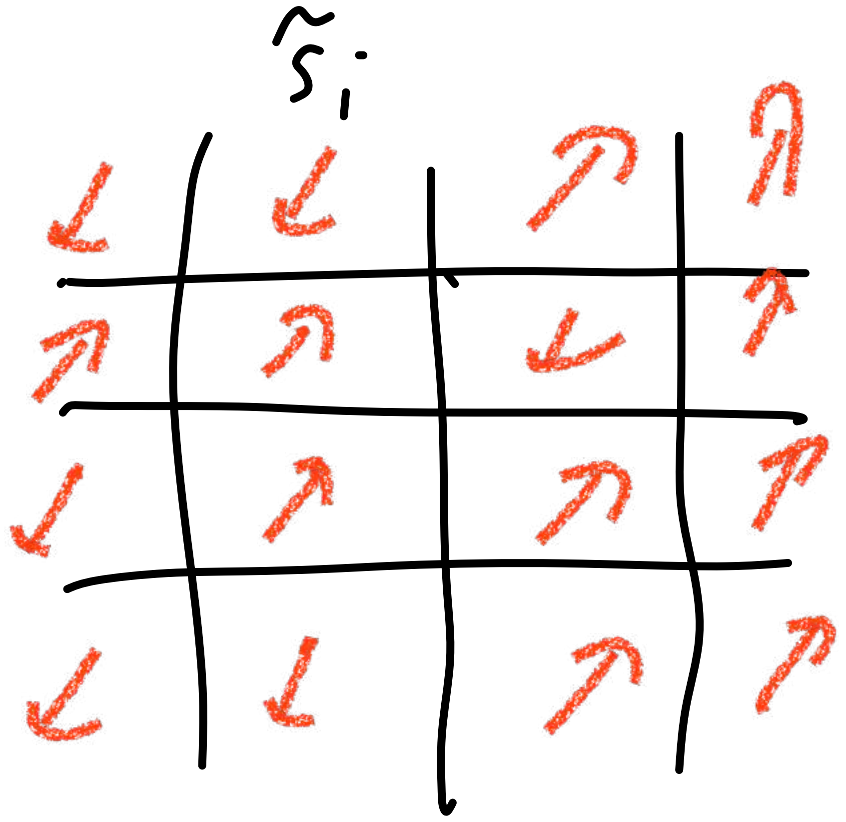
More general fusion rule: "noninvertible" (or "categorical") symmetry



Kramers - Wannier duality of 2d Ising model



Dual $\longleftrightarrow$



$e^{J \sum s_i s_j}$ $\qquad$ $e^{\tilde{J} \sum \tilde{s}_i \tilde{s}_j}$

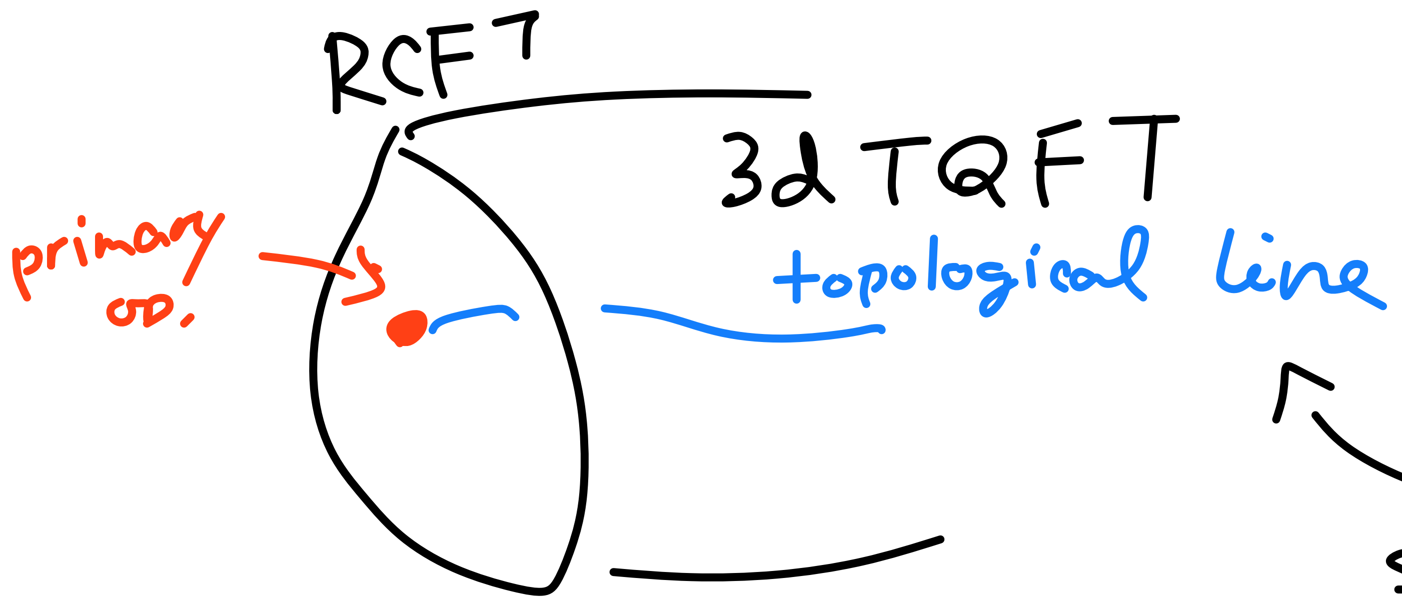
This exchanges the original spin & dual spin.

$$s(x) \longleftrightarrow \tilde{s}(x).$$

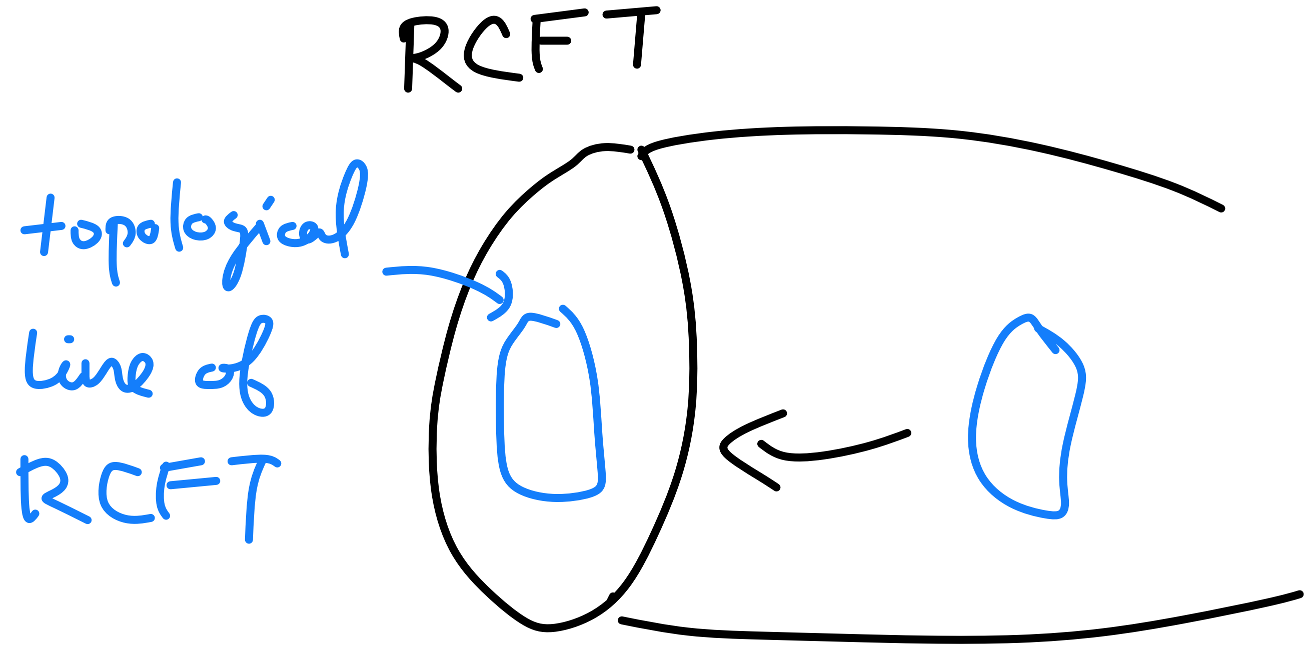
Not mutually local

what's this trans?

A. Non-invertible symmetry! (← described by fusion category)



same object
in higher-dim viewpoint



(cf. Verlinde '88)

2d Ising CFT

Primaries : $\underset{\text{trivial}}{1}$, $\underset{\text{energy}}{\varepsilon}$, $\underset{\text{spin}}{\sigma}$

Operator Product Expansion
$$\begin{cases} \sigma \times \sigma = 1 + \varepsilon \\ \sigma \times \varepsilon = \sigma \\ \varepsilon \times \varepsilon = 1 \end{cases}$$

Topological line : 1 , $\underset{\mathbb{Z}_2 \text{ sym}}{\eta}$, $\underset{\text{KW duality}}{D}$

$$\begin{cases} D \times D = 1 + \eta \\ D \times \eta = D \\ \eta \times \eta = 1 \end{cases}$$

(Lattice construction : Aasen, Fendley, Mong $\xrightarrow{\text{extension to 4d}}$ Koide, Nagoya, Yamaguchi; Choi, Cordova, et al. ; Kaidi, Ohmori, Zheng ...)

Non-invertible 1-form symmetry

(Nguyen, YT, Ünsal ; Heidenreich, McNamara, Montero, Reece, Rudelius, Valenzuela, . . .)

Pure $U(1)$ gauge theory has $U(1)^{[1]}$ and $(\mathbb{Z}_2)_c$ (charge conjugation).

$\xRightarrow{\text{Gauging } (\mathbb{Z}_2)_c}$ Gauge group becomes $\underline{O(2) = U(1) \rtimes \mathbb{Z}_2}$.

↑ Center of $O(2)$ is just $\mathbb{Z}_2$, which is much smaller than $U(1)$.

Do we lose most part of 1-form symmetry by gauging $(\mathbb{Z}_2)_c$?

A. They survive as non-invertible symmetry!

$U_\alpha(M_{d-2})$: $U(1)$ 1-form symmetry generator in $U(1)$ gauge theory

$\Downarrow$ $(\mathbb{Z}_2)_c$ gauging

$U_\pi(M_{d-2})$: $\mathbb{Z}_2^{[1]}$ defect for $O(2)$ gauge theory

$U'_\alpha(M_{d-2}) = U_\alpha + U_{-\alpha}$: Non-invertible 1-form symmetry generators of $O(2)$ gauge theory.

$$\left(U'_\alpha U'_\beta = U'_{\alpha+\beta} + U'_{\alpha-\beta} \right)$$

[In 3d $O(2)$ gauge theory with monopoles, selection rule of confining strings obey this non-inv. symmetry.]

Summary

Take-home message

$$\boxed{\text{Symmetry} = \text{Topological defect operators}}$$

$\Rightarrow$ New aspects of strongly-coupled QFTs.