National Synchrotron Light Source II

Brookhaven National Laboratory

U.S. DEPARTMENT OF ENERGY

Coupled-bunch Instability for Arbitrary Multibunch Configurations

Gabriele Bassi

Mini-workshop on Impedance Modeling and Impedance Effects at SuperKEKB and Future Colliders, December 15-16, 2022, Tokyo, Japan

Outlook

- Theoretical Analysis
- Impedance Model
- Closed Formulas
- Benchmarking Theory with Measurements

PHYSICAL REVIEW ACCELERATORS AND BEAMS **25,** 014402 (2022)

Coupled-bunch instability for arbitrary multibunch configurations

Gabriele Bassi,* Alexei Blednykh,† and Victor Smaluk‡
Brookhaven National Laboratory, Upton, New York 11973-5000, USA

(Received 22 March 2021; accepted 6 December 2021; published 25 January 2022)

The coupled-bunch instability for arbitrary multibunch configurations is investigated in its generality. The theoretical framework adopted is based on the general eigenvalue analysis based on the known formulas of the complex frequency shifts for the uniform multibunch configuration case. Closed formulas are derived for special cases. For the configuration consisting of a uniform filling pattern with a gap of missing bunches, the theoretical results are found to be in good agreement with measurements of the transverse coupled-bunch instability driven by the resistive wall impedance performed at the NSLS-II storage ring.

DOI: 10.1103/PhysRevAccelBeams.25.014402

TRANSVERSE EIGENANALYSIS FROM h-COUPLED VLASOV EQNS*

The transverse phase space densities $\Psi_m(\tau, \delta, x, p_x, t)$, associated to the $m-th$ bunch with N_m particles satisfy the following system of $h-$ coupled Vlasov equations for $0 \le m \le h-1$

$$\frac{\partial \Psi_m}{\partial t} - \eta\delta\frac{\partial \Psi_m}{\partial \tau} + \frac{\omega_s^2}{\eta}\tau\frac{\partial \Psi_m}{\partial \delta} + p_x\frac{\partial \Psi_m}{\partial x} - \omega_\beta^2 x\frac{\partial \Psi_m}{\partial p_x}$$

$$-A_x\sum_{k=0}^{+\infty}\sum_{m'=0}^{h-1} N_{m'}\left[\int_{-\infty}^{\tau} d\tau' \bar{W}_1\left(\tau-\tau'+a_{m'm}^k T_0\right)\int_{-\infty}^{+\infty} dx' x' \rho_{m'}\left(\tau', x', t-a_{m'm}^k T_0\right)\right]\frac{\partial \Psi_m}{\partial p_x} = 0,$$

where $\rho_m(\tau, x, t) = \int d\delta \int dp_x \Psi_m(\tau, \delta, x, p_x, t)$, $a_{m'm}^k = k + (m'-m)/h$ and $A_x = e\,\omega_\beta/(T_0 E_0)$.

The equation for the evolution of the dipole moments $\langle x_m \rangle = \int d\tau\, d\delta\, dx\, dp_x\, x\Psi_m$ and $\langle p_{x_m} \rangle = \int d\tau\, d\delta\, dx\, dp_x\, p_x \Psi_m$ can be found by integrating by parts the Vlasov equations using the boundary conditions for Ψ_m

$$\langle \ddot{x}_m \rangle + \omega_\beta^2 \langle x_m \rangle = -A_x \sum_{k=0}^{+\infty}\sum_{m'=0}^{h-1} f\left(a_{m'm}^k T_0\right)\, N_{m'}\langle x_{m'} \rangle\left(t - a_{m'm}^k T_0\right),$$

where $f(x) \equiv \int d\tau \int d\tau' \bar{W}_1(\tau - \tau' + x)\lambda(\tau)\lambda(\tau')$.

* G. Bassi et al., PRAB **19** 024401 (2016)

Eigenvalue Equation for Arbitrary Filling Patterns

Omit the brackets, i . e . understand x_m as $\langle x_m \rangle$, and define the coupled − bunch mode $\tilde{x}_\mu$ by

$$\tilde{x}_\mu(t) = \sum_{m=0}^{h-1} x_m(t) e^{-i2\pi m\mu/h}, \quad x_m(t) = \frac{1}{h}\sum_{\mu=0}^{h-1} \tilde{x}_\mu(t) e^{i2\pi m\mu/h},$$

it follows the modes $\tilde{x}_\mu$ are coupled and satisfy the equations of motion

$$\ddot{\tilde{x}}_\mu(t) + \omega_\beta^2 \tilde{x}_\mu(t) = -\frac{A_x}{h}\sum_{k=0}^{\infty} f\left(k\frac{T_0}{h}\right) e^{i2\pi\mu k/h} \sum_{\mu'=0}^{h-1} \tilde{x}_{\mu'}\left(t - k\frac{T_0}{h}\right) \sum_{m=0}^{h-1} N_m e^{i2\pi m(\mu'-\mu)/h}.$$

Assume a perturbative solution of the form: $\tilde{x}_\mu(t) = a_\mu e^{-i(\omega_\beta+\Omega)t}, \quad |\Omega| \ll \omega_\beta, \quad \Omega \in \mathbb{C}$.

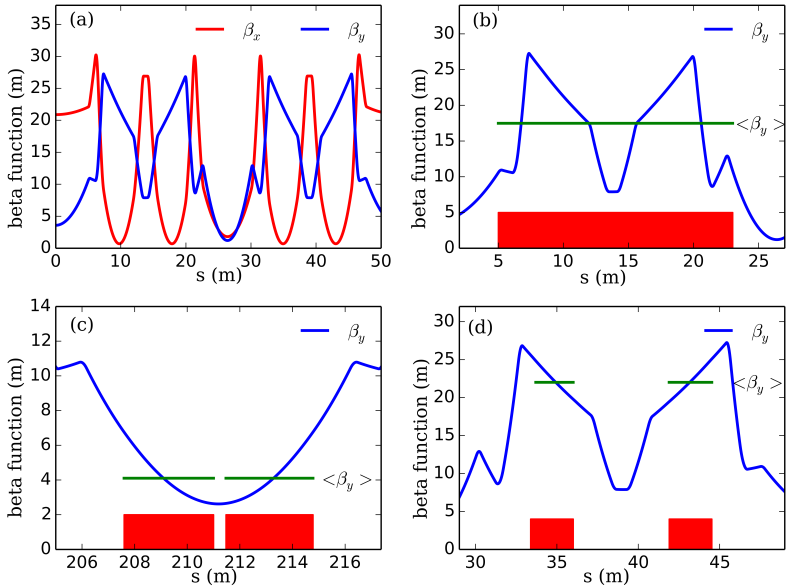
It follows the following eigenvalue equation for arbitrary multibunch configurations

$$(\mathbf{B} - \Omega\mathbf{I})\mathbf{a} = 0, \quad B_{\mu\mu'} = \frac{\Omega_\mu^U}{N_T}\sum_{m=0}^{h-1} N_m e^{i2\pi m(\mu'-\mu)/h}, \quad \mathbf{a} = [a_0, \ldots, a_{h-1}]^T,$$

where Ω_μ^U are the eigenvalues of the uniform filling patten case ($B_{\mu\mu'} = \Omega_\mu^U$ if $\mu' = \mu$, 0 otherwise)

$$\Omega_\mu^U = -i\frac{I_0}{2T_0E_0}\sum_{p=-\infty}^{+\infty} \left|\tilde{\lambda}\left((ph+\mu)\omega_0 + \omega_\beta\right)\right|^2 \bar{Z}_1^\perp\left[(ph+\mu)\omega_0 + \omega_\beta\right].$$

Resistive Wall Impedance



One superperiod of the NSLS-II DBA lattice consisting of two cells with short and long straight sections. Vertical beta function (magenta trace) variation and its average (green trace) over the length of the impedance element (red rectangle): (b) aluminum vacuum chamber in the first half superperiod; (c) DWs in cell 8; (d) bending magnet chamber in cell 1.

TABLE I. NSLS-II resistive wall impedance.

Material	Symbol	σ_c (MS/m)
Electrical conductivity of resistive wall components		
Stainless steel	SS	1.35
Aluminum	Al	31.6
Inconel	Inc	0.775

Main contributions from arcs

Element	Material	L (m)	$\bar{\beta}_y$ (m)	b (mm)	α/K (%)
CH1	Al	536.4	17.6	12.5	70.8
CH2	SS	11.5	8	12.5	3.3

Main contributions from straight sections

Cell	Element	Material	L (m)	$\bar{\beta}_y$ (m)	b (mm)	α/K (%)
1	CH	Al	5.2	22	12.5	6.2[a]
1	FC	Inc	0.37	8.3	12.5	4.4[b]
6	CH	Al	5.2	22	12.5	3.4[c]
6	FC	Inc	0.37	8.7	12.5	4.7[d]
8	DW	Al	7.5	4	5.75	7.2[e]

[a]Including the contribution from cells 9, 13, 15, 25, 27, and 29, which are identical to cell 1.
[b]Total contribution from all odd cells, which have two FCs each.
[c]Including the contribution from cells 14, 20, and 26, which are identical to cell 6.
[d]Total contribution from all even cells, which have two FCs each.
[e]Including the contribution from cells 18 and 28, which are identical to cell 8.

For one component (thick wall formula)

$$Z_1^{\perp}(\omega) = (1 - i\,\mathrm{sgn}(\omega))\frac{Z_0 s_0 L\sqrt{s_0 c\,|\omega|}}{2\pi\omega b^4},$$

$$s_0 = \left(\frac{2b^2}{Z_0\sigma_c}\right)^{\frac{3}{2}},$$

The contribution from N elements is weighted by the average of the beta function over the length of the impedance structure

Total average impedance

$$\bar{Z}_1^{\perp}(\omega) = (1 - i\,\mathrm{sgn}(\omega))\frac{\sqrt{2Z_0 c\,|\omega|}}{2\pi\omega}K, \quad K = \sum_i \alpha_i,$$

where $\alpha_i = f_i L_i \bar{\beta}_{y,i}/(\sqrt{\sigma_{c,i}} b_i^3)$, $\bar{\beta}_{y,i} = \frac{1}{L_i}\int_{s_i - L_i/2}^{s_i + L_i/2} ds'\beta_y(s'), \quad i = 1, \cdots, N,$

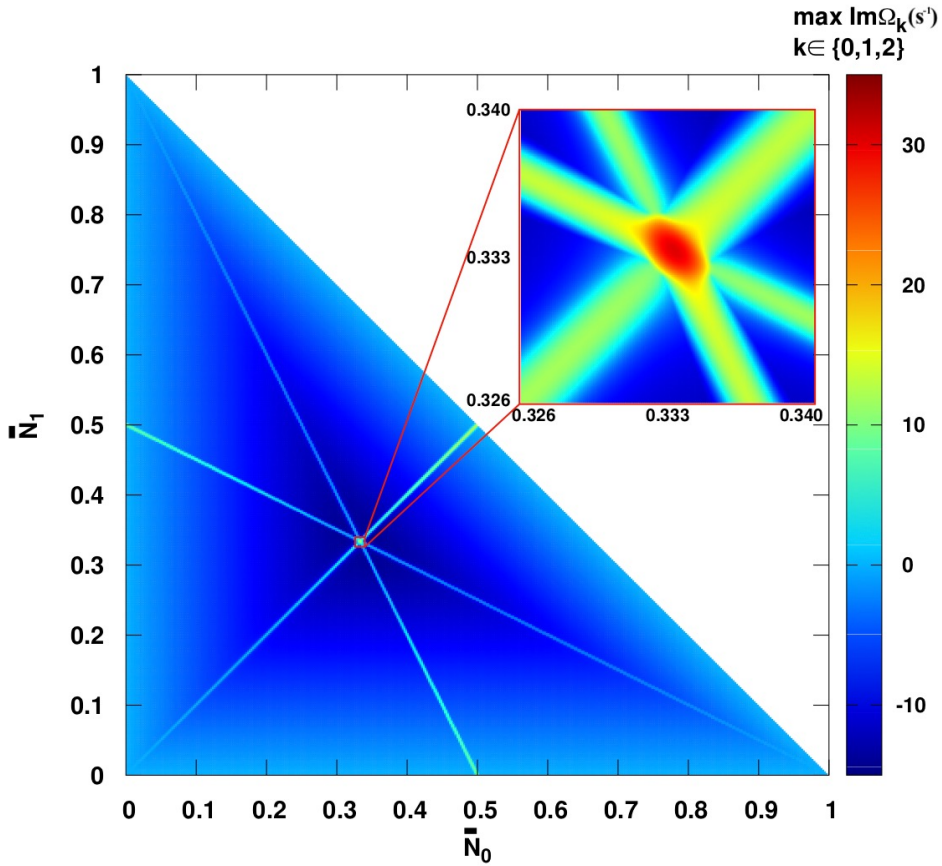
and $f_i = 1$ for circular geometry, $f_i = \pi^2/12$ for parallel geometry .

CLOSED FORMULAS

A. Elementary case: $M = 3$

The complexity of the solution of the eigenvalue problem for arbitrary multibunch configurations is illustrated in the case of $M = 3$ equidistant bunches, in which case the eigenvalues can be expressed in closed form. The corresponding eigenvalue problem defined by Eq. (13) reads

$$|\mathbf{B} - \Omega\mathbf{I}| = \begin{vmatrix} \Omega_0^U - \Omega & a\Omega_0^U & a^*\Omega_0^U \\ a^*\Omega_1^U & \Omega_1^U - \Omega & a\Omega_1^U \\ a\Omega_2^U & a^*\Omega_2^U & \Omega_2^U - \Omega \end{vmatrix} = 0, \quad (17)$$



where $a = \bar{N}_0 - \alpha\bar{N}_1 - \alpha^*\bar{N}_2$, $\alpha = (1 - \sqrt{3}i)/2$, $\bar{N}_m = N_m/N_T$ and * denotes complex conjugate. Eliminating $\bar{N}_2$ with the use of $\bar{N}_0 + \bar{N}_1 + \bar{N}_2 = 1$, it follows that Eq. (17) reads

$$|\mathbf{B} - \Omega\mathbf{I}| = \Omega^3 + b\Omega^2 + c\Omega + d = 0, \quad (18)$$

with coefficients given by

$$b = -\mathrm{Tr}\,\mathbf{B} = -(\Omega_0^U + \Omega_1^U + \Omega_2^U), \quad (19)$$

$$c = -3(\Omega_0^U\Omega_1^U + \Omega_0^U\Omega_2^U + \Omega_1^U\Omega_2^U) \times [\bar{N}_0(\bar{N}_0 - 1) + \bar{N}_0\bar{N}_1 + \bar{N}_1(\bar{N}_1 - 1)], \quad (20)$$

$$d = 27\bar{N}_0\bar{N}_1(\bar{N}_0 + \bar{N}_1 - 1)\Omega_0^U\Omega_1^U\Omega_2^U. \quad (21)$$

The general solution of Eq. (18), which is cubic, is written in the following compact form

$$\Omega_\mu = -\frac{1}{3}\left(b + \xi^\mu C + \frac{\Delta_0}{\xi^\mu C}\right), \qquad \mu \in \{0, 1, 2\}, \quad (22)$$

where $C = [(\Delta_1 - \sqrt{\Delta_1^2 - 4\Delta_0^3})/2]^{1/3}$, $\Delta_0 = b^2 - 3c$, $\Delta_1 = 2b^3 - 9bc + 27d$, and $\xi = \alpha^*$.

CLOSED FORMULAS

C. Gap in the equidistant filling pattern: $\bar{N}_m = 0$ for $m > M - g$

With the use of the Fadded-LeVerrier algorithm [20], we construct now an explicit form of $p(\Omega)$ for the case $g = M - 2$, namely

$$p(\Omega) = \Omega^{M-2}(c_M\Omega^2 + c_{M-1}\Omega + c_{M-2}), \qquad (28)$$

where $c_M = 1$, $c_{M-1} = -\mathrm{Tr}\mathbf{B}$, and $c_{M-2} = -(\mathrm{Tr}\mathbf{B}^2 - (\mathrm{Tr}\mathbf{B})^2)/2$. Using the identities

$$(\mathrm{Tr}\,\mathbf{B})^2 = \sum_{\mu=0}^{M-1}\Omega_\mu^{U2} + 2\sum_{\mu=0}^{M-1}\sum_{\mu'=\mu+1}^{M-1}\Omega_\mu^U\Omega_{\mu'}^U, \qquad (29)$$

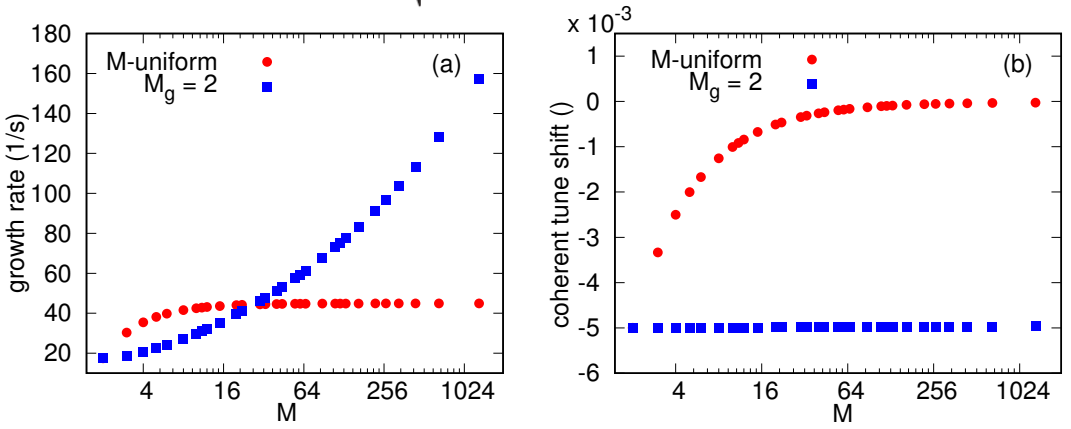
$$\begin{aligned}\mathrm{Tr}\,\mathbf{B^2} &= \sum_{\mu=0}^{M-1}\sum_{\mu'=0}^{M-1}\Omega_\mu^U\Omega_{\mu'}^U|H_{\mu,\mu'}|^2 \\ &= \sum_{\mu=0}^{M-1}\Omega_\mu^{U2} + 2\sum_{\mu=0}^{M-1}\sum_{\mu'=\mu+1}^{M-1}\Omega_\mu^U\Omega_{\mu'}^U|H_{\mu,\mu'}|^2, \qquad (30)\end{aligned}$$

it follows that

$$\begin{aligned}\mathrm{Tr}\,\mathbf{B^2} &= (\mathrm{Tr}\,\mathbf{B})^2 + 2\sum_{\mu=0}^{M-1}\sum_{\mu'=\mu+1}^{M-1}\Omega_\mu^U\Omega_{\mu'}^U(|H_{\mu,\mu'}|^2 - 1) \\ &= (\mathrm{Tr}\,\mathbf{B})^2\left(1 + \frac{1}{2}\sum_{\mu=0}^{M-1}\sum_{\mu'=\mu+1}^{M-1} g_{\mu\mu'}\Omega_\mu^U\Omega_{\mu'}^U\right), \qquad (31)\end{aligned}$$

where $g_{\mu\mu'} = 4(|H_{\mu\mu'}|^2 - 1)/((\mathrm{Tr}\,\mathbf{B})^2)$, thus the eigenvalues of Eq. (28) read $\Omega_k = 0$ for $k \geq 2$ and

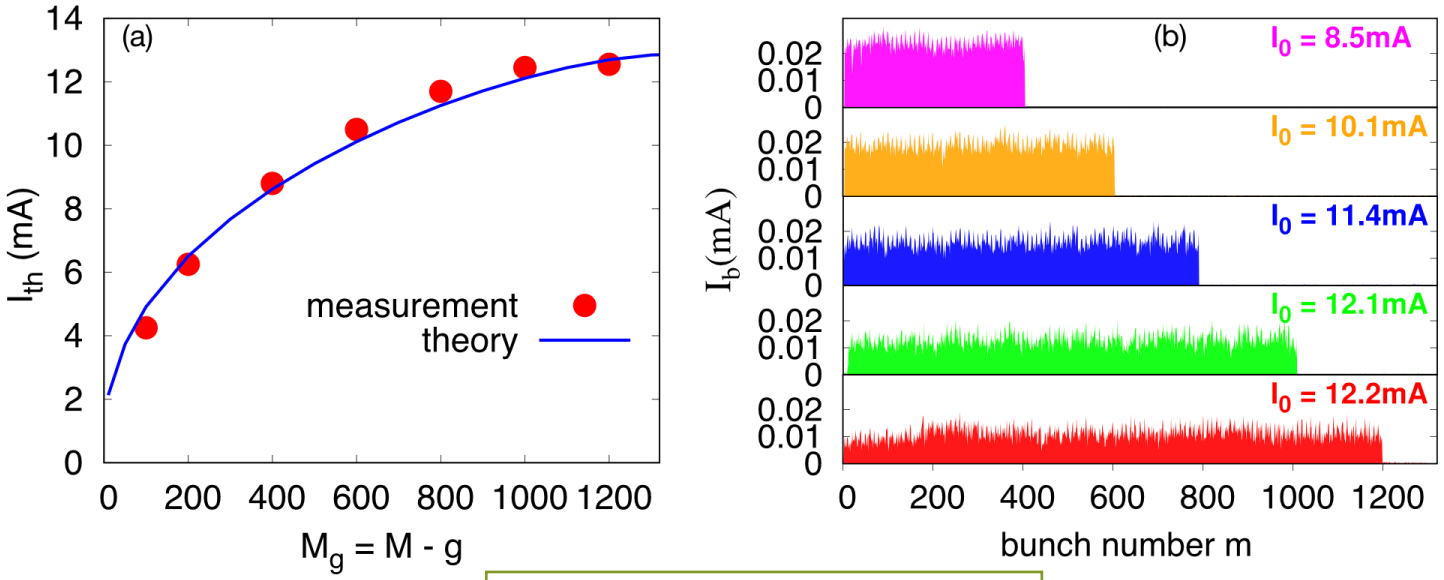
$$\Omega_{0,1} = \frac{\mathrm{Tr}\,\mathbf{B}}{2}\left(1 \pm \sqrt{1 + \sum_{\mu=0}^{M-1}\sum_{\mu'=\mu+1}^{M-1} g_{\mu\mu'}\Omega_\mu^U\Omega_{\mu'}^U}\right). \qquad (32)$$



Benchmarking Theory with Measurements

Measurements with a variable gap in the maximal uniform filling pattern

(Operational NSLS-II lattice with 3 damping wigglers and zero linear chromaticity)

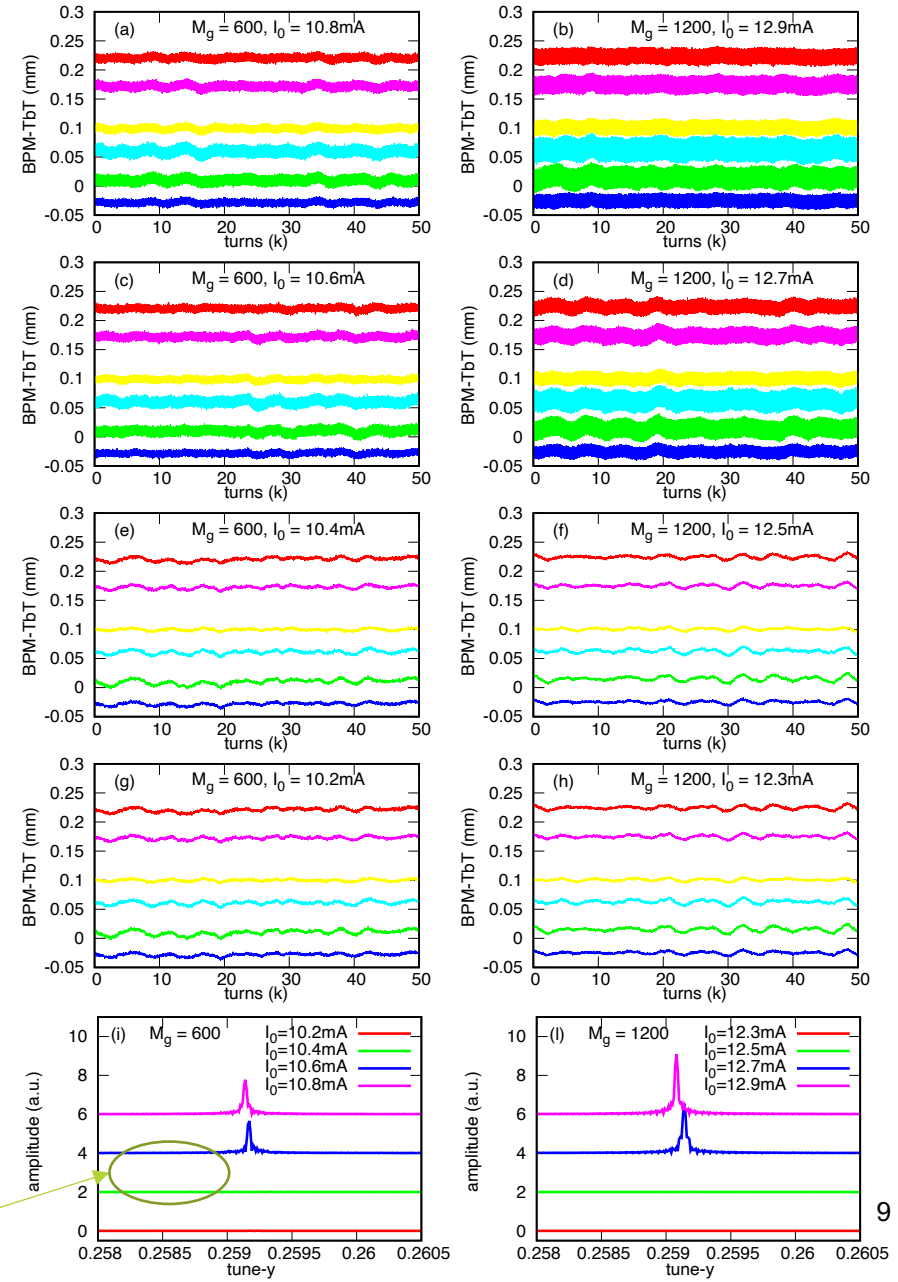


τ_y is vertical radiation damping time

Analytical instability threshold

$$I_{\rm th} = I_{\rm th}^U \frac{\tau_g}{\tau_y}, \qquad (33)$$

where τ_g is the growth time of the fastest growing mode, determined by the eigenvalue of $\mathbf{B}(I_{\rm th}^U)$ with the largest imaginary part.



Measured instability threshold: arithmetic mean of the currents slightly below and above threshold

Conclusion

- Coupled-bunch instability (CBI) theory for arbitrary multibunch configurations successfully benchmarked against measurements at the NSLS-II Storage Ring.
- Impedance model used: resistive wall impedance with contributions weighted by the local beta function.
- CBI threshold shown to decrease with the increase of the gap in the uniform filling pattern.

Thank You for You Attention!