ISAB23

Theory for Magnets in accelerators

POSTECH
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Today's agenda

- Maxwell's equations
- Dipole magnet
- Quaddupole magnet
- Sextupole magnet

Laplace's equations for the Magnetostatic fields

The *Magnetostatic system* is a system only with the *steady currents*, which produces the time-independent magnetic field.

We know the two of the Maxwell's equations are as follows;

- Gauss's law for the magnetic field $\nabla \cdot \mathbf{B} = 0$ (1)
- Ampere law in the Magnetostatic system

$$\nabla \times \mathbf{H} = \mathbf{J}_f \qquad (2) \qquad \Rightarrow \qquad \oint \mathbf{H} \cdot d\mathbf{l} = I$$

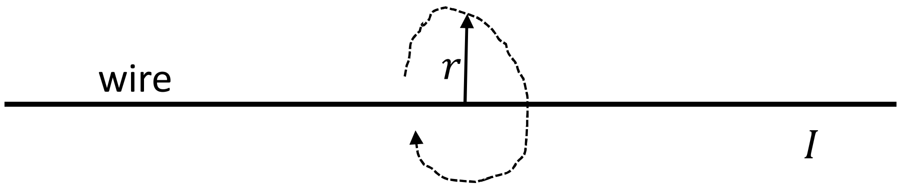
$$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

μ: permeability of the material

Exercise 1

$$\oint \mathbf{H} \cdot d\mathbf{l} = I \qquad \mathbf{H} = \frac{1}{\mu}\mathbf{B} \Rightarrow \mathbf{H} = \frac{1}{\mu_0}\mathbf{B}$$

μ_0: permeability of the air



$$\oint \mathbf{H} \cdot d\mathbf{l} = \oint \frac{1}{\mu_0}\mathbf{B} \cdot d\mathbf{l} = \frac{1}{\mu_0}\mathbf{B} \times 2\pi r = I \Rightarrow \mathrm{B} = \frac{\mu_0 I}{2\pi r}$$

Laplace's equations for the Magnetostatic fields (cont'd)

From the Gauss's law, we can define the *vector potential* **A** s.t.

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad (3)$$

Substitute (3) to the Ampere law in the free space ($\mathbf{J} = 0$)

$$\nabla \times \mathbf{B} = 0 \qquad (4)$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = 0$$

By choosing the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, the Laplace equation is derived.

$$\nabla^2 \mathbf{A} = 0 \qquad (5)$$

Laplace's equations for the Magnetostatic fields (cont'd)

Another Laplace's equation can be derived in the free space, by defining the *magnetostatic scalar potential* Φ_m s.t.

$$\mathbf{B} = -\nabla\Phi_m \qquad (6)$$

Apply Gauss's law (1),

$$\nabla\cdot(-\nabla\Phi_m) = 0 \qquad \therefore\ \ \nabla^2\Phi_m = 0 \qquad (7)$$

hence the Laplace's equation is derived.

Magnetostatic fields in the Accelerator

In the accelerator, we control the motion of the charged particle beam passing through the vacuum chamber by letting the beam experiences external magnetic field.

We can solve the Laplace's equations

$$\nabla^2 \mathbf{A} = 0 \qquad (5)$$

$$\nabla^2 \Phi_m = 0 \qquad (7)$$

by applying boundary conditions to obtain potential fields and the magnetic fields.

Of course, two Laplace's equations give the same solutions, however, treating the scalar potential is easier than treating the vector potential.

Solving the Laplace's equation via Separation of Variables

Laplace's equation in the fixed polar coordinate

$$\nabla^2\Phi_m(r,\theta) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi_m}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi_m}{\partial\theta^2} = 0$$

Separation of variables as $\Phi_m(r,\theta) = R(r)\Theta(\theta)$, then

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = 0$$

The first term only includes r, and the second term only includes θ, i.e. two terms must be constant.

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = k^2, \qquad \frac{d^2\Theta}{d\theta^2} = -k^2\Theta$$

Solving the Laplace's equation via Separation of Variables (cont'd)

$k = n \neq 0$ case

$$r^2 \frac{d^2R}{dr^2} + r\frac{dR}{dr} - n^2 R = 0 \quad \rightarrow \quad R_n(r) = A_n r^n + B_n r^{-n} \quad \left(\text{by using the trial solution } R \sim r^\lambda\right)$$

$$\frac{d^2\Theta}{d\theta^2} = -n^2\Theta \quad \rightarrow \quad \Theta_n(\theta) = C_n \cos n\theta + D_n \sin n\theta$$

The general solution

$$\Phi_m(r,\theta) = \sum_{n=1}^{\infty} (A_n r^n + B_n r^{-n})(C_n \cos n\theta + D_n \sin n\theta)$$

Boundary condition : Φ_m is $'0'$ at $r = 0 \quad \rightarrow \quad B_k = 0 \ \ (k \geq 1)$

$$\therefore \boxed{\Phi_m(r,\theta) = \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta) \qquad (11)} \qquad a_n = A_n C_n, b_n = A_n D_n$$

Solving the Laplace's equation via Separation of Variables (cont'd)

Let the eigenmode of the Laplace's equation

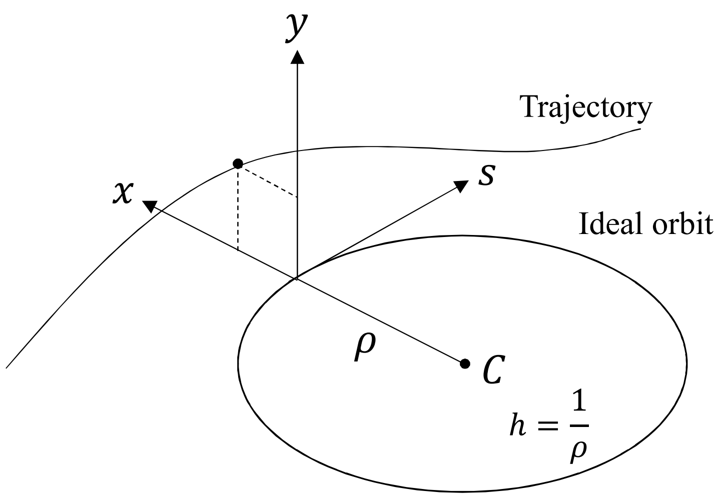
$$\Phi_m^{(n)}(r,\theta) = r^n(a_n \cos n\theta + b_n \sin n\theta) \qquad (12)$$

The *magnet* is a magnetic system consists of a single eigenmode

- The *normal* $2n$-pole magnet is represented by $a_n = 0$: $\Phi_{m,\mathrm{normal}}^{(n)}(r,\theta) = b_n r^n \sin n\theta \qquad (16)$
- The *skew* $2n$-pole magnet is represented by $b_n = 0$: $\Phi_{m,\mathrm{skew}}^{(n)}(r,\theta) = a_n r^n \cos n\theta \qquad (17)$

Vector potential in the Frenet-Serret coordinate

The motion of a particle in the accelerator is described in the Frenet-Serret coordinate (x, y, s), thereby we need to solve the Laplace's equations in the Frenet-Serret coordinate.



- The Hamiltonian of a charged particle in the Frenet-Serret coordinate

$$\mathcal{H} = c\sqrt{(p_x - qA_x)^2 + \left(p_y - qA_y\right)^2 + \frac{(p_z - qA_s)^2}{h_s^2} + m^2c^2} + qV(x, y, z)$$

where $h_s = 1 + \dfrac{x}{\rho(s)}$: scaling factor

- Expansion of the curl operator in the Frenet-Serret coordinate

$$\mathbf{B} = \frac{1}{h_s}\left[\frac{\partial(h_s A_s)}{\partial y} - \frac{\partial A_y}{\partial s}\right]\hat{\mathbf{x}} + \frac{1}{h_s}\left[\frac{\partial A_x}{\partial s} - \frac{\partial(h_s A_s)}{\partial x}\right]\hat{\mathbf{y}} + \left(\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y}\right)\hat{\mathbf{s}} \qquad (8)$$

Magnetostatic fields in the Frenet-Serret coordinate (cont'd)

- In the previous expansion of curl operator (8), we only considered the transverse rotation around the center, i.e. did not consider the vertical torsion of the coordinate.

We only consider the transverse magnetic field, which means

- Laplace's equation for the magnetostatic scalar potential reduces to two-dimensional boundary problem of x and y.
- The vector potential only has the longitudinal component, i.e. $A_x = A_y = 0$ or $\mathbf{A} = (0, 0, A_s)$. Then, (8) can be reduced as

$$\mathbf{B} = \frac{1}{h_s}\left[\frac{\partial(h_s A_s)}{\partial y} - \frac{\partial A_y}{\partial s}\right]\hat{\mathbf{x}} + \frac{1}{h_s}\left[\frac{\partial A_x}{\partial s} - \frac{\partial(h_s A_s)}{\partial x}\right]\hat{\mathbf{y}} + \left(\frac{\partial A_x}{\partial x} - \frac{\partial A_y}{\partial y}\right)\hat{\mathbf{s}} \qquad h_s = 1 + \frac{x}{\rho}$$

$$B_x = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{1}{h_s}\frac{\partial(h_s A_s)}{\partial x}, \qquad B_s = 0 \qquad (9)$$

Magnetostatic fields in the Frenet-Serret coordinate (cont'd)

$$B_x = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{1}{h_s}\frac{\partial (h_s A_s)}{\partial x}, \qquad B_s = 0 \qquad (9)$$

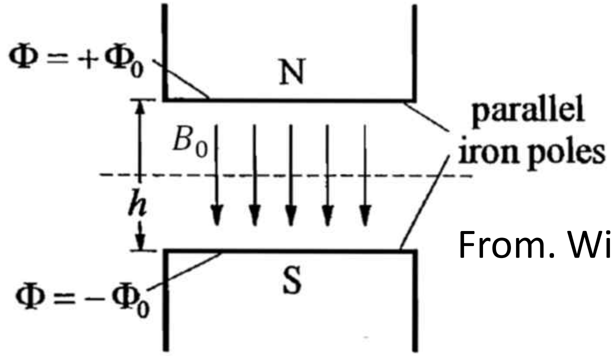
As $\rho \to \infty$, the scaling factor $h_s = 1 + \frac{x}{\rho} \to 1$ and the above equations (9) reduces as

$$B_x = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{\partial A_s}{\partial x}, \qquad B_s = 0 \qquad (10)$$

which are same with the definition of curl operator in the Cartesian coordinate.

This coincides with the fact that the Frenet-Serret coordinate can be approximated to the Cartesian coordinate as $\rho \to \infty$.

Normal dipole magnet ($n = 1$)



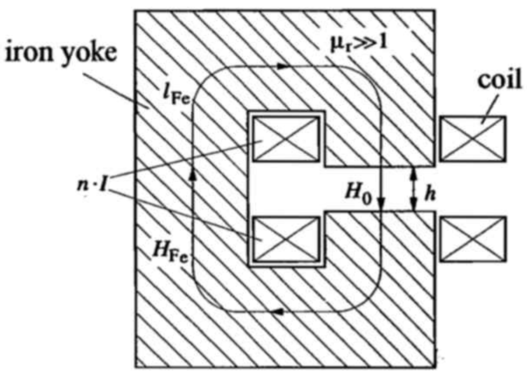
From. Wille

- Magnetostatic scalar potential : From (16)

$$\Phi_m = b_1 r \sin\theta = b_1 y = -B_0 y$$

- Magnetic field

$$\mathbf{B} = -\nabla\Phi_m = -b_1\hat{\mathbf{y}} \equiv B_0\hat{\mathbf{y}}$$



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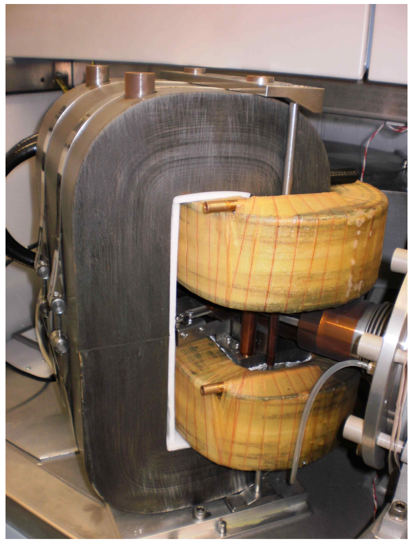
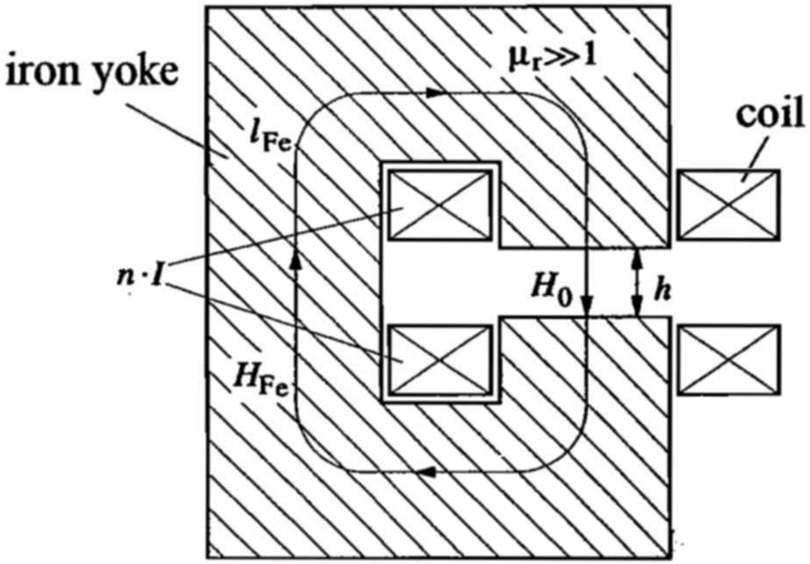


From. Wikipedia

Normal dipole magnet ($n = 1$) (cont'd)

Calculation of the characteristic value B_0



- Boundary condition : $\mu_{Fe} H_{Fe} = \mu_0 H_0$, while $\mu_r = \frac{\mu_{Fe}}{\mu_0} \sim 5000 \gg 1$
- Ampere's law

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_0 h + H_{Fe} l_{Fe} \approx H_0 h = NI$$

- magnetic field strength

$$B_0 = \mu_0 H_0 = \frac{\mu_0 NI}{h}$$

Normal dipole magnet ($n = 1$) (cont'd)

In the uniform magnetic field, a charged particle moves in a uniform circular motion due to the Lorentz force

$$|\mathbf{F}| = qv_0B_0 = \frac{\gamma m\, v_0^2}{\rho}\;, \qquad v_0\text{: velocity of the ideal particle}$$

$$\therefore\;\; \rho B_0 = \frac{\gamma m\, v_0}{q} = \frac{p_0}{q} \qquad (19) \qquad \text{: magnetic rigidity}$$

- Vector potential

From (9),

$$B_x = \frac{\partial A_s}{\partial y} = 0$$
$$B_y = -\frac{1}{h_s}\frac{\partial (h_s A_s)}{\partial x} = B_0$$

Normal dipole magnet ($n = 1$) (cont'd)

Assume $h_s A_s = a + bx + cx^2$, then the last equation becomes

$$b + 2cx = -h_s B_0 = -B_0 - \frac{B_0}{\rho} x \quad \underset{\forall x}{\rightarrow} \quad b = -B_0, \qquad c = -\frac{B_0}{2\rho}$$

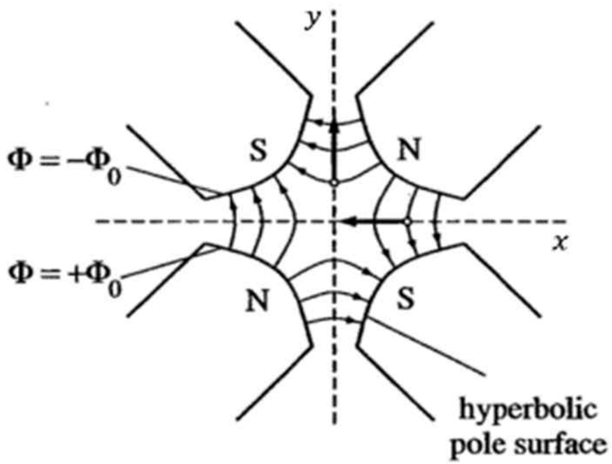
$$\therefore \; A_s = -\frac{B_0}{1 + \frac{x}{\rho}} \left(x + \frac{x^2}{2\rho} \right)$$

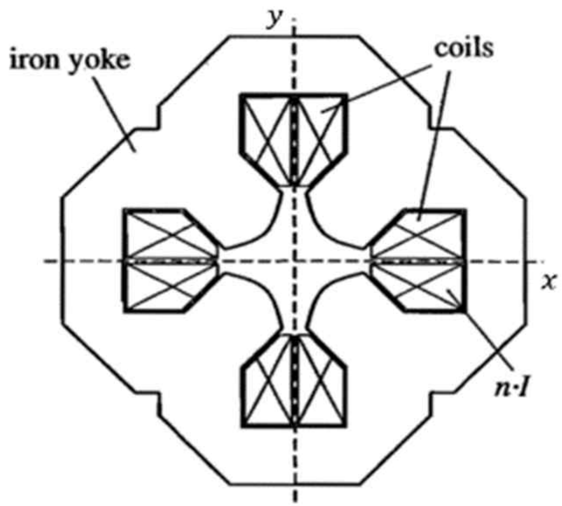
Note that $h_s = 1 + \frac{x}{\rho}$ is a function of x, since $\rho \neq \infty$ from (19)

- As $\rho \to \infty$, $A_s \approx -B_0 x$: the vector potential at Cartesian coordinate, satisfying (10).

$$B_x = \frac{\partial A_s}{\partial y} = 0 \,, \qquad B_y = -\frac{\partial A_s}{\partial x} = B_0,$$

Normal quadrupole magnet ($n = 2$) (cont'd)





From. Wille

From. TRIUMF

Normal quadrupole magnet ($n = 2$)

- Magnetostatic scalar potential

$$\Phi_m = b_2 r^2 \sin 2\theta = b_2 r^2 (2 \sin\theta \cos\theta) = 2b_2 (r\sin\theta)(r\cos\theta) = 2b_2 xy = -gxy$$

- Magnetic field

$$\mathbf{B} = -\nabla\Phi_m = -\frac{\partial\Phi_m}{\partial x}\hat{\mathbf{x}} - \frac{\partial\Phi_m}{\partial y}\hat{\mathbf{y}} = -2b_2 y\hat{\mathbf{x}} - 2b_2 x\hat{\mathbf{y}} \equiv gy\hat{\mathbf{x}} + gx\hat{\mathbf{y}} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}$$

$$= -2b_2 r(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) \equiv B_{am}(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) \qquad B_{am} = -2b_2 r$$

- Magnetic field gradient

$$g \equiv \frac{\partial B_{am}}{\partial r} = -2b_2 \quad \Rightarrow \quad \Phi_m = 2b_2 xy = -gxy$$

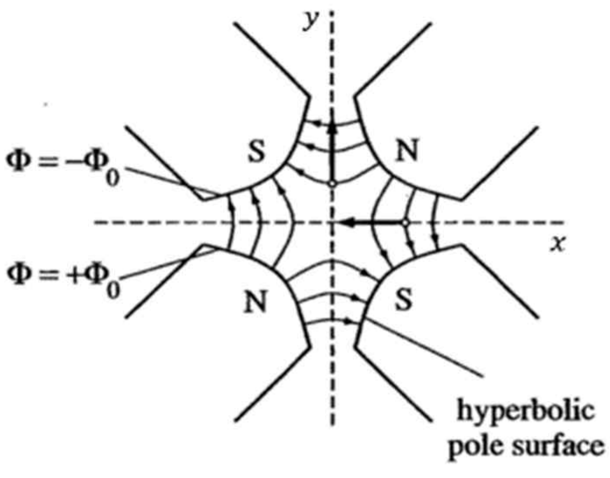
Normal quadrupole magnet ($n = 2$) (cont'd)

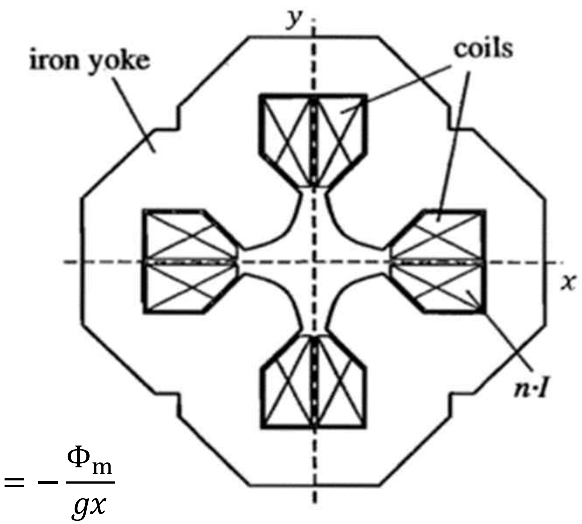
- Characteristic value on x −axis ($\theta = 0$)

$$\mathbf{B} \equiv B_{am}(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) = B_{am}\hat{\mathbf{y}} = B_y\hat{\mathbf{y}} \quad \rightarrow \quad g = \frac{\partial B_{am}}{\partial r} = \frac{\partial B_y}{\partial x} = -2b_2$$

- Characteristic value on y −axis $\left(\theta = \frac{\pi}{2}\right)$

$$\mathbf{B} \equiv B_{am}(\sin\theta\,\hat{\mathbf{x}} + \cos\theta\,\hat{\mathbf{y}}) = B_{am}\hat{\mathbf{x}} = B_x\hat{\mathbf{x}} \quad \rightarrow \quad g = \frac{\partial B_{am}}{\partial r} = \frac{\partial B_x}{\partial y} = -2b_2$$



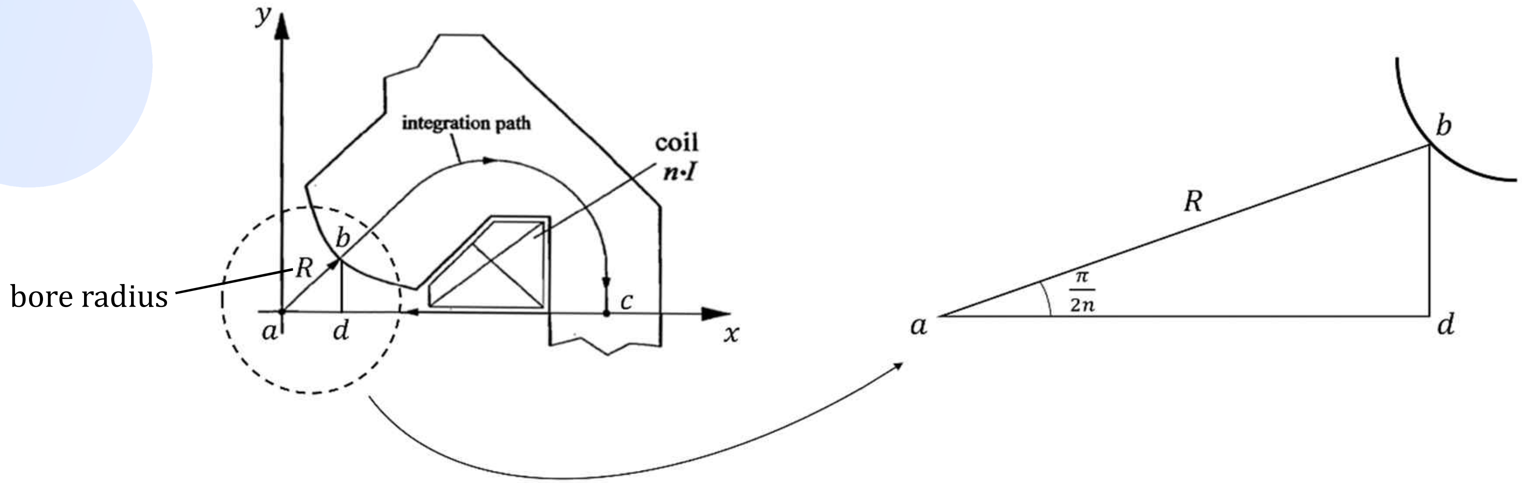


$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{s}} \\ 0 & 0 & -v \\ B_x & B_y & 0 \end{vmatrix} = qvgx\,\hat{\mathbf{x}} - qvgy\,\hat{\mathbf{y}}$$

if $q < 0$, focusing on x −direction,

defocusing on y −direction

Calculation of the Magnetic field gradient ($n \geq 2$) (cont'd)

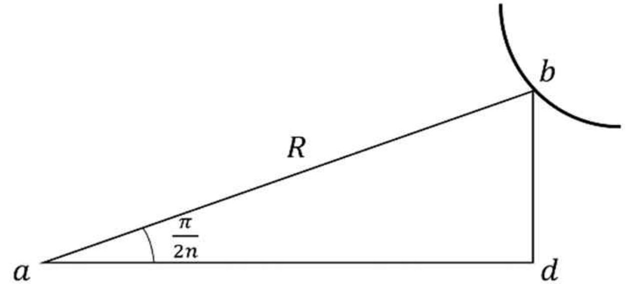


Ampere's law for $a \rightarrow b \rightarrow c \rightarrow a$

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_a^b \mathbf{H}_0 \cdot d\mathbf{l} + \underbrace{\int_b^c \mathbf{H}_{\mathrm{Fe}} \cdot d\mathbf{l}}_{\substack{=0 \\ (\because \text{inside the iron yoke})}} + \underbrace{\int_c^a \mathbf{H} \cdot d\mathbf{l}}_{\substack{=0 \\ (\because \mathbf{H} \perp d\mathbf{l})}} = NI$$

$$\therefore \oint \mathbf{H} \cdot d\mathbf{l} = \frac{1}{\mu_0} \int_a^b \mathbf{B} \cdot d\mathbf{l} = NI \quad \Rightarrow \quad \int_a^b \mathbf{B} \cdot d\mathbf{l} = \mu_0 NI$$

Calculation of the Magnetic field gradient ($n \geq 2$) (cont'd)



To calculate the last integral, apply Ampere's law again for $a \to b \to d \to a$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int_a^b \mathbf{B} \cdot d\mathbf{l} + \int_b^d B_y dy + \int_d^a B_x dx = 0$$

$$\int_a^b \mathbf{B} \cdot d\mathbf{l} = \mu_0 N I$$

$$\rightarrow \int_a^b \mathbf{B} \cdot d\mathbf{l} = -\int_d^a B_x dx - \int_b^d B_y dy = \int_a^d B_x dx + \int_d^b B_y dy = \int_0^{R\cos\frac{\pi}{2n}} B_x\Big|_{y=0} dx + \int_0^{R\sin\frac{\pi}{2n}} B_y\Big|_{x=R\cos\frac{\pi}{2n}} dy$$

$$\therefore \boxed{\int_0^{R\cos\frac{\pi}{2n}} B_x\Big|_{y=0} dx + \int_0^{R\sin\frac{\pi}{2n}} B_y\Big|_{x=R\cos\frac{\pi}{2n}} dy = \mu_0 N I \qquad (21)}$$

Normal quadrupole magnet ($n = 2$) (cont'd)

$$\mathbf{B} = gy\hat{\mathbf{x}} + gx\hat{\mathbf{y}} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}}$$

- Magnetic field gradient of the quadrupole magnet

Apply (21) with $n = 2$,

$$\mu_0 NI = \int_0^{R\cos\frac{\pi}{2n}} B_x\Big|_{y=0} dx + \int_0^{R\sin\frac{\pi}{2n}} B_y\Big|_{x=R\cos\frac{\pi}{2n}} dy = \int_0^{\frac{1}{\sqrt{2}}R} gy\Big|_{y=0} dx + \int_0^{\frac{1}{\sqrt{2}}R} gx\Big|_{x=\frac{1}{\sqrt{2}}R} dy$$

$$= 0 + g\frac{1}{\sqrt{2}}R\int_0^{\frac{1}{\sqrt{2}}R} dy = \frac{1}{2}gR^2$$

$$\therefore \quad g = \frac{2\mu_0 NI}{R^2}$$

Normal quadrupole magnet ($n = 2$) (cont'd)

- Vector potential

The quadrupole magnet, there is no bending component, so $\rho \to \infty$ (or $h_s = 1$).

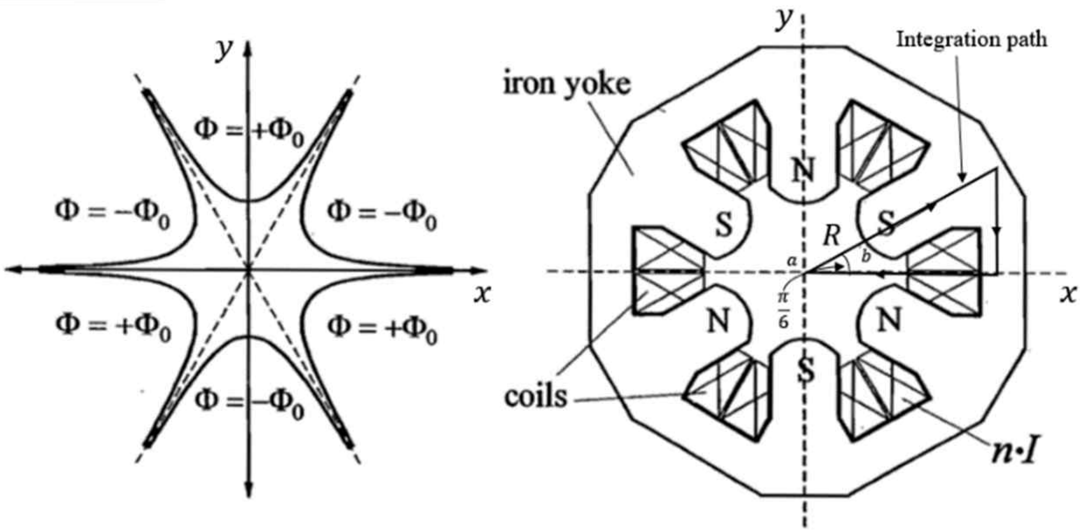
Therefore, the Frenet-Serret coordinate is approximated to the Cartesian coordinate, so the magnetic field and the vector potential has the relation (10)

$$B_x = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{\partial A_s}{\partial x}, \qquad B_s = 0 \qquad (10)$$

Guessing $A_s = ax^2 + by^2$, then $B_x = \frac{\partial A_s}{\partial y} = 2by = gy$, $B_y = -\frac{\partial A_s}{\partial x} = -2ax = gx \;\therefore\; a = -\frac{g}{2}, b = \frac{g}{2}$

$$\therefore \; A_s = -\frac{1}{2} g(x^2 - y^2)$$

Normal sextupole magnet ($n = 3$) (cont'd)



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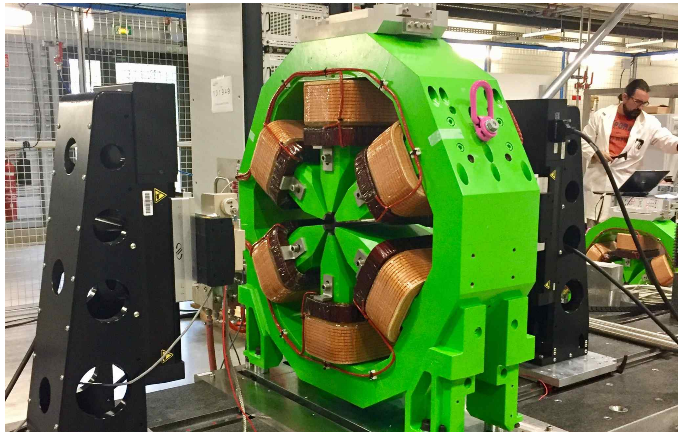


From. ESRF

Normal sextupole magnet ($n = 3$)

- Magnetostatic scalar potential

$$\Phi_m = b_3 r^3 \sin 3\theta = b_3 r^3(-4\sin^3\theta + 3\sin\theta) = -4b_3 r^3 \sin^3\theta + 3b_3 r^3 \sin\theta = -4b_3 y^3 + 3b_3 r^2 r \sin\theta$$
$$= -4b_3 y^3 + 3b_3(x^2+y^2)y = 3b_3 x^2 y - b_3 y^3$$

- Magnetic field

$$\mathbf{B} = -\nabla\Phi_m = -\frac{\partial\Phi_m}{\partial x}\hat{\mathbf{x}} - \frac{\partial\Phi_m}{\partial y}\hat{\mathbf{y}} = -6b_3 xy\,\hat{\mathbf{x}} - 3b_3(x^2-y^2)\hat{\mathbf{y}} \quad = -3b_3 r^2(\sin 2\theta\,\hat{\mathbf{x}} + \cos 2\theta\,\hat{\mathbf{y}})$$
$$\equiv B_{am}(\sin 2\theta\,\hat{\mathbf{x}} + \cos 2\theta\,\hat{\mathbf{y}})$$

- Magnetic 2nd field gradient : g'

$$g = \frac{\partial B_{am}}{\partial r} = -6b_3 r\,, \qquad g' \equiv \frac{\partial g}{\partial r} = \frac{\partial^2 B_{am}}{\partial r^2} = -6b_3 \quad \Rightarrow \quad \Phi_m = -\frac{1}{6}g'(3x^2y - y^3)$$

$$\mathbf{B} \equiv B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} = g'xy\,\hat{\mathbf{x}} + \frac{1}{2}g'(x^2-y^2)\hat{\mathbf{y}}$$

Normal sextupole magnet ($n = 3$) (cont'd)

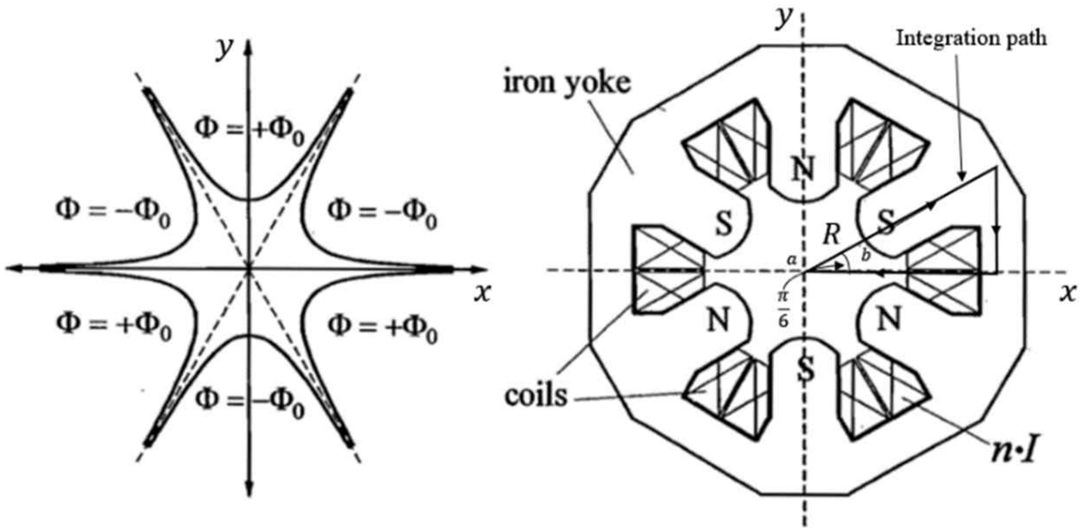
- Characteristic value of the sextupole magnet

Apply (21) with $n = 3$,

$$\mu_0 NI = \int_0^{R\cos\frac{\pi}{2n}} B_x\Big|_{y=0} dx + \int_0^{R\sin\frac{\pi}{2n}} B_y\Big|_{x=R\cos\frac{\pi}{2n}} dy = \int_0^{\frac{\sqrt{3}}{2}R} g'xy\Big|_{y=0} dx + \int_0^{\frac{1}{2}R} \frac{1}{2}g'(x^2-y^2)\Big|_{x=\frac{\sqrt{3}}{2}R} dy$$

$$= \frac{1}{6}gR^3$$

$$\therefore \quad g' = \frac{6\mu_0 NI}{R^3}$$



Normal sextupole magnet ($n = 3$) (cont'd)

- Vector potential

Guessing $A_s = ax^3 + bx\,y^2$ and using the relation (10)

$$B_x = \frac{\partial A_s}{\partial y}, \qquad B_y = -\frac{\partial A_s}{\partial x}, \qquad B_s = 0 \qquad (10)$$

Comparing with $\mathbf{B} = g' xy\,\hat{\mathbf{x}} + \frac{1}{2} g'(x^2 - y^2)\hat{\mathbf{y}}$ gives $a = -\frac{g'}{6}, b = \frac{g'}{2}$

$$\therefore\ A_s = -\frac{1}{6} g'(x^3 - 3xy^2)$$

Summary

- In general, magnetic field can be Taylor expanded in any magnetic system, e.g. on the x-axis,

$$B_y(x,0) = B_y\Big|_{y=0} + \frac{\partial B_y}{\partial x}\Big|_{y=0} x + \frac{1}{2!}\frac{\partial^2 B_y}{\partial x^2}\Big|_{y=0} x^2 + \frac{1}{3!}\frac{\partial^3 B_y}{\partial x^3}\Big|_{y=0} x^3 + \cdots$$

$$= B_0 + gx + \frac{1}{2}g'x^2 + \frac{1}{6}g''x^3 + \cdots$$

Multiply $\frac{e}{p}$ to each side,

$$\frac{e}{p}B_y = \frac{e}{p}B_0 + \frac{e}{p}gx + \frac{1}{2}\frac{e}{p}g'x^2 + \frac{1}{6}\frac{e}{p}g''x^3 + \cdots$$

$$\equiv \frac{1}{\rho} + kx + \frac{1}{2!}mx^2 + \frac{1}{3!}ox^3 + \cdots$$

$\frac{1}{\rho} = \frac{e}{p}B_0$: Dipole strength

$k = \frac{e}{p}g$: Quadrupole strength

$m = \frac{e}{p}g'$: Sextupole strength

$o = \frac{e}{p}g'$: Octupole strength

Backup slides

Normal octupole magnet ($n = 4$)

- Magnetostatic scalar potential

$$\Phi_m = b_4 r^4 \sin 4\theta = 4b_4 r^4 (2\sin\theta\cos^3\theta - \sin\theta\cos\theta) = 4b_4 x^3 y - 4b_4 x y^3 = -\frac{1}{6} g''(x^3 y - x y^3)$$

- Magnetic field

$$\mathbf{B} = -\nabla\Phi_m = -4b_4(3x^2 y - y^3)\hat{\mathbf{x}} - 4b_4(x^3 - 3xy^2)\hat{\mathbf{y}} = \frac{1}{6} g' \left[(3x^2 y - y^3)\hat{\mathbf{x}} + (x^3 - 3xy^2)\hat{\mathbf{y}}\right]$$

$$= -4b_4 r^3 (\sin 3\theta\, \hat{\mathbf{x}} + \cos 3\theta\, \hat{\mathbf{y}}) \equiv B_{am}(\sin 3\theta\, \hat{\mathbf{x}} + \cos 3\theta\, \hat{\mathbf{y}})$$

- Characteristic value

$$g'' \equiv \frac{\partial^3 B_{am}}{\partial r^3} = -24 b_4$$

Normal octupole magnet ($n = 3$) (cont'd)

- Characteristic value of the octupole magnet

Apply (21) with $n = 4$,

$$\mu_0 NI = \int_0^{R\cos\frac{\pi}{2n}} B_x\Big|_{y=0} dx + \int_0^{R\sin\frac{\pi}{2n}} B_y\Big|_{x=R\cos\frac{\pi}{2n}} dy$$

$$= \frac{1}{6}g'' \int_0^{R\cos\frac{\pi}{8}} (3x^2y - y^3)\Big|_{y=0} dx + \frac{1}{6}g'' \int_0^{R\sin\frac{\pi}{8}} (x^3 - 3xy^2)\Big|_{x=R\cos\frac{\pi}{8}} dy$$

$$= \frac{1}{24}g''R^3$$

$$\therefore \quad g' = \frac{24\mu_0 NI}{R^4}$$

Normal octupole magnet ($n = 3$) (cont'd)

- Vector potential

Consider $A_s = \frac{1}{6} g' \,(ax^4 + bx^2y^2 + cy^4)$ and using the relation (10)

$$B_x = \frac{\partial A_s}{\partial y}\,, \qquad B_y = -\frac{\partial A_s}{\partial x}, \qquad B_s = 0 \qquad (10)$$

Comparing with $\mathbf{B} = \frac{1}{6} g' \,[(3x^2y - y^3)\hat{\mathbf{x}} + (x^3 - 3xy^2)\hat{\mathbf{y}}]$ gives $a = -\frac{1}{4}, b = \frac{3}{2}, c = -\frac{1}{4}$

$$\therefore \; A_s = -\frac{1}{24} g''(x^4 - 6x^2y^2 + y^4)$$