A Semi-classical Spacetime Region with Maximum Entropy

RIKEN iTHEMS

Yuki Yokokura

[arXiv: 2309.00602]

2023 Nov 29 @ KEK Theory Workshop 2023

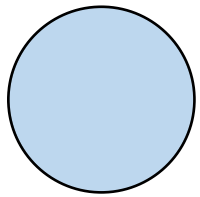
iTHEMS RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program

Motivation

Q1: What is the maximum entropy that can be given to a finite region?

Q2: What is the structure of such a spacetime $g^{*}_{\mu\nu}$?

⇒Important for finding the fundamental d.o.f. in quantum gravity



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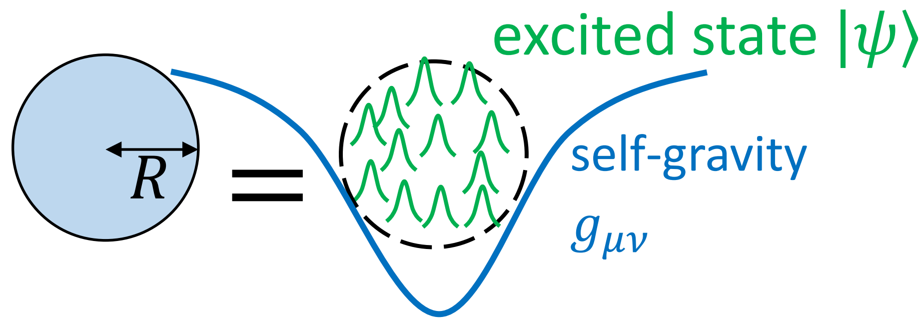

4D spherical static spacetime region $=$ a collection of excited quanta in $(|\psi\rangle, g_{\mu\nu})$

semi-classical Einstein eq

$$\boxed{G_{\mu\nu} = 8\pi G \langle\psi|T_{\mu\nu}|\psi\rangle}$$

"weight of information"

←self-consistent eq
in a mean field approximation of QG
gravity = classical $g_{\mu\nu}$
matter = quantum $\hat{\phi}_i(x)$

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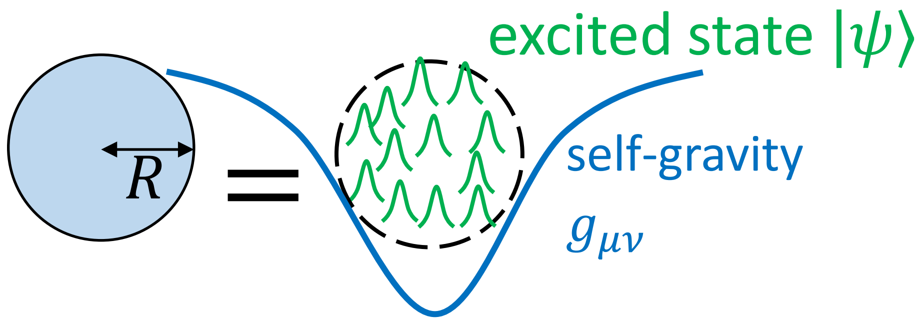

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- We will estimate the entropy including self-gravity

$$S[g_{\mu\nu}] = \int_\Sigma d\Sigma_\mu \, s^\mu$$

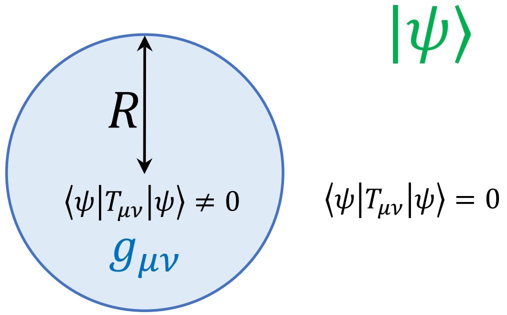
- many local d.o.f. with local interactions
- sufficiently excited states $\{|\psi\rangle\}$
- $\nabla_\mu s^\mu = 0$

and find $g^*_{\mu\nu}$ s.t.

$$S[g_{\mu\nu}] \leq S_{max} \equiv S[g^*_{\mu\nu}]$$

Setup: 1-bit quantum

- Consider $(g_{\mu\nu}, |\psi\rangle)$ satisfying $G_{\mu\nu} = 8\pi G\langle\psi|T_{\mu\nu}|\psi\rangle$ self-consistently.



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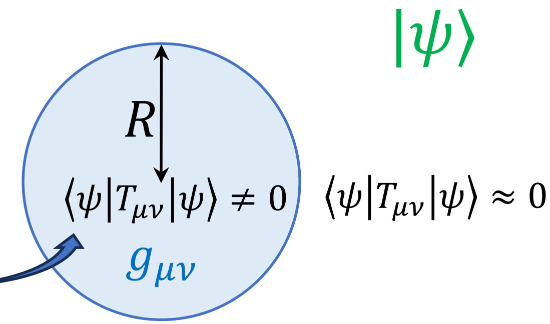
<u>metric</u>

$$ds^2 = \begin{cases} -\left(1-\dfrac{a_0}{r}\right)dt^2 + \dfrac{1}{1-\dfrac{a_0}{r}}dr^2 + r^2 d\Omega^2 & \text{for } R \le r \\ -\left(1-\dfrac{a(r)}{r}\right)e^{A(r)}dt^2 + \dfrac{1}{1-\dfrac{a(r)}{r}}dr^2 + r^2 d\Omega^2 & \text{for } r \le R \end{cases}$$

$\frac{a(r)}{2G} \equiv m(r)$: Misner-Sharp energy at r

$\xrightarrow{r \ge R} \frac{a_0}{2G} \equiv M_0$:total ADM energy



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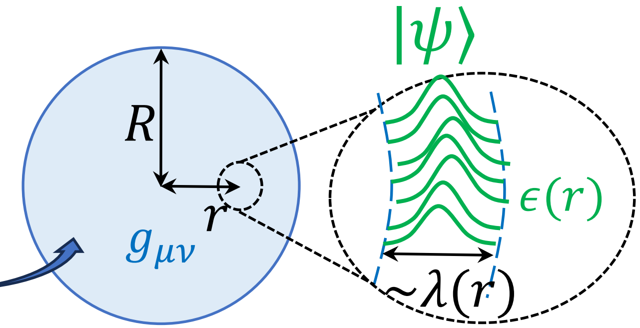

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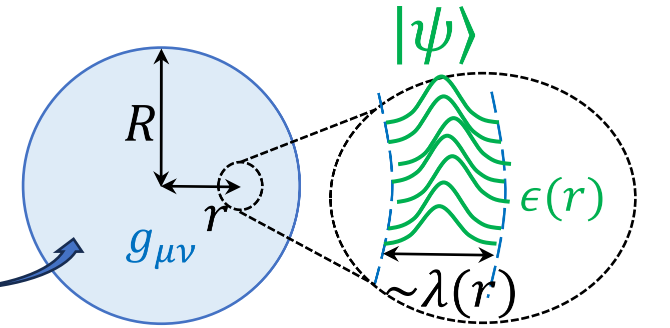

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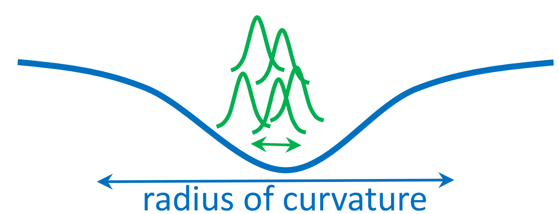

⇒WKB-like particle without feeling gravity

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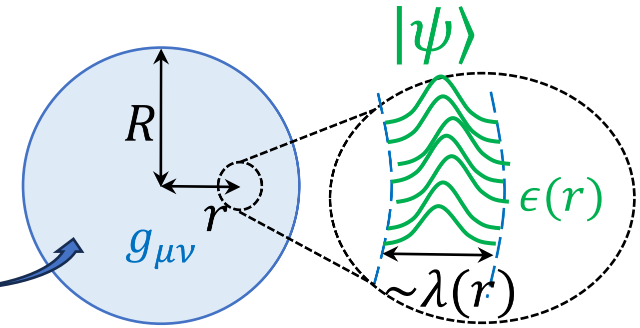

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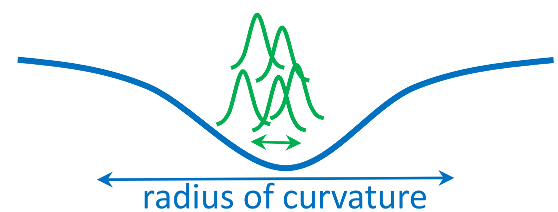

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- Many such quanta behave typically like a local thermal equilibrium:

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Thermodynamical Typicality
e.g. [Goldstein, et al. 2006]

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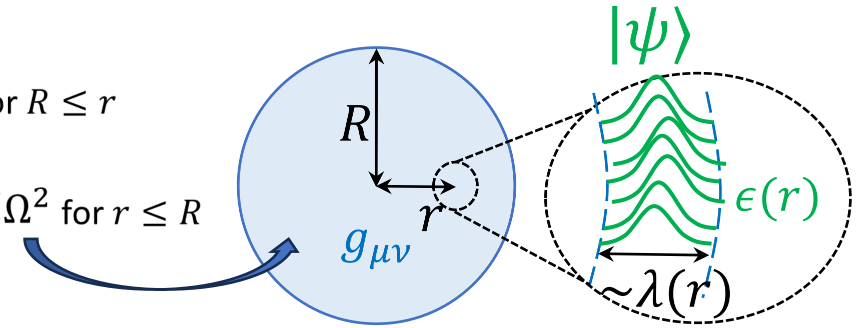

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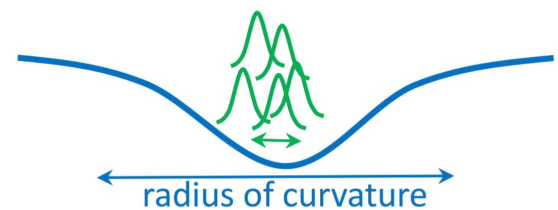

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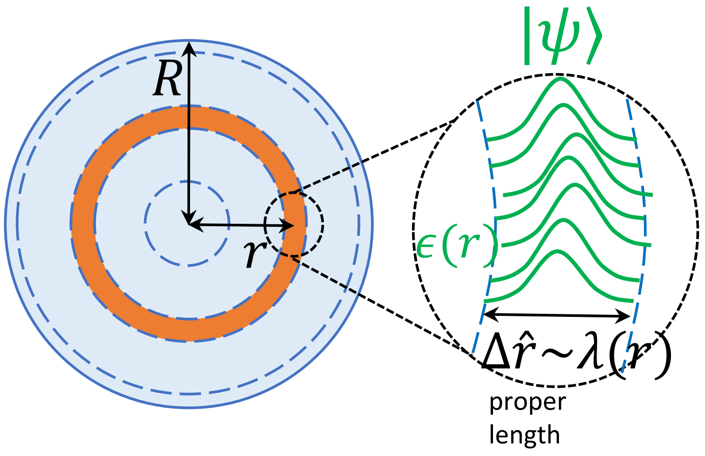
⇒ Such a quantum has 1-bit of entropy. $\boxed{T = \frac{dE}{dS} = \text{energy}/1\text{bit}}$

"1-bit quantum"

WKB-like estimation of $S[g_{\mu\nu}]$

- Decompose the system into the smallest spherical subsystems.



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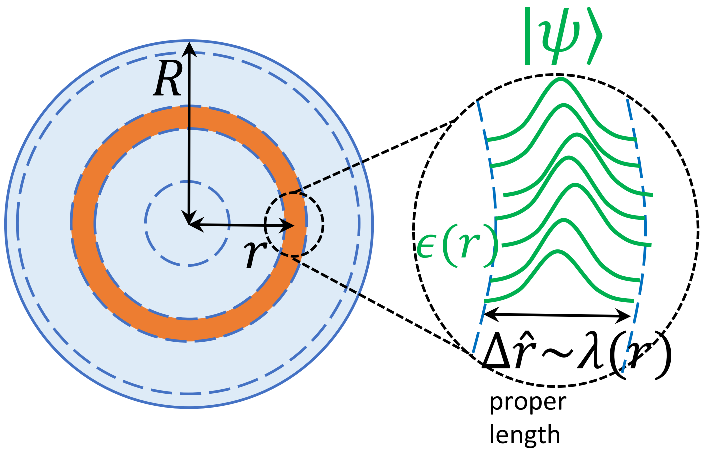

- Define a quasi-local function

$$\boxed{N(r) \equiv \frac{4\pi r^2 \langle T^{\hat{t}\hat{t}}(r)\rangle \Delta\hat{r}}{\epsilon(r)} \quad \text{for } \Delta\hat{r} \sim \lambda(r)}$$

$\sim$ # of 1-bit quanta in the subsystem for $N(r) \gg 1$

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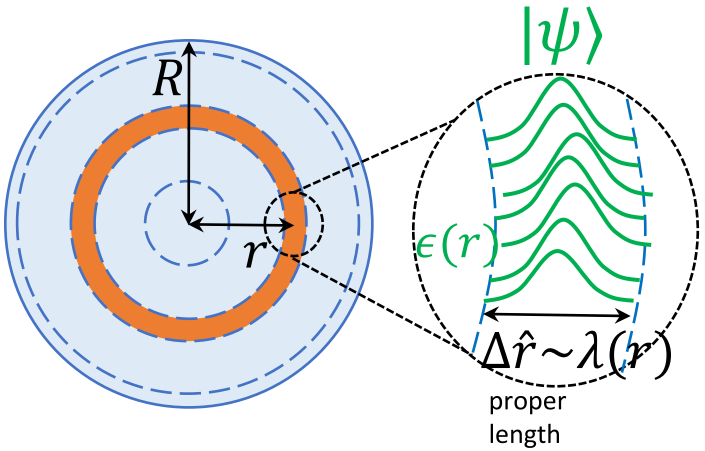

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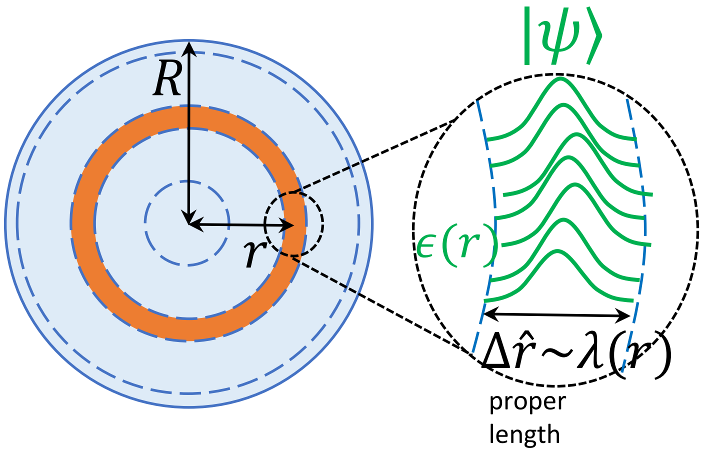

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- Thus, the entropy $S[g_{\mu\nu}]$ can be estimated as

$$\boxed{S[g_{\mu\nu}] \sim \frac{1}{\hbar}\int_0^R dr\, N(r)\epsilon(r)\left[1 - \frac{2}{r}\int_0^r dr'\, N(r')\frac{\epsilon(r')^2}{m_p^2}\right]^{-\frac{1}{2}}}$$

self-gravity

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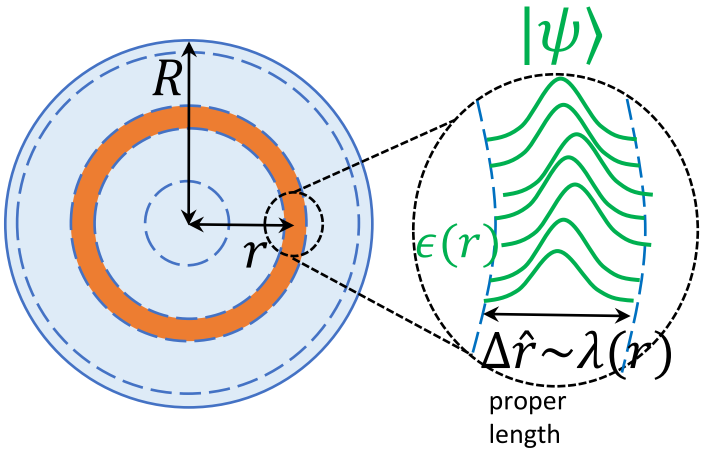

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$\Rightarrow$For a given $N(r)$, ϵ_{max} provides S_{max}.

For the semi-classical description,
$$\epsilon(r) \le \epsilon_{max} \sim \frac{m_p}{\sqrt{n}} \qquad (n\text{: O(1) constant >>1 })$$

Upper bound

- A static spacetime has a timelike Killing vector globally.

$\Rightarrow$ No trapped surface exists. [Mars-Senovilla 2003]

Ex. $k = \partial_t$ in Schwarzschild metric

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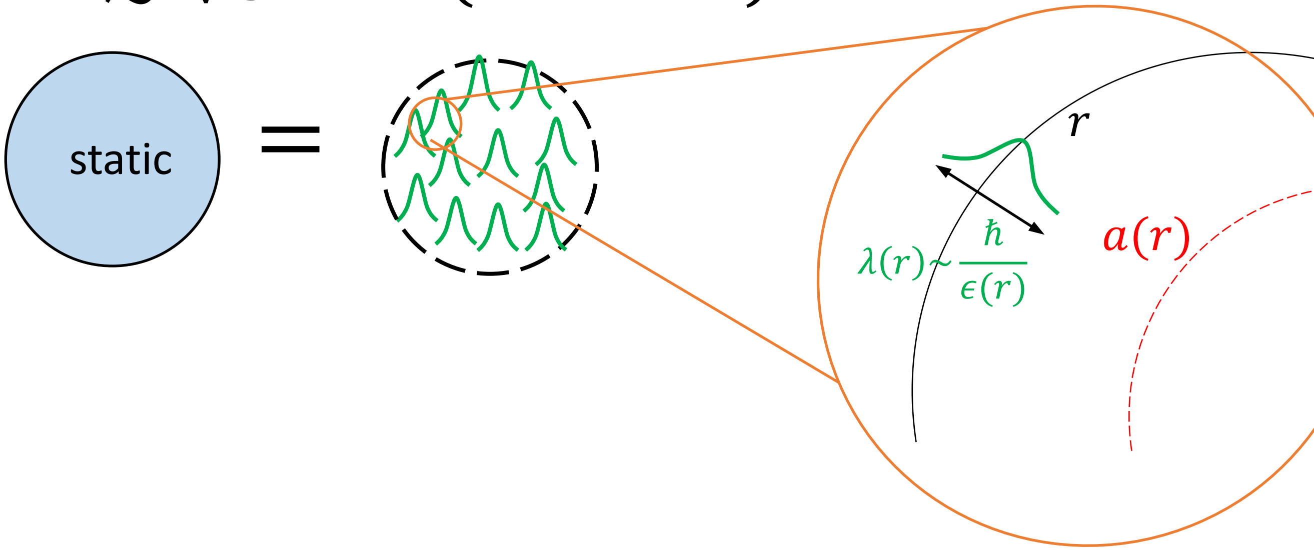

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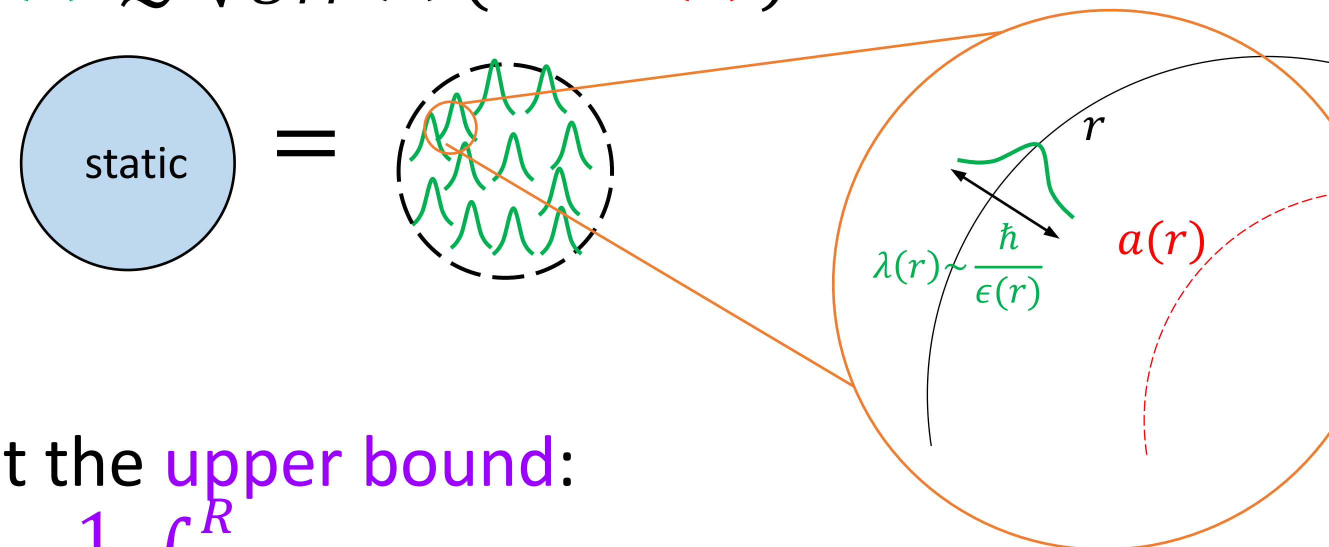

- Then, we can get the upper bound:

$$S \lesssim \frac{1}{l_p^2} \int_0^R dr\, r\, \underbrace{\partial_r a(r)}_{= 8\pi G r^2 \langle -T_t^t(r) \rangle}$$

Entropy-maximized spacetime (1/2)

- To get the saturating configuration $g^{*}_{\mu\nu}$, we solve

$$\lambda(r) \sim \sqrt{g_{rr}(r)}\big(r - a(r)\big) \text{ for } \epsilon(r) = \epsilon_{max} \sim \frac{m_p}{\sqrt{n}}$$

and use two consistencies:

- thermodynamics,
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- Thus, we reach uniquely the physical saturating configuration:

$$ds^2 = -\frac{\sigma\eta^2}{2r^2} e^{-\frac{R^2 - r^2}{2\sigma\eta}} dt^2 + \frac{r^2}{2\sigma} dr^2 + r^2 d\Omega^2.$$

$\sigma = O(nl_p^2)$, $1 \le \eta < 2$

- Can be obtained in various ways and robust.

[Kawai-Matsuo-Yokokura 2013, Kawai-Yokokura 2014, 2015, 2016,2020,2021, Yokokura 2022, Ho-Kawai-Liyao-Yokokura 2023]

$\rightarrow \lambda(r) \lesssim \sqrt{g_{rr}(r)}\big(r - a(r)\big)$ can be verified in a dynamical model.

- Non-perturbative solution of $G_{\mu\nu} = 8\pi G \langle\psi|T_{\mu\nu}|\psi\rangle$ for $\hbar$ [Kawai-Yokokura 2020]
- n = # of d.o.f. in the theory

Entropy-maximized spacetime (2/2)

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represents a dense configuration without horizon or singularity.

- $\mathcal{R} = O\left(\frac{1}{n l_p^2}\right) \ll O\left(\frac{1}{l_p^2}\right)$
- almost flat around $r = 0$

$\Rightarrow$ No singularity

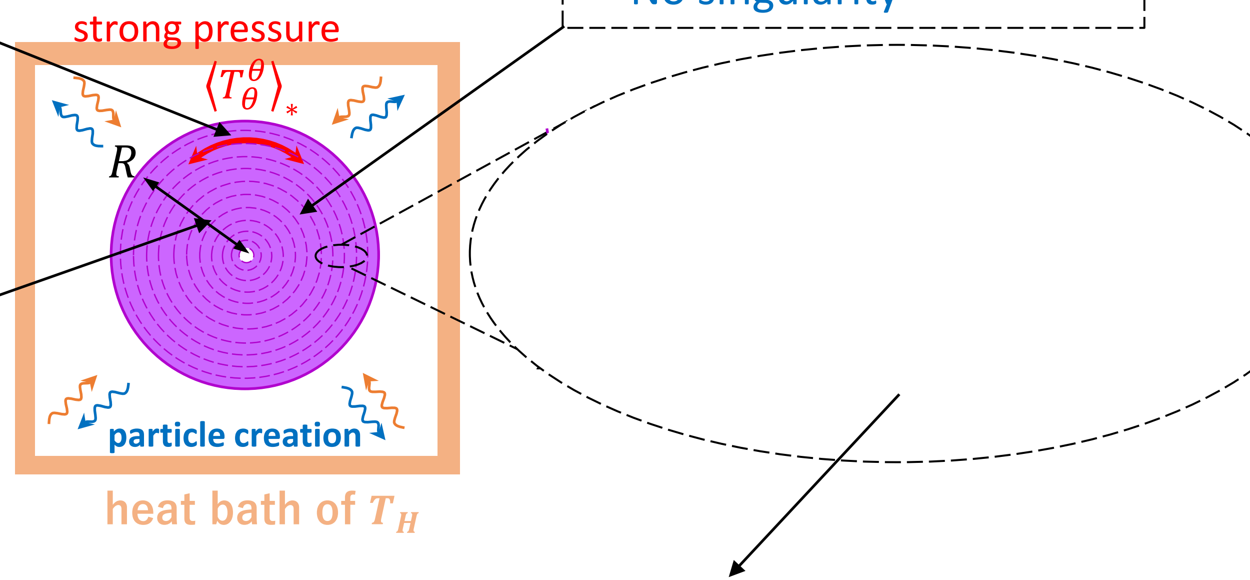
$\langle T^\theta{}_\theta \rangle = O(1)$

- Stabilize the system
- Not fluid

Surface from thermodynamics

$$R = a_0 + \frac{\sigma\eta^2}{2a_0} \ (> a_0)$$

$(a_0 \equiv 2GM_0)$



- Maximum entropy

$$S_{max} =$$

[Yokokura 2022]

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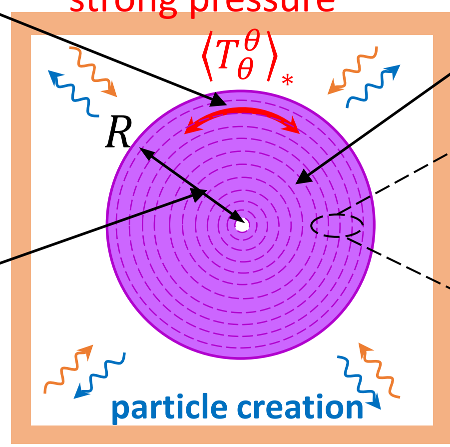

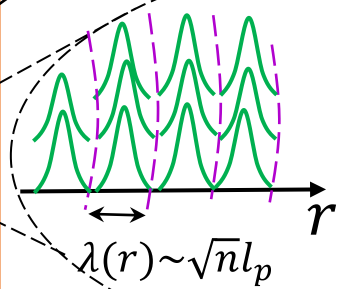

uniform in r direction

$\Rightarrow N(r) = const. \sim n$

⇒entropy density

$$s(r) = \frac{2\pi\sqrt{2\sigma}}{l_p^2}$$

- Maximum entropy

$$S_{max} = \int_0^R dr \sqrt{g_{rr}}\, s(r) = \int_0^R dr \sqrt{\frac{r^2}{2\sigma}} \frac{2\pi\sqrt{2\sigma}}{l_p^2} = \frac{A}{4 l_p^2}$$

[Yokokura 2022]

$$A \equiv 4\pi R^2 = 4\pi a_0^2 + O(1)$$

Verification of Bousso bound

- We have obtained

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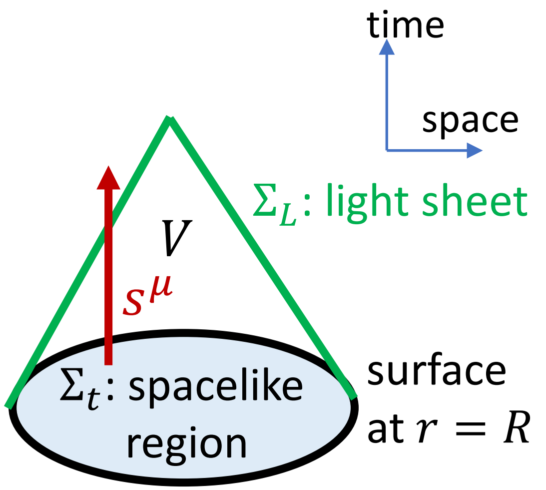
ours   Bousso's



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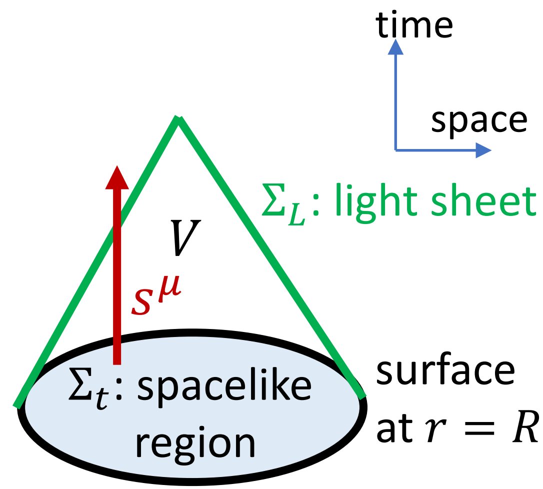


- $g^*_{\mu\nu}$ saturates local sufficient conditions for the Bousso bound:

$$\begin{cases} \left|-s_\mu k^\mu\right| \leq \dfrac{1}{\hbar} \langle T_{\mu\nu} \rangle k^\mu k^\nu \Delta\zeta, & \text{[Flanagan-Marolf-Wald 2000],} \\ \left|k^\mu k^\nu \nabla_\mu s_\nu\right| \leq \dfrac{2\pi}{\hbar} \langle T_{\mu\nu} \rangle k^\mu k^\nu & \text{[Bousso-Flanagan-Marolf 2003]} \end{cases}$$

- Thus, we have verified the Bousso bound by constructing explicitly the saturating configuration.

Conclusions

[Yokokura, arXiv: 2309.00602]

- Considered 4D spherical static spacetime for highly excited states $\{|\psi\rangle\}$.
- Estimated the entropy $S[g_{\mu\nu}]$ including the self-gravity.
- Found the entropy-maximized spacetime uniquely:

$$ds^2 = -\frac{\sigma\eta^2}{2r^2} e^{-\frac{R^2-r^2}{2\sigma\eta}} dt^2 + \frac{r^2}{2\sigma} dr^2 + r^2 d\Omega^2.$$

Non-perturbative solution of $G_{\mu\nu} = 8\pi G \langle\psi|T_{\mu\nu}|\psi\rangle$ for $\hbar$

Dense configuration
(without horizon or singularity)

- Verified the Bousso bound:

$$S \leq S_{max} = \frac{A}{4l_p^2},$$

← the result of the self-gravity

where the information is stored inside.

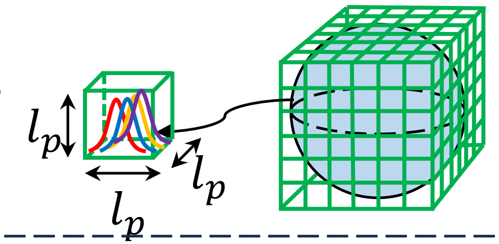
- *Q1: What is the maximum entropy S_{max} that can be given a finite region?*
  $\Rightarrow S_{max} = \frac{A}{4l_p^2}$ (in this class)
- *Q2: What is the structure of such a spacetime?*
  $\Rightarrow$ Necessarily, the above metric (in this class)

Future directions

- Origin of Holography?

A naïve estimation of entropy density

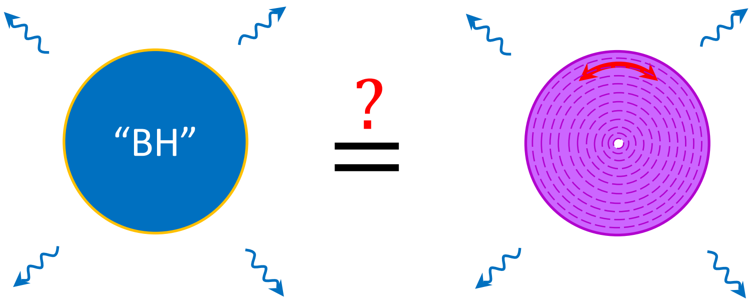
Full excitations of n local d.o.f.



$$S_{max} = 4\pi \int_0^R dr r^2 \sqrt{g_{rr}}\, \bar{s}(r) = \frac{A}{4 l_p^2} \text{ from } \bar{s}(r) \sim \frac{\sqrt{n}}{l_p r^2} \ll \bar{s}_{naive}(x) \sim \frac{n}{l_p^3}$$

⇒Self-gravity suppresses excitations of local d.o.f. ? [work in progress]

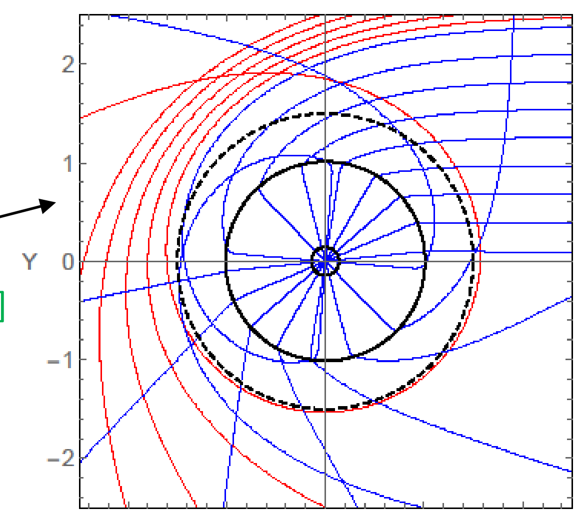
- Quantum BH = the dense configuration?



Both have

$$T = \frac{\hbar}{4\pi a_0}, \qquad S = \frac{A}{4 l_p^2}, \qquad R \approx a_0$$

- Phenomenology as a BH mimicker?
  - BH shadow image [work in progress with C.Y. Chen (iTHEMS)]
  - Gravitational Waves [work in progress with N. Oshita (Hakubi-YITP)]



Thank you!