Exact WKB Analysis and TBA Equations for the Stark Effect

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Outline of the viewpoints

- **Exact WKB analysis (EWKB)** provides an analytic and non-perturbative approach for ODEs in physics.
- **Exact quantization condition (EQC)** is formulated as functional equality in terms of the Borel-resummed quantum WKB periods.
- **Thermodynamic Bethe Ansatz (TBA) equations** encapsulate the quantum periods to satisfy integral equations.

Stark effect

- Hydrogen in a uniform electrostatic field F oriented z axis

$$\left(-\frac{\hbar^2}{2}\nabla^2 - \frac{1}{r} + Fz\right)\Psi = E\Psi$$

- This system exhibits a **discrete** spectrum E_n of the Hamiltonian with properly chosen boundary conditions.
- **Complex** resonant frequency $E = E_c - i\frac{\Gamma}{2}$ is composed of energy peak E_c and ionization rate Γ.
- Not exactly solvable, EWKB provides a systematic approach.

- Adapt the Schrödinger equation with wave ansatz $\Psi = \sqrt{\xi\eta}\psi_1(\xi)\psi_2(\eta)e^{im\varphi}$ under **parabolic coordinates** (ξ, η, φ):

$$\left(-\hbar^2\frac{d^2}{d\xi^2} + \frac{F}{4}\xi - \frac{E}{2} - \frac{A_1}{\xi} + \frac{\hbar^2(m^2-1)}{4\xi^2}\right)\psi_1(\xi) = 0$$
$$\left(-\hbar^2\frac{d^2}{d\eta^2} - \frac{F}{4}\eta - \frac{E}{2} - \frac{A_2}{\eta} + \frac{\hbar^2(m^2-1)}{4\eta^2}\right)\psi_2(\eta) = 0$$

with A_1 and A_2 satisfying $A_1 + A_2 = 1$.

- Langer modification is essential in the WKB analysis.

$$\hbar^2\ell(\ell+1) \to (\ell + \frac{1}{2})^2 - \frac{\hbar^2}{4}$$
$$\frac{\hbar^2(m^2-1)}{4} \to \frac{m^2}{4} - \frac{\hbar^2}{4}.$$

WKB solutions

- Consider the generic equation

$$\left(-\hbar^2\frac{d^2}{dx^2}+Q_0(x)+Q_2(x)\hbar^2\right)\psi(x)=0,$$

$$Q_0(x)=u_0x+u_1+\frac{u_2}{x}+\frac{u_3}{x^2},\quad Q_2(x)=-\frac{1}{4x^2}.$$

- x is a complex coordinate variable, and $\hbar$ is regarded as an expansion parameter setting to 1 in the end.
- Formal WKB solutions

$$\psi_a^{\pm}(x)=\frac{1}{\sqrt{P(x)}}\exp\left(\pm\frac{1}{\hbar}\int_a^x P(x)\mathrm{d}x\right),\quad P(x,\hbar)=\sum_0^{\infty}p_n(x)\hbar^n.$$

- $p_0(x)=\sqrt{Q_0(x)}$, the explicit form for $p_n(x)$ determined recursively.

Stokes graph

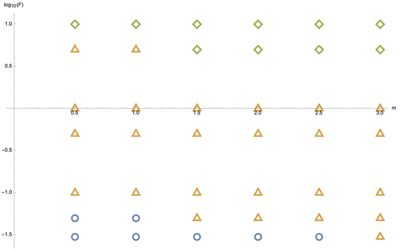
- WKB solutions are well-defined for each Stokes region after the **Borel resummation (Borel resum)**.
- Analytic continuation of the Borel-resummed WKB solutions encounters obstacles in the **Stokes line**

$$\Im \frac{1}{\hbar} \int_a^x \sqrt{Q_0(x)} \mathrm{d}x = 0.$$

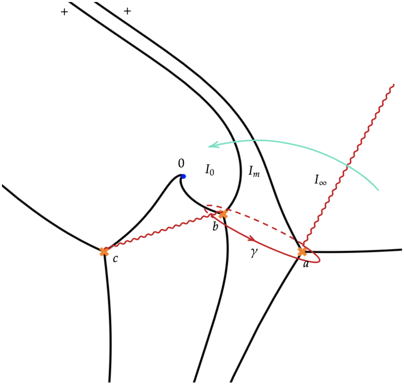
turning points a satisfies $Q_0(a) = 0$

- Stokes lines delimit the complex coordinate plane into adjacent **Stokes regions** I.
- Collection of all Stokes lines forms the **Stokes graph**, which depends on the parameters in classical potential $Q_0(x)$.

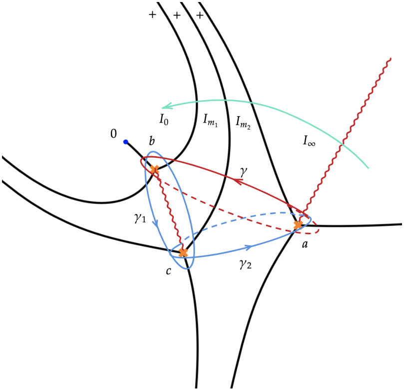
- Configuration of the Stokes graph changes as F and m change.



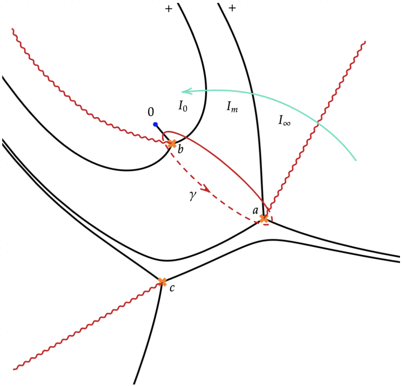
Three distinct regions of parameters (F, m): I_ξ, II_ξ and III_ξ from below to above.



Stokes graph for ξ-equation in region III_ξ



Stokes graph for ξ-equation in region II_ξ



Stokes graph for ξ-equation in region I_ξ

Connection formulas

- WKB solutions are discontinuously changed when crossing a Stokes line.
- Connection formulas [Iwaki and Nakanishi, arXiv:1409.4641]

$$\begin{cases} \psi^+ \to \psi^+ + i\psi^-, \quad \psi^- \to \psi^-, \text{ for anticlockwise across Stokes line,} \\ \psi^+ \to \psi^+ - i\psi^-, \quad \psi^- \to \psi^-, \text{ for clockwise across Stokes line.} \end{cases}$$

- Analytic continuation of the wave basis from infinity to the origin

$$\begin{pmatrix} \iota\psi^-_{a,\mathrm{I}_\infty}(\xi) \\ \iota\psi^+_{a,\mathrm{I}_\infty}(\xi) \end{pmatrix} = \begin{cases} \begin{pmatrix} \psi^+_{a,\mathrm{I}_0}(\xi) + i(1+\mathcal{V}_\gamma)\psi^-_{a,\mathrm{I}_0}(\xi) \\ \psi^-_{a,\mathrm{I}_0}(\xi) \end{pmatrix}, \text{ for region } \mathrm{I}_\xi, \mathrm{III}_\xi \\ \begin{pmatrix} \psi^+_{a,\mathrm{I}_0}(\xi) + i(1+\mathcal{V}_{\gamma_2} + \mathcal{V}_{\gamma_1}\mathcal{V}_{\gamma_2})\psi^-_{a,\mathrm{I}_0}(\xi) \\ \psi^-_{a,\mathrm{I}_0}(\xi) \end{pmatrix}, \text{ for region } \mathrm{II}_\xi. \end{cases}$$

- Quantum WKB periods: $\Pi_\gamma(\hbar) = \oint_\gamma P(x)\mathrm{d}x$, its asymptotic expansions in $\hbar$ computable.
- Voros symbol: $\mathcal{V}_\gamma = \exp\left(\frac{1}{\hbar}\oint_\gamma P(x)\mathrm{d}x\right)$

Exact quantization conditions

- Boundary conditions:
  - Around the origin

$$\psi_\pm(\xi) \sim \frac{1}{\sqrt{p_0(\xi)}} \exp(\pm \int^\xi p_0(\xi) d\xi) \sim \xi^{\frac{1}{2} \pm \frac{|m|}{2}} \sim \begin{cases} \xi^{\ell+1}, & \text{for} \quad \psi_+, \\ \xi^{-\ell}, & \text{for} \quad \psi_-. \end{cases}$$

  - Near positive infinity
    $\psi_\pm(\xi) \sim \xi^{-\frac{1}{4}} \exp\left(\pm \frac{F}{3} \xi^{\frac{3}{2}}\right)$, $-$ component required.

Exact quantization conditions

$$\begin{cases} \Pi_\gamma(\hbar) = \oint P(\xi) d\xi = 2\pi i \hbar \left(n_\xi + \frac{1}{2}\right), \text{ for region } \mathrm{I}_\xi, \mathrm{III}_\xi. \\ \Pi_\gamma(\hbar) + \log\left(1 + e^{-\frac{1}{\hbar}\Pi_{\gamma_1}(\hbar)}\right) = 2\pi i \hbar \left(n_\xi + \frac{1}{2}\right), \text{ for region } \mathrm{II}_\xi. \end{cases}$$

- Quantum periods here are understood as Borel-resummed quantities.
- EQCs are exact in the sense of Borel resum, the EQCs for η-equation can be established similarly.

- We derived EQCs for different regions of parameters, but the EQC itself must be continuous as F changes. This continuity is guaranteed by the **Delabaere-Dillinger-Pham (DDP) formula**

$$\mathcal{Q}_{\mathrm{I}_\xi}(\mathcal{V}_\gamma) := 1 + \mathcal{V}_\gamma = 0.$$

$$\mathcal{Q}_{\mathrm{II}_\xi}(\mathcal{V}_\gamma, \mathcal{V}_{\gamma_1}) := 1 + \mathcal{V}_\gamma + \mathcal{V}_{\gamma_2} = 1 + \mathcal{V}_\gamma \left(1 + \mathcal{V}_{\gamma_1}^{-1}\right) = 0.$$

DDP formula

$$\mathcal{S}_+ \left[\mathcal{V}_\gamma\right] = \mathcal{S}_- \left[\mathcal{V}_\gamma\right] \left(1 + \mathcal{S}\left[\mathcal{V}_{\gamma_1}^{-1}\right]\right)$$

- $\mathcal{S}_\pm$ represent two lateral Borel resummations which are continuously deformed to the quantum periods (or Voros symbols) for region I_ξ and II_ξ respectively.

Complex resonant frequency and computation

- EQCs for ξ and η variables involving E and A_1, sorting out discrete values of them labelled by quantum number (n_ξ, n_η, m) for fixed F and m.
- Complex resonant frequencies E_n and separation constant A_1^n resolved by the simultaneous solutions of two EQCs

F	0.005
$\mathcal{O}(\hbar^0)$	-0.5000265658500433
$\mathcal{O}(\hbar^2)$	-0.5000562669655335
$\mathcal{O}(\hbar^4)$	-0.5000562847698887
$\mathcal{O}(\hbar^6)$	-0.5000562847937314
$\mathcal{O}(\hbar^8)$	-0.5000562847937927
RPM	-0.5000562847937930
F	1
$\mathcal{O}(\hbar^0)$	$-0.6697556295027800 - 0.5359677715861759i$
RPM	$-0.6243365071054226 - 0.6468208995063122i$

Ground state energy level E_0 for $F = 0.005$ and 1 with $m = 0$

TBA equations

- Quantum periods correspond to the Y functions which follow the TBA equations.
$$Y(\theta) \sim e^{\Pi(\hbar)}, \quad \hbar = e^{-\theta}$$
- Unfortunately, we cannot write down the TBA equations in the presence of Langer's modification by now.
- TBA equations for bare potential are partially established for $-1 < \ell < 1$ in the minimal chamber. [Ito and Shu, arxiv:1910.09406]

$$Q(x) = u_0 x + u_1 + \frac{u_2}{x} + \frac{\ell(\ell+1)\hbar^2}{x^2}.$$

$$\log Y_1(\theta) = -|\Pi_1^{(0)}|e^{\theta} + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\log\left(1 - e^{2\pi i \ell}\hat{Y}(\theta')\right)\left(1 - e^{-2\pi i \ell}\hat{Y}(\theta')\right)}{\cosh(\theta - \theta')}$$

$$\log \hat{Y}(\theta) = -|\hat{\Pi}^{(0)}|e^{\theta} + \int_{\mathbb{R}} \frac{d\theta'}{2\pi} \frac{\log(1 + Y_1(\theta'))}{\cosh(\theta - \theta')}.$$

Continuation of the TBA equations

- TBA equations encode the all-order expansions and the Borel singularity of quantum periods.
- parameters u_i and ℓ for the Stark effect live outside the minimal chamber of the TBA system, the continuation for these two kinds of parameters is required.
- Continuation in ℓ introduces **logarithmic corrections** for Y-function. [Barak Gabai and Xi Yin, arxiv:2109.07516]

$$\log\left(\frac{e^{\theta}+ie^{\alpha_1(\ell)}}{e^{\theta}-ie^{\alpha_1(\ell)}}\right)$$

- Continuation in u_i results in the **wall-crossing** of TBA equations. The explicit form is still unknown, but it can be mapped to the TBA equations for the quartic potential by coordinate transform $x \to z^2$ for special $\ell = -\frac{1}{4}, -\frac{3}{4}$.

Summary and outlook

- EWKB analysis is applied to a couple of ODEs reduced from the Stark effect to provide an **analytic and exact** formulation.
- EQCs derived from Stokes graphs and connection formulas, determine the complex resonant spectrum for E.
- TBA equations are applied to determining quantum WKB periods for special cases without taking Langer's modification into account.
- The strategy here is capable of being applied in more fruitful areas, for example, (A_1, D_r)-type Argyres-Douglas theory and black hole quasinormal modes and so on.

Thanks for your attention!

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