# Research on QFT beyond the high-symmetry regime -resurgence, lattice and graph-

Tatsuhiro MISUMI (Kindai U.)

KEK-TH 2023 @KEK, Tsukuba 11/29/23

# Research on QFT without the aid of symmetry
## -resurgence, lattice and graph-

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# My plan of talk

- "Symmetry" is one of the most powerful tools to explore QFT as well as nature theoretically.
- After the establishment of Standard model, we have searched for more and more "symmetric" theories as BSM and theoretical tools (GUT, SUSY, CFT, etc.).
- In this talk, after discussing the power of symmetry, I turn to discussing how we tackle QFT without aid of symmetry.

# 0. Saga of symmetry

# What is symmetry in QFT

## We had paradigm shift on symmetry in the last decade:

See also Tanizaki-san's talk KEK-TH 22

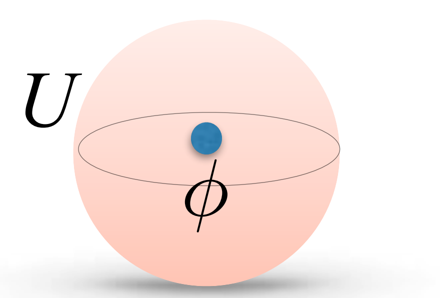
Symmetry was characterized and classified by groups.

$d * J = 0$ $\quad Q = \int_{M_3} *J$ $\Rightarrow$ $U = e^{i\alpha Q}$ group element

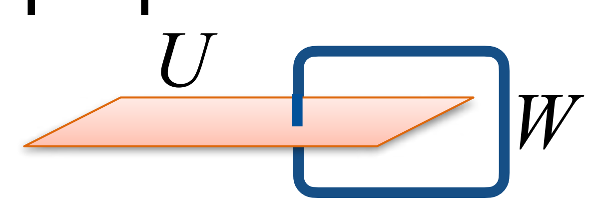
⇓

Symmetry is characterized and classified by **topological operators.**

ex.) 3d top. operator



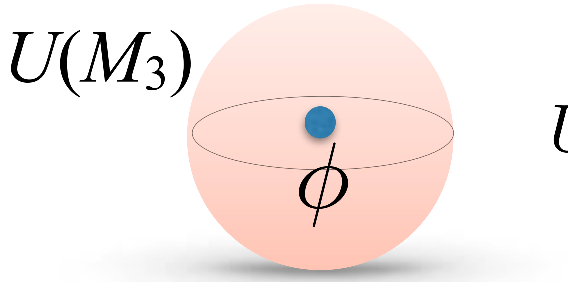

ex.) 2d top. operator



# Higher-form symmetry

Gaiotto, Kapustin, Seiberg, Willett (14)

- 0-form symmetry in 4d : usual global symmetry
  3d top. operator $U(M_3)$ and 0d charged operator $\varphi$

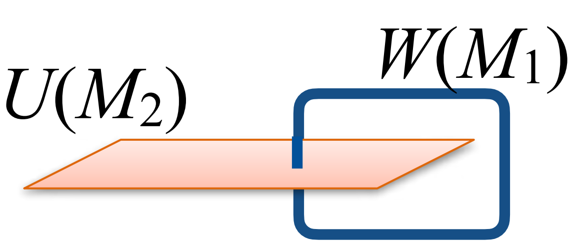



$$U_g \phi U_g^{-1} = R_g \phi$$

ex.) $\phi \to e^{i\theta}\phi$ U(1) 0-form symmetry

$\phi \to e^{i\frac{2\pi}{N}}\phi$ $Z_N$ 0-form symmetry

- 1-form symmetry in 4d :
  2d top. operator $U(M_2)$ and 1d line charged operator $W(M_1)$




$$U_g W U_g^{-1} = R_g W$$

ex.) $W \to e^{i\frac{2\pi}{N}} W$ $Z_N$ 1-form symmetry

center symmetry in pure $SU(N)$ YM

cf.) Abe-san's talk on 30th

- $p$-form symmetry in $D$+1 dimensions :
  ($D$-$p$)-dim top. operator $U(M_{D\text{-}p})$ and $p$-dim operator $W(M_p)$

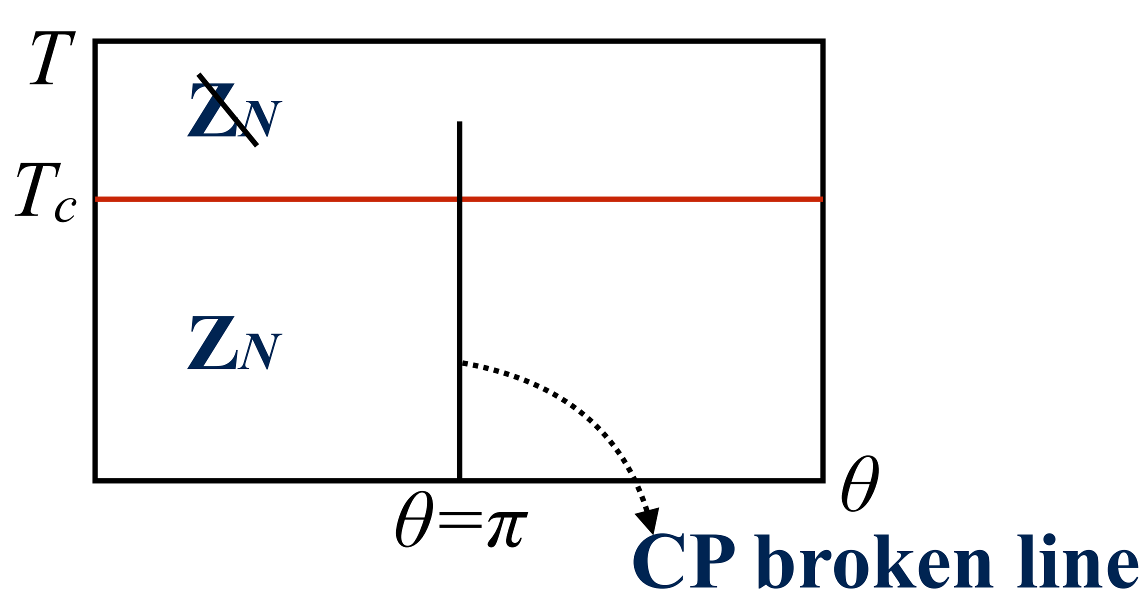
# SU($N$) Yang-Mills theory with $\theta$=$\pi$ on $R_3 \times S_1$

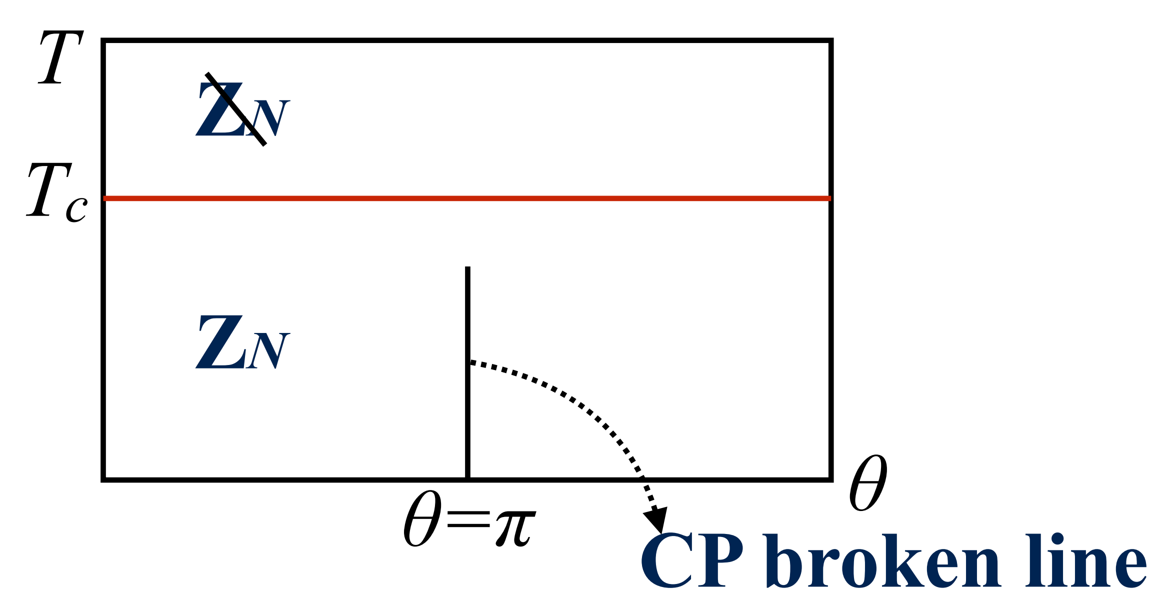
Gaiotto, Kapustin, Komargodski, Seiberg (17)

$$S = \frac{1}{2g^2}\int \mathrm{tr}[G \wedge *G] + \frac{i\theta}{8\pi^2}\int \mathrm{tr}[G \wedge G] \qquad G = da + ia \wedge a$$

- $Z_N$ 1-form symmetry (center symmetry)
- Time reversal invariance (CP symmetry) at $\theta = 0, \pi$

**’t Hooft anomaly indicates SSB of CP or $Z_N$ 1-form symmetry even at finite-T, prohibiting trivially gapped phase!**



# SU($N$) Yang-Mills theory with $\theta=\pi$ on $R_3 \times S_1$

Gaiotto, Kapustin, Komargodski, Seiberg (17)

$$S = \frac{1}{2g^2}\int \mathrm{tr}[G \wedge *G] + \frac{i\theta}{8\pi^2}\int \mathrm{tr}[G \wedge G] \qquad G = da + ia \wedge a$$

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- Time reversal invariance (CP symmetry) at $\theta = 0, \pi$

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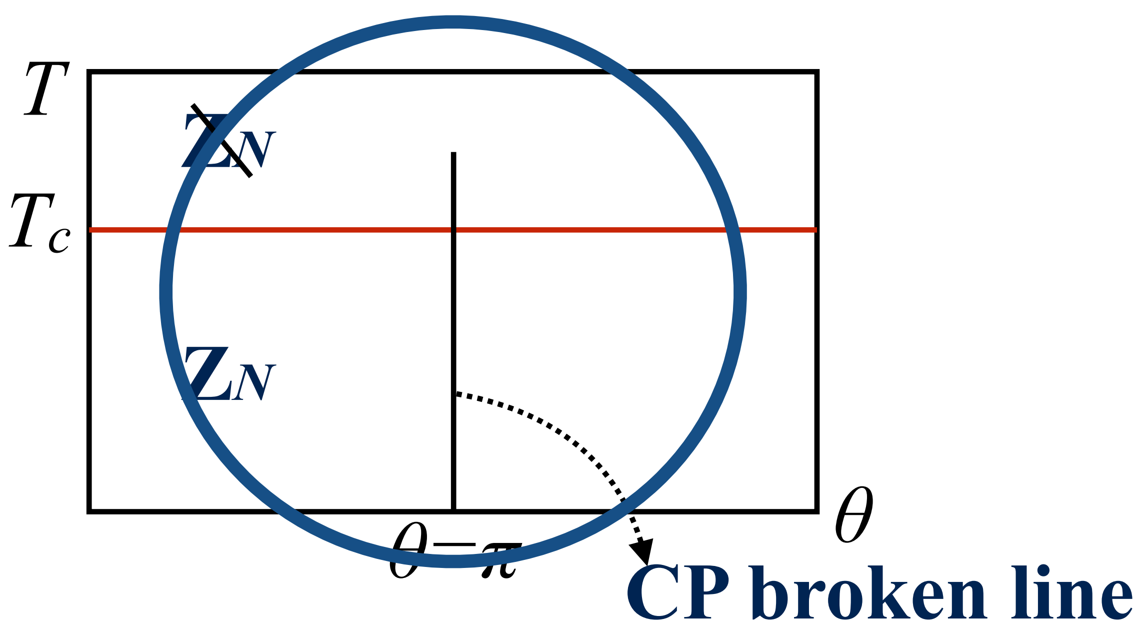
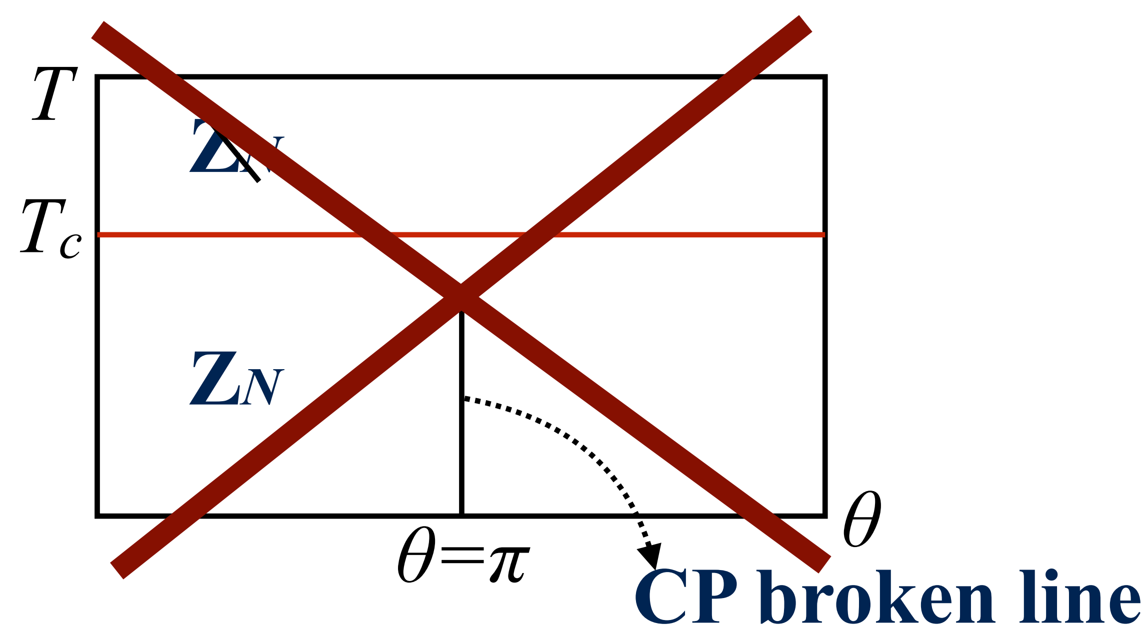
→

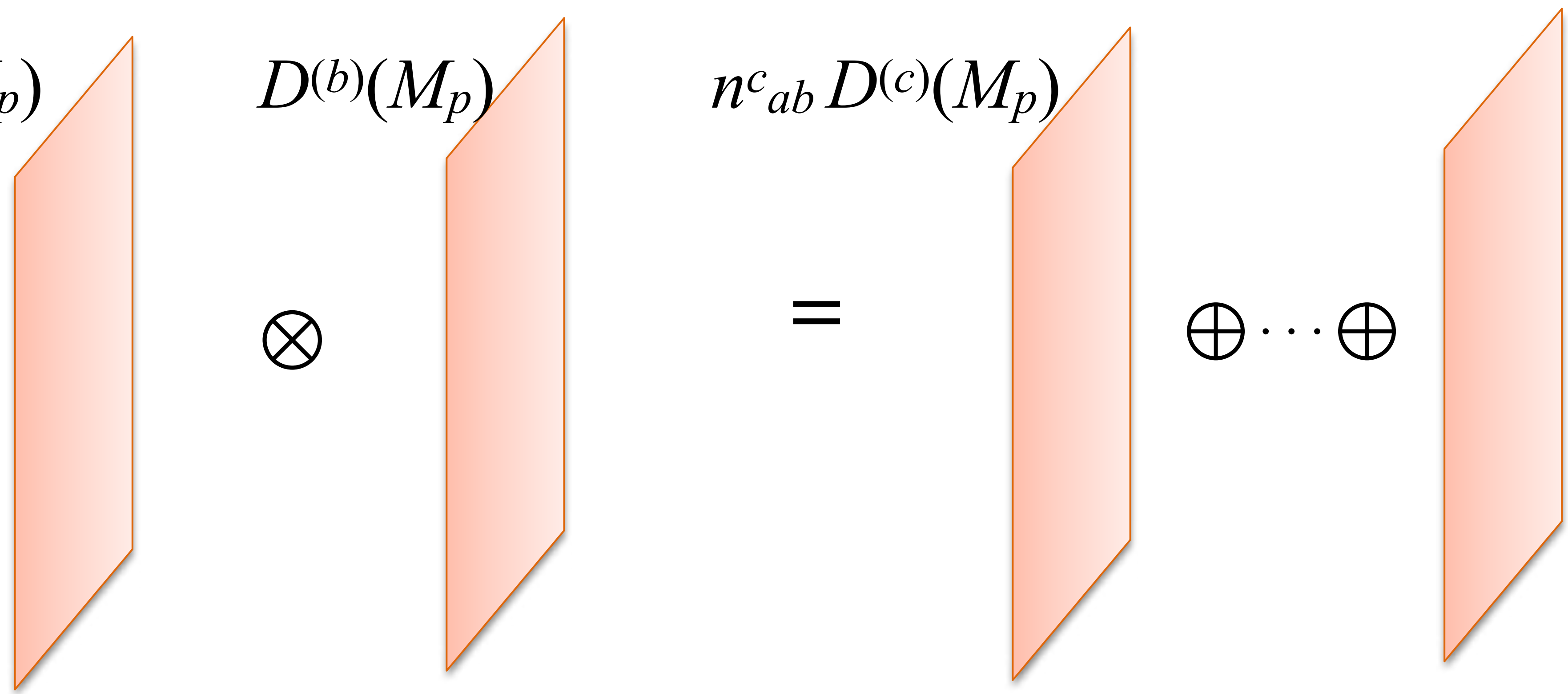


# Non-invertible (categorical) symmetry

Bhardwaj, Tachikawa(17), Chang, et.al. (18), Thorngren, Wang(19), Komargodski, et.al. (20), Nguyen, et.al. (21)

Some topological operator behaves not as group but fusion category
(O(2) gauge theory, SO(3) YM with θ=π, etc. )



$$D^{(a)}(M_p) \otimes D^{(b)}(M_p) = n^c{}_{ab}\, D^{(c)}(M_p) \oplus \cdots \oplus$$

$$a \otimes b = \sum_{c \in \mathcal{S}} n^c_{ab}\, c\,, \qquad a, b \in \mathcal{S}$$

Not group,
but fusion category

cf.) Shimamori-san's, Wada-san's and Yokokura-san's talks on 30th

cf.)2d critical Ising (c=1/2 rational CFT)

primary operators $\mathrm{id}, \eta, D$ ⇒ $\eta \otimes \eta = \mathrm{id}\,, \quad \eta \otimes D = D \otimes \eta = D\,, \quad \underline{D \otimes D = \mathrm{id} \oplus \eta}$

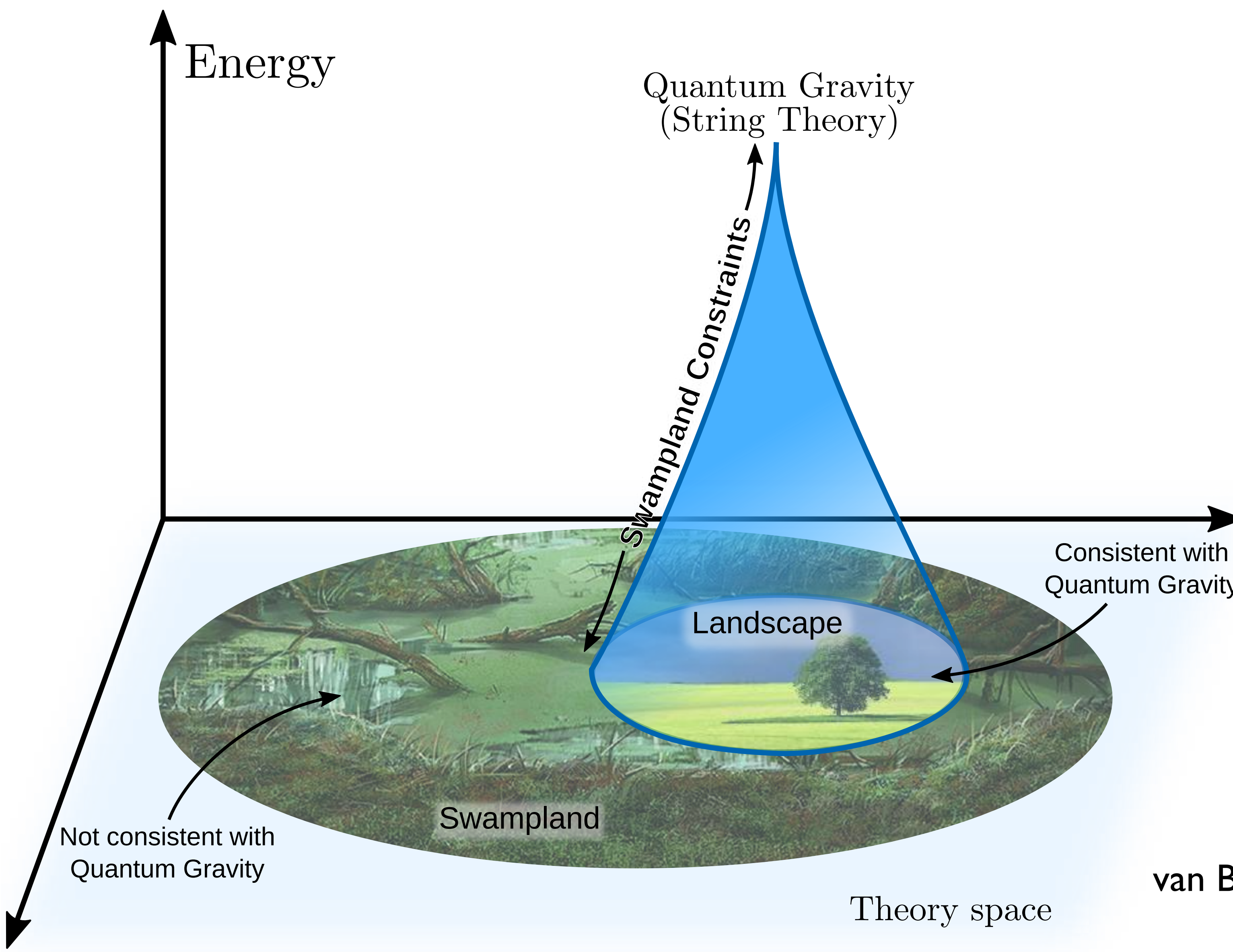
fusion rule

Although the notion of symmetry has been generalized well, all of QFT with global symmetry may be in Swampland….

# Swampland and no global symmetry conjecture

Banks-Seiberg 10, ..., Harlow-Ooguri 18
cf.) Hamada-san's talk on 30th

Swampland QFT: Self-consistent and well-defined QFT that is inconsistent with quantum gravity




van Beest, et.al.(21)

# Swampland and no global symmetry conjecture

Banks-Seiberg 10, ..., Harlow-Ooguri 18

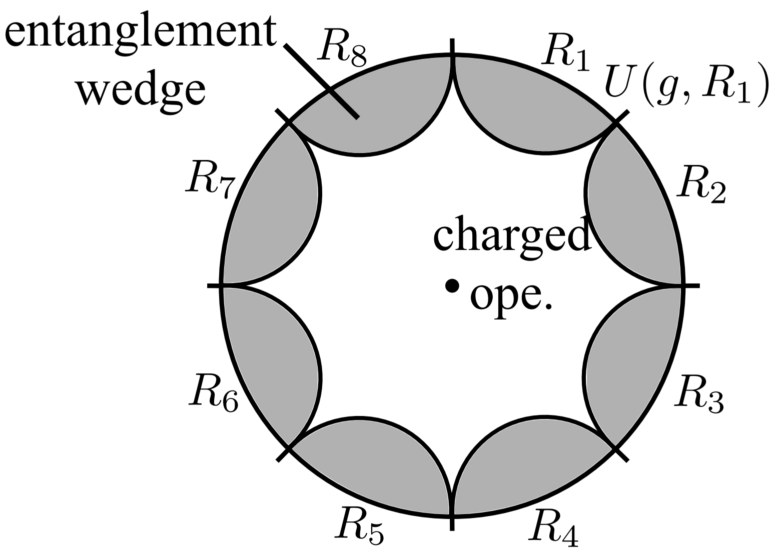
No global symmetry conjecture:
Any internal symmetry in quantum gravity is either broken or gauged.

Blackhole argument (Banks-Seiberg 10):

Hawking radiation is only sensitive to gauge charges measurable from outside.
After evaporation, blackhole with global charge can have large entropy, inconsistent to BH entropy.

AdS/CFT proof (Harlow-Ooguri 18):

Assumption of global symmetry in bulk leads to contradiction in boundary:
$U_g = \Pi_n U_g(R_n)$ ($R_n$: small boundary region), and
$U_g(R_n)$ trivially commutes with ope. in center of bulk.



Indeed, Standard model has no symmetry but B-L symmetry which may also be approximate symmetry…

Although the conjecture does not necessarily forbid approximate symmetry,
it is now worth searching for ways of analyzing QFT without aid of symmetry…

# Examples of no-symmetry-aid tools for QFT

## I. Resurgence theory

Topology, differential and algebraic geometry, etc. has been applied to QFT.
In that sense, Resurgence is "Algebraic analysis" for QFT.

## II. Lattice field theory and graph theory

Probability theory, computer science and analysis has been used for lattice QFT.
In that sense, Graph theory is "Discrete geometry" for QFT.

# I. Resurgence theory

# Path integral and Saddle points

$$Z = \int \mathcal{D}\phi \, \exp(-S[\phi]) = \sum_{\sigma \in \text{saddles}} Z_\sigma$$

: stationary points

**Trivial (perturbative) saddle**

$$Z_0 = \sum_{q=0}^{\infty} a_q g^{2q}$$

**Perturbative series**

**Nontrivial saddles** $\frac{\delta S}{\delta \phi} = 0$

$$Z_\sigma \propto e^{-S_{\rm sol}} \sim e^{-\frac{A}{g^2}}$$

**Non-perturbative contribution**

# Path integral and Saddle points

$$\sum_{q=0} a_q g^{2q}$$

$$\exp\left[-\frac{A}{g^2}\right]$$

**Perturbative series**

**Non-perturbative contribution**

"They are not connected ?
We just have independent contributions ?"

**No, it is not correct !**

# Perturbation and Borel resummation

$$\left[H_0 + g^2 H_{\rm pert}\right]\psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$ Perturbative series is often divergent factorially $a_q \propto q!$

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$ Borel transform

$$\mathbb{B}(g^2) = \int_0^\infty \frac{dt}{g^2} e^{-t/g^2} BP(t)$$ Borel resummation

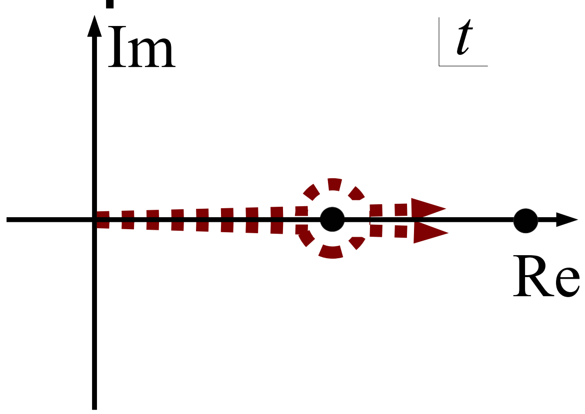
# Perturbation and Borel resummation

$$\left[H_0 + g^2 H_{\rm pert}\right] \psi(x) = E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$ Perturbative series is often divergent factorially $$a_q \propto q!$$

**Borel transform can have singularities on positive real axis**

$$BP(t) := \sum_{q=0}^{\infty} \frac{a_q}{q!} t^q$$




$$\Rightarrow \quad \mathbb{B}(g^2 e^{\mp i\epsilon}) = \int_0^{\infty e^{\pm i\epsilon}} \frac{dt}{g^2} e^{-\frac{t}{g^2}} BP(t)$$

Singularities on positive real axis leads to ambiguity

# Perturbation and Borel resummation

$$\left[H_0 \ + \ g^2 H_{\rm pert}\right] \psi(x) \ = \ E\psi(x)$$

$$P(g^2) = \sum_{q=0}^{\infty} a_q g^{2q}$$ Perturbative series is often divergent factorially $$a_q \propto q!$$

$$\mathbb{B}(g^2 e^{\mp i\epsilon}) = {\rm Re}[\mathbb{B}(g^2)] \pm i{\rm Im}[\mathbb{B}(g^2)]$$

$${\rm Im}[\mathbb{B}(g^2)] \ \approx \ e^{-\frac{A}{g^2}}$$

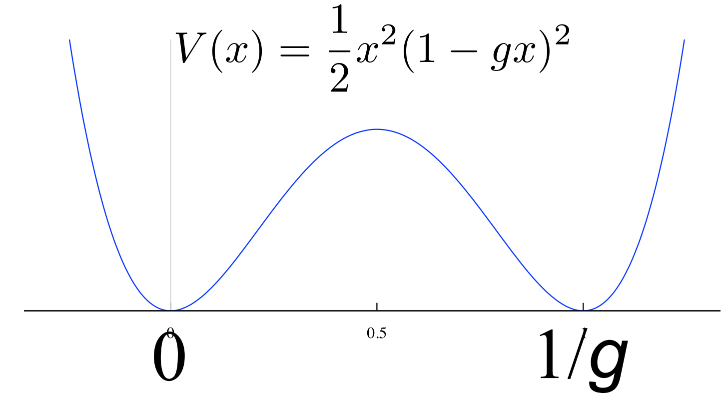
This ambiguity is cancelled by that from non-perturbative contribution!

Non-pert. cont. shows up in perturbative calculation through imaginary ambiguity : resurgent structure

# Resurgence in quantum mechanics

Zinn-Justin, Bogomolny, Voros,.. .(80's)

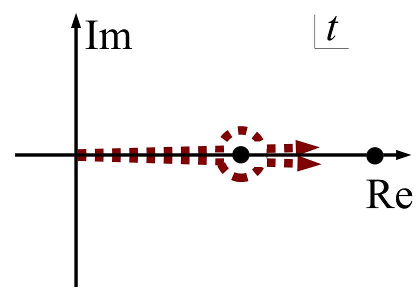
## Ex.) Double-well potential QM




$$E = \frac{1}{2} - g^2 - \frac{9}{2}g^4 - \frac{89}{2}g^6 - \frac{5013}{8}g^8 \dots \rightarrow -\frac{3^{q+1}q!}{\pi}g^{2q} \quad (q\rightarrow\infty)$$

Borel transform

$$BE(t) = \sum_{q=0}^{\infty} \frac{a_q}{q!}t^q = \frac{1}{\pi}\frac{1}{t-1/3}$$

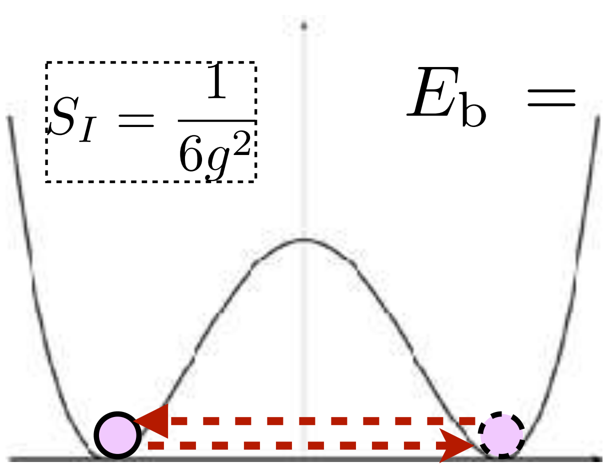



Borel resummation

$$\mathbb{E}(g^2 e^{\pm i\epsilon}) = \mathrm{Re}\mathbb{E} \mp i\frac{e^{-2S_I}}{g^2}$$

Imaginary ambiguity emerges in pert. Borel resum.

cancelled out




$$E_\mathrm{b} = \mathrm{Re}E_\mathrm{b} \pm i\frac{e^{-2S_I}}{g^2}$$

Im. ambiguity emerges in non-pert. tunneling effect

bion = instanton + antiinstanton

Unsal (07)

Non-pert. cont. shows up in perturbative calculation through imaginary ambiguity : resurgent structure

# Resurgence in quantum mechanics

Zinn-Justin, Bogomolny, Voros,.. .(80's)

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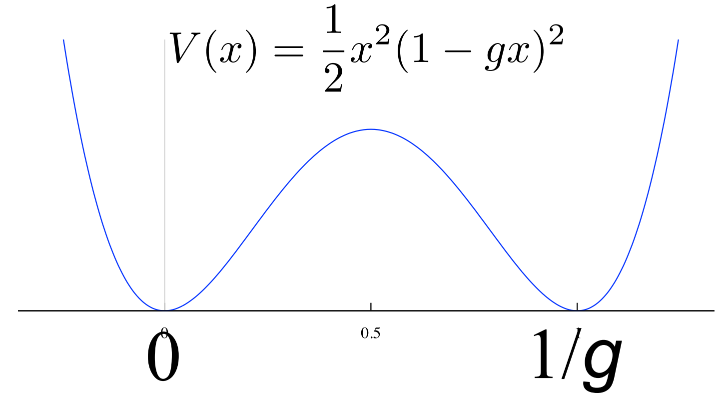
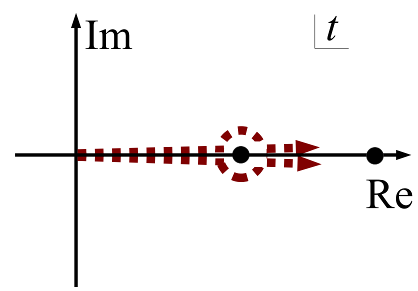



$$E = \frac{1}{2} - g^2 - \frac{9}{2}g^4 - \frac{89}{2}g^6 - \frac{5013}{8}g^8 \ldots \rightarrow -\frac{3^{q+1}q!}{\pi}g^{2q} \quad (q\to\infty)$$

Borel transform

$$BE(t) = \sum_{q=0}^{\infty} \frac{a_q}{q!}t^q = \frac{1}{\pi}\frac{1}{t-1/3}$$




Borel resummation

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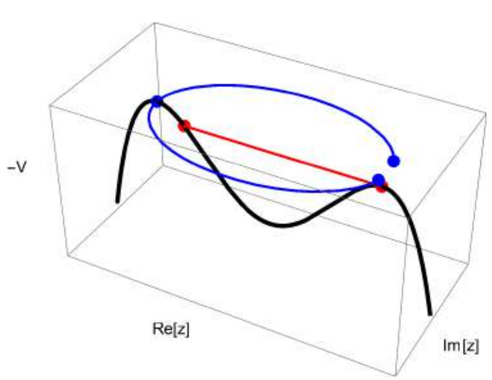
cancelled out

- Contribution from complex bion solution to $E$




$$E_{cb} = \frac{e^{-\frac{1}{3g^2}}}{\pi g^2}\left(\frac{g^2}{2}\right)^{\epsilon}\left[-\cos(\epsilon\pi)\Gamma(\epsilon) \pm \frac{i\pi}{\Gamma(1-\epsilon)}\right]$$

Behtash, et.al. (15)
Fujimori, et.al. (16)(17)

- The imaginary ambiguity from bion cancels that from perturbative series
- it disappears for ε=1 (SUSY) while real part consistent with SSB of SUSY

# Resurgence in integral

Integral is decomposed into saddle contributions
in steepest descent method (thimble decomposition)

cf.) Chou-san's and Tripathi-san's talks today

· Airy integral

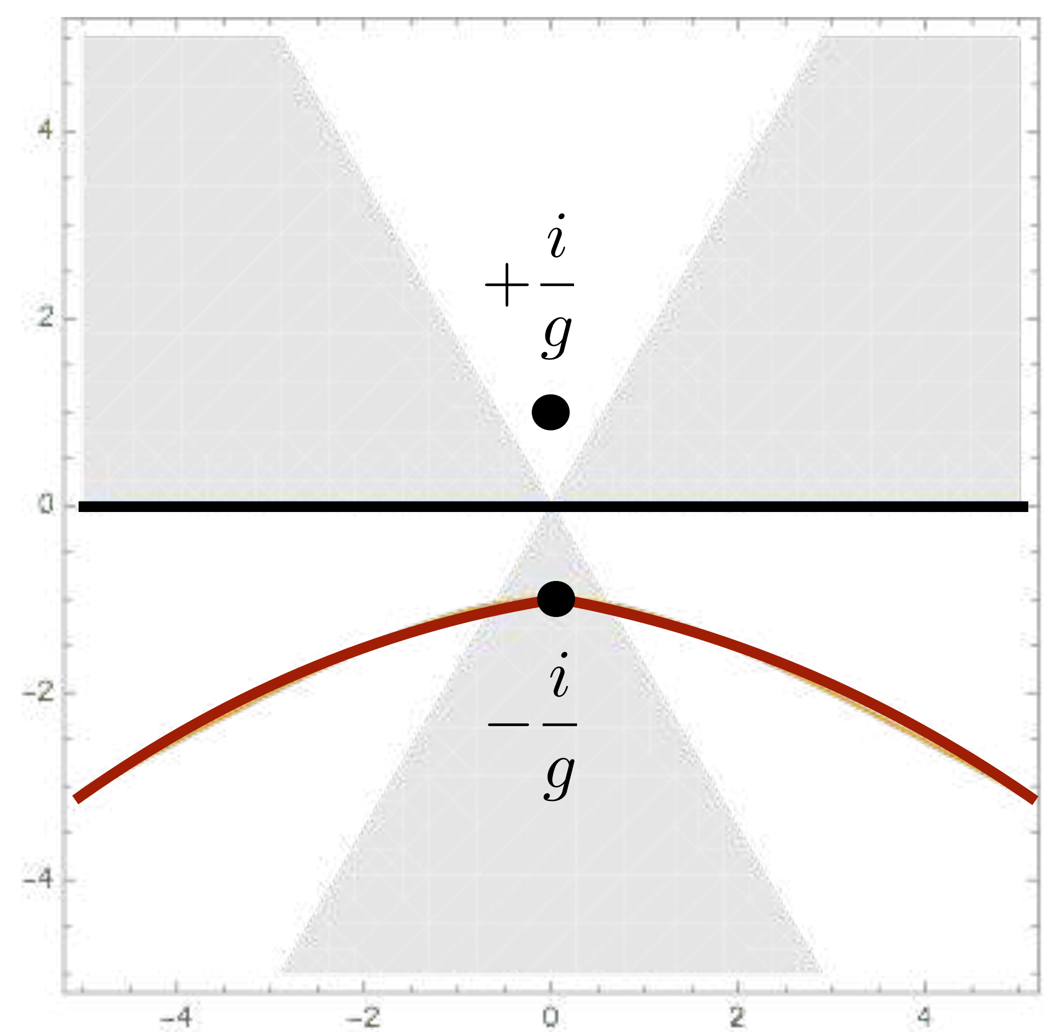
$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

complex saddle points $\phi = \pm \frac{i}{g}$

Steepest descent method :
Original contour is decomposed into
contours associated with saddle points.

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$$

Steepest descent contour
= Thimble




$\arg[g^2] = 0+$

# Resurgence in integral

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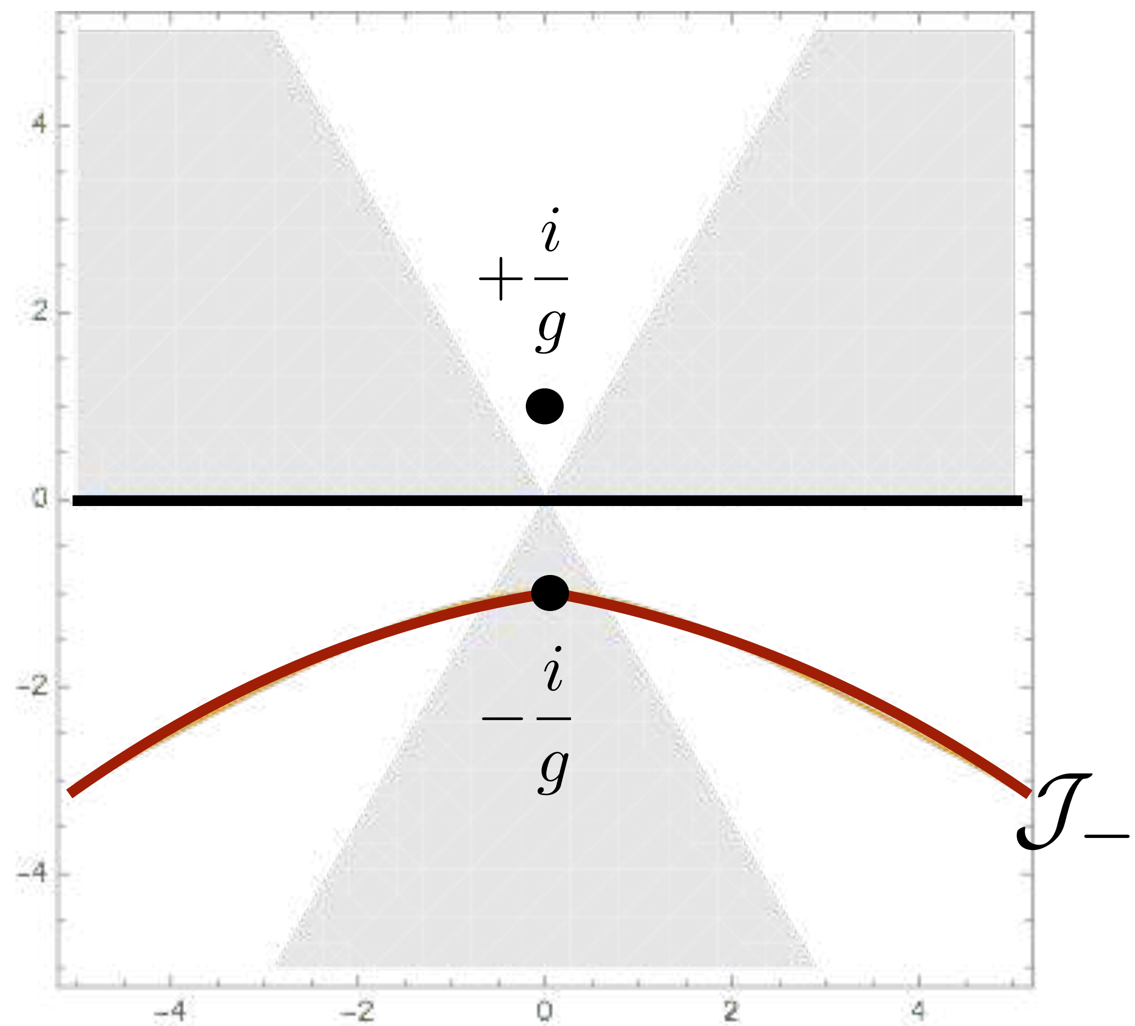
· Airy integral

$$\mathrm{Ai}(g^{-2}) = \int_{-\infty}^{\infty} d\phi \exp\left[-i\left(\frac{\phi^3}{3} + \frac{\phi}{g^2}\right)\right]$$

- $\mathcal{J}_\sigma$ $\quad \mathrm{Im}[S] = \mathrm{Im}[S_0]$, $\quad \mathrm{Re}[S] \leq \mathrm{Re}[S_0]$ — Thimble
- $n_\sigma = \langle \mathcal{K}_\sigma, \mathcal{C} \rangle$ — Intersection number of dual thimble $\mathcal{K}$ and original contour

$\Rightarrow$ $$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma$$




$\arg[g^2] = 0+$

# Resurgence in integral

Integral is decomposed into saddle contributions
in steepest descent method (thimble decomposition)

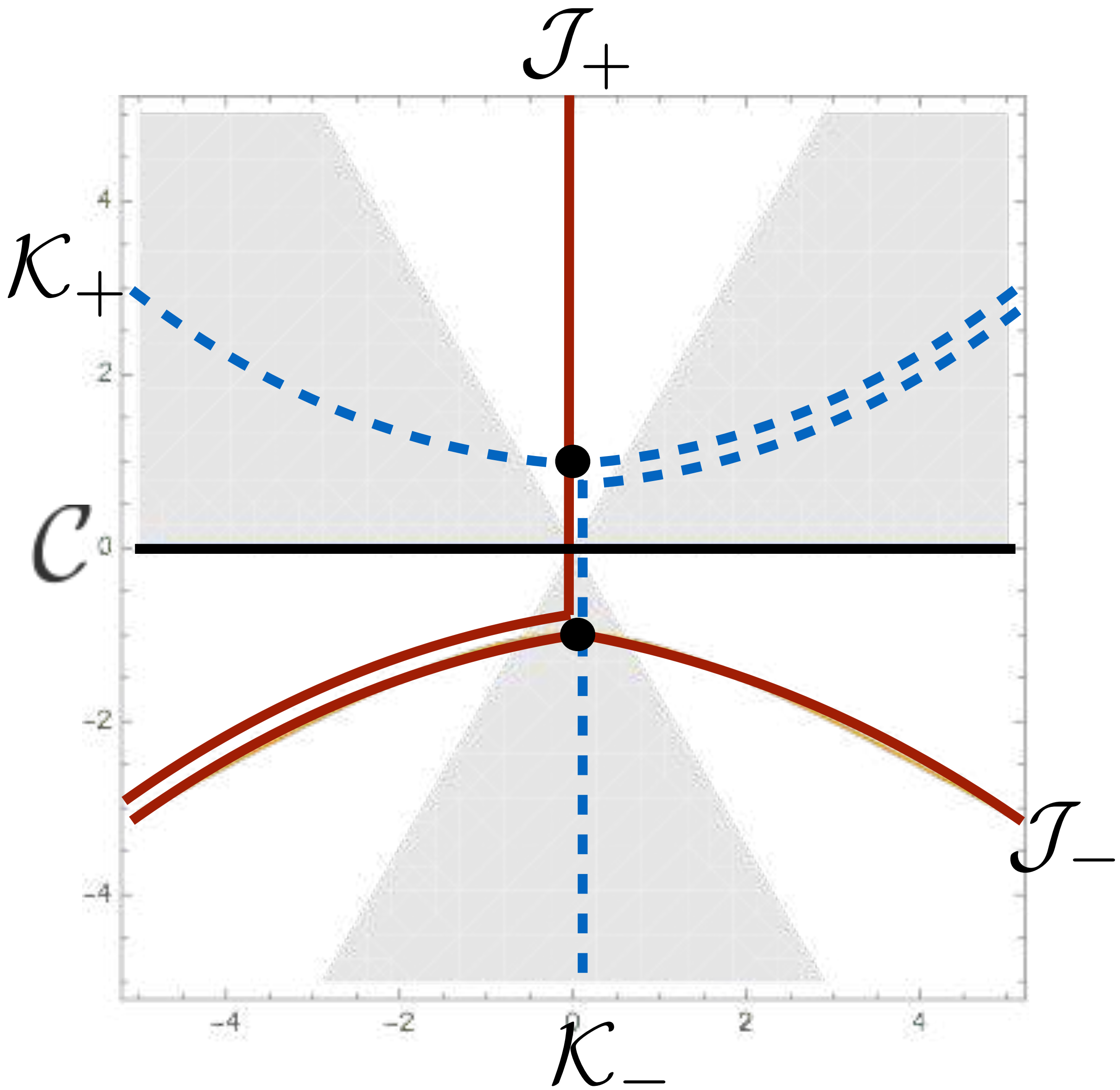
· Airy integral $\arg[g^2] = 0+$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 0$$

$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \Rightarrow \boxed{\mathcal{C} = \mathcal{J}_-}$$

valid decomposition till $\arg[g^2] = \frac{2\pi}{3}-$




$\arg[g^2] = 0+$

# Resurgence in integral

Integral is decomposed into saddle contributions
in steepest descent method (thimble decomposition)

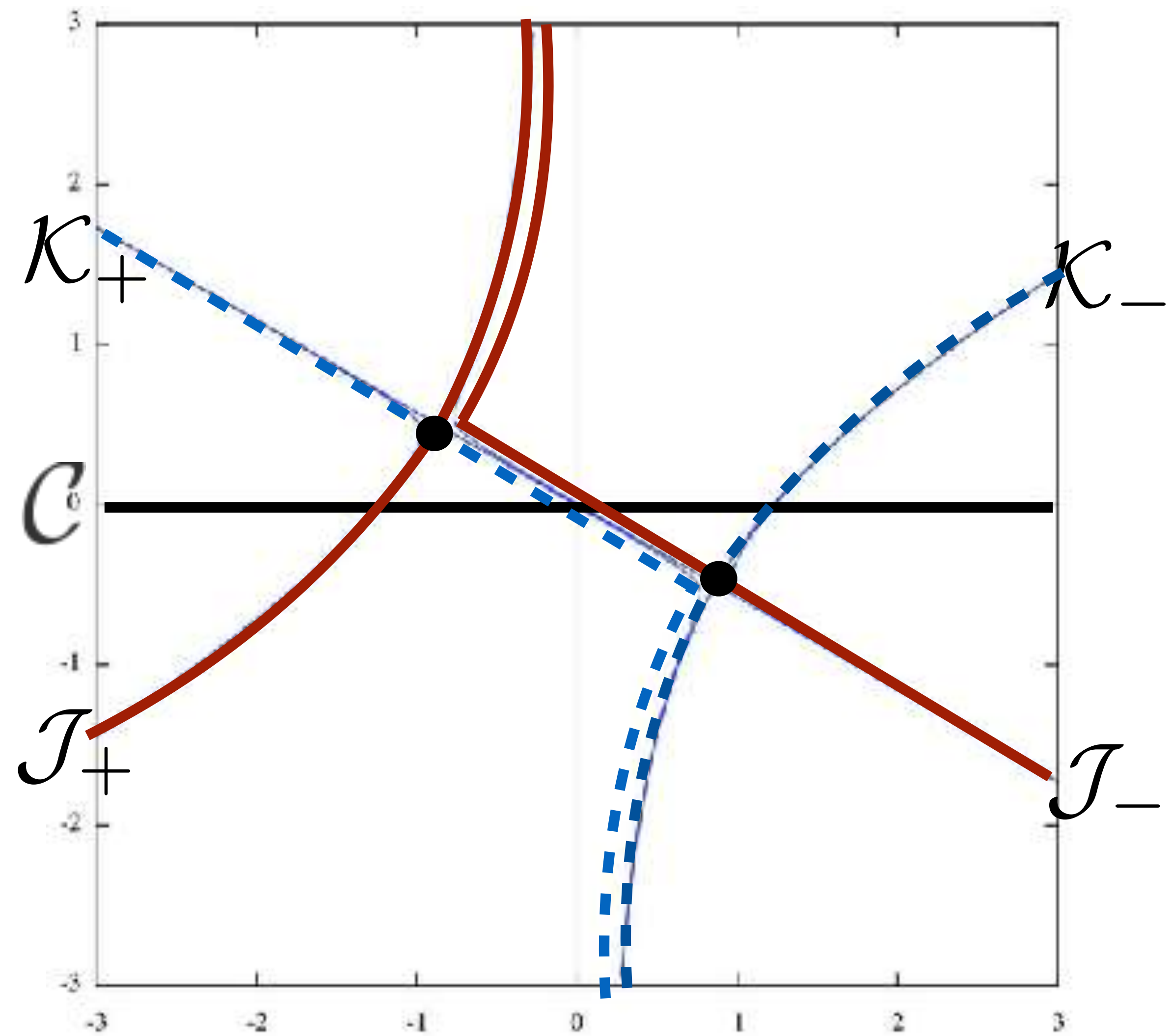
· Airy integral $\quad \arg[g^2] = \frac{2\pi}{3}+$

$$n_+ = \langle \mathcal{K}_+, \mathcal{C} \rangle = 1$$

$$n_- = \langle \mathcal{K}_-, \mathcal{C} \rangle = 1$$

$$\mathcal{C} = \sum_\sigma n_\sigma \mathcal{J}_\sigma \quad \Rightarrow \quad \boxed{\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+}$$

Stokes phenomenon : at special $\arg[g^2]$,
thimble decomposition discretely changes




$$\arg[g^2] = \frac{2\pi}{3}+$$

# Resurgence in integral

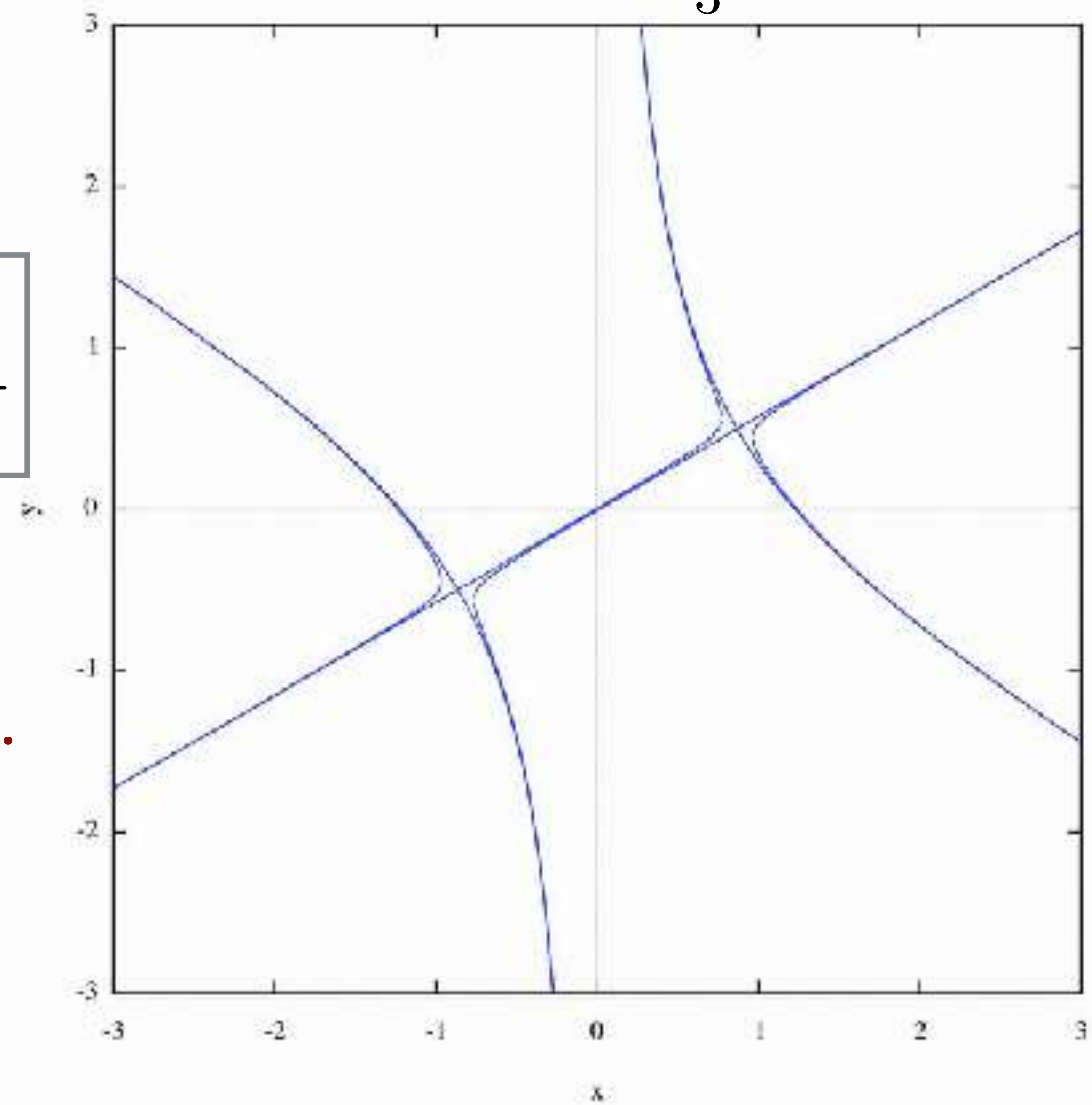
Integral is decomposed into saddle contributions
in steepest descent method (thimble decomposition)

· Airy integral

$$\arg[g^2] = \frac{2\pi}{3}^- \qquad\qquad \arg[g^2] = \frac{2\pi}{3}^+$$

$$\mathcal{C} = \mathcal{J}_- \quad \Longleftrightarrow \quad \mathcal{C} = \mathcal{J}_- + \mathcal{J}_+$$

$$\arg[g^2] = -\frac{2\pi}{3} \to \pi$$




* Decomposition is changed at Stokes line.
* Airy function is continuous even at Stokes line.

$$\mathcal{J}_-\left[\frac{2\pi}{3}^-\right] - \mathcal{J}_-\left[\frac{2\pi}{3}^+\right] = \mathcal{J}_+$$

# Resurgence in integral

Integral is decomposed into saddle contributions
in steepest descent method (thimble decomposition)

· Airy integral

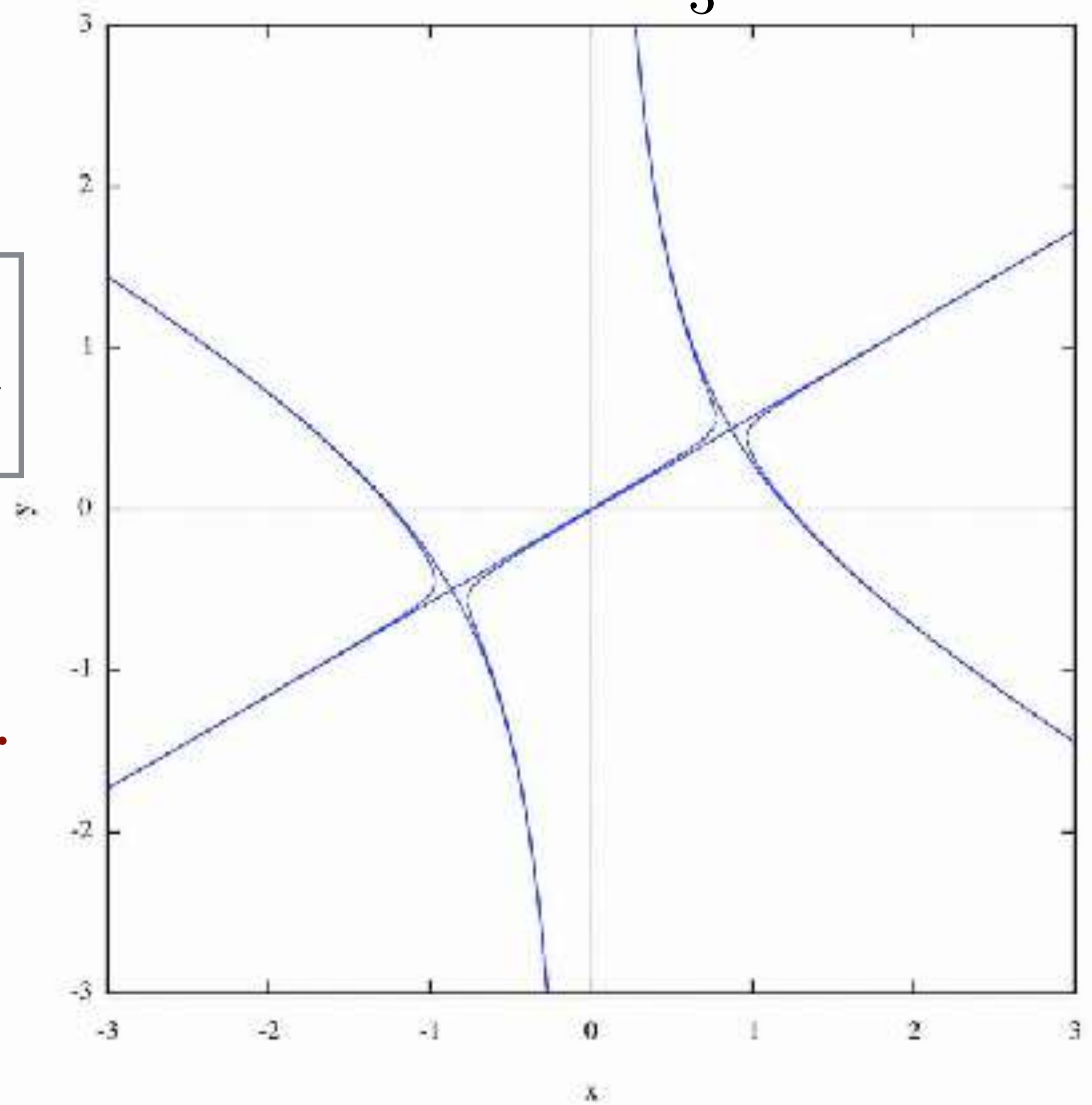
$$\arg[g^2] = \frac{2\pi}{3}-$$

$$\mathcal{C} = \mathcal{J}_-$$

$$\arg[g^2] = \frac{2\pi}{3}+$$

$$\mathcal{C} = \mathcal{J}_- + \mathcal{J}_+$$

$$\arg[g^2] = -\frac{2\pi}{3} \to \pi$$



- Decomposition is changed at Stokes line.
- Airy function is continuous even at Stokes line.

**Two saddle contributions are non-trivially related due to Stokes phenomenon !**

# How to use resurgent structure

$$\mathcal{S}_{+}\Phi_0(z) - \mathcal{S}_{-}\Phi_0(z) \approx \mathfrak{s} e^{-Az} \mathcal{S}\Phi_1(z)$$

Perturbative imaginary ambiguity

Non-perturbative effect

Perturbative series could include nonpert. information !

→ We may be able to derive non-perturbative result from perturbation and define QFT as trans-series.

$$F(g^2) \approx \sum_{q=0}^{\infty} c_{(0,q)} g^{2q} + \sum_{n=1}^{\infty} e^{-nA/g^2} \sum_{q=0}^{\infty} c_{(n,k)} g^{2q}$$

trans-series

# How to use resurgent structure

Imaginary ambiguities and their cancellation gets more clearer for theories with smaller symmetry.

$$\mathbb{B}(g^2 e^{\mp i\epsilon}) = \mathrm{Re}[\mathbb{B}(g^2)] \pm i\mathrm{Im}[\mathbb{B}(g^2)]$$ large symmetry

↓

$$\mathbb{B}(g^2 e^{\mp i\epsilon}) = \mathrm{Re}[\mathbb{B}(g^2)] \pm i\mathrm{Im}[\mathbb{B}(g^2)]$$ small symmetry

ex.)Supersymmetric quantum theory : less resurgent structure

Symmetry for exact-solvability : less resurgent structure

This empirical fact implies that resurgence theory suits QFT with small symmetry

# Applications

- Mathematics and QM
  Ecalle(80's), Zinn-Justin, Voros, Berry, Howls, Sauzin, Fauvet, Kawai, Takei, Aoki, Iwaki, Kamimoto,…
- SUSY QFT and String theories
  Marino(07~), Schiappa, Aniceto, Vonk, Honda, Grassi…
- Aymptotically-free field theory
  Dunne, Unsal(12~), Cherman, Dorigoni, Basar, Fujimori, Kamata, TM, Nitta, Sakai(14~),… Morikawa, Nakayama, Shibata, Suzuki, Takaura (19) Yamazaki, Yonekura(18)(19)
- Integrable models & High-$T_c$ superconductor
  Marino, Reis(19~),….
- Hydrodynamics, Fluid dynamics
  Heller, Spalinski(15~), Behtash, Kamata, Martinez, Shi(19),…
- Schwinger mechanism
  Taya, Fujimori, TM, Nitta, Sakai (20~), Enomoto, Matsuda(20)
- Quantization conditions, exact-WKB, TBA equations
  Kashani-poor, Troost (15), Hollands, Neitzke(19), Ito, Shu (19) Sueishi, Kamata, TM, Unsal(20~)…
- Phase transition
  Kanazawa, Tanizaki (15), Dunne, et.al. (16)(17)(18) Yoda, Honda, Fujimori, Kamata, TM, Sakai (21)…

# 1. Resurgent structure in 2D sigma model

What is resurgent structure in asymptotically-free QFT?

# ◆2D O(N) and CP$^{N-1}$ sigma models in large N

Nishimura, Fujimori, TM, Nitta, Sakai (21)

$$\mathrm{Im}\langle \delta D^2 \rangle = \pm\pi \left[ \left( \mu^2 e^{-\frac{4\pi}{\lambda_\mu}} \right)^2 \Lambda^0 - 2\Lambda^4 + \left( \mu^2 e^{-\frac{4\pi}{\lambda_\mu}} \right)^{-2} \Lambda^8 \right] \theta(\Lambda - a) = 0$$

$\lambda_\mu$: 't Hooft coupling, $a$: IR cutoff, $\Lambda$: Dynamical scale

$$\Lambda = \mu \exp\left( -\frac{2\pi}{\lambda_\mu} \right)$$

(1) Renormalon ambiguity is cancelled by those at several nonpert. sectors, where the cancellation structure is binomial-expansion-like.

(2) The ambiguities (or the resurgent structure) emerge only for $a<\Lambda$, due to analytic continuation from $a>\Lambda$ to $a<\Lambda$.

(3) The renormalon and binomial-expansion-like resurgent structure is deformed due to infinite-time Stokes phenomena during compactification

## ◆2D O(N) and $CP^{N-1}$ sigma models in large N

Nishimura, Fujimori, TM, Nitta, Sakai (21)

$\Lambda^4$ renormalon $\quad\quad\quad\quad$ $\Lambda^{-4}$

$$\mathrm{Im}\langle \delta D^2 \rangle = \pm\pi \left[ \underbrace{\left(\mu^2 e^{-\frac{4\pi}{\lambda_\mu}}\right)^2 \Lambda^0}_{\text{perturbative}} - \underbrace{2\Lambda^4 + \left(\mu^2 e^{-\frac{4\pi}{\lambda_\mu}}\right)^{-2} \Lambda^8}_{\text{non-perturbative}} \right] \theta(\Lambda - a) = 0$$

$\lambda_\mu$: 't Hooft coupling, $a$: IR cutoff, $\Lambda$: Dynamical scale

$$\Lambda = \mu \exp\left(-\frac{2\pi}{\lambda_\mu}\right)$$

(1) Renormalon ambiguity is cancelled by those at several nonpert. sectors, where the cancellation structure is binomial-expansion-like.

(2) The ambiguities (or the resurgent structure) emerge only for $a<\Lambda$, due to analytic continuation from $a>\Lambda$ to $a<\Lambda$.

(3) The renormalon and binomial-expansion-like resurgent structure is deformed due to infinite-time Stokes phenomena during compactification

## ◆2D O(N) and CP$^{N-1}$ sigma models in large N

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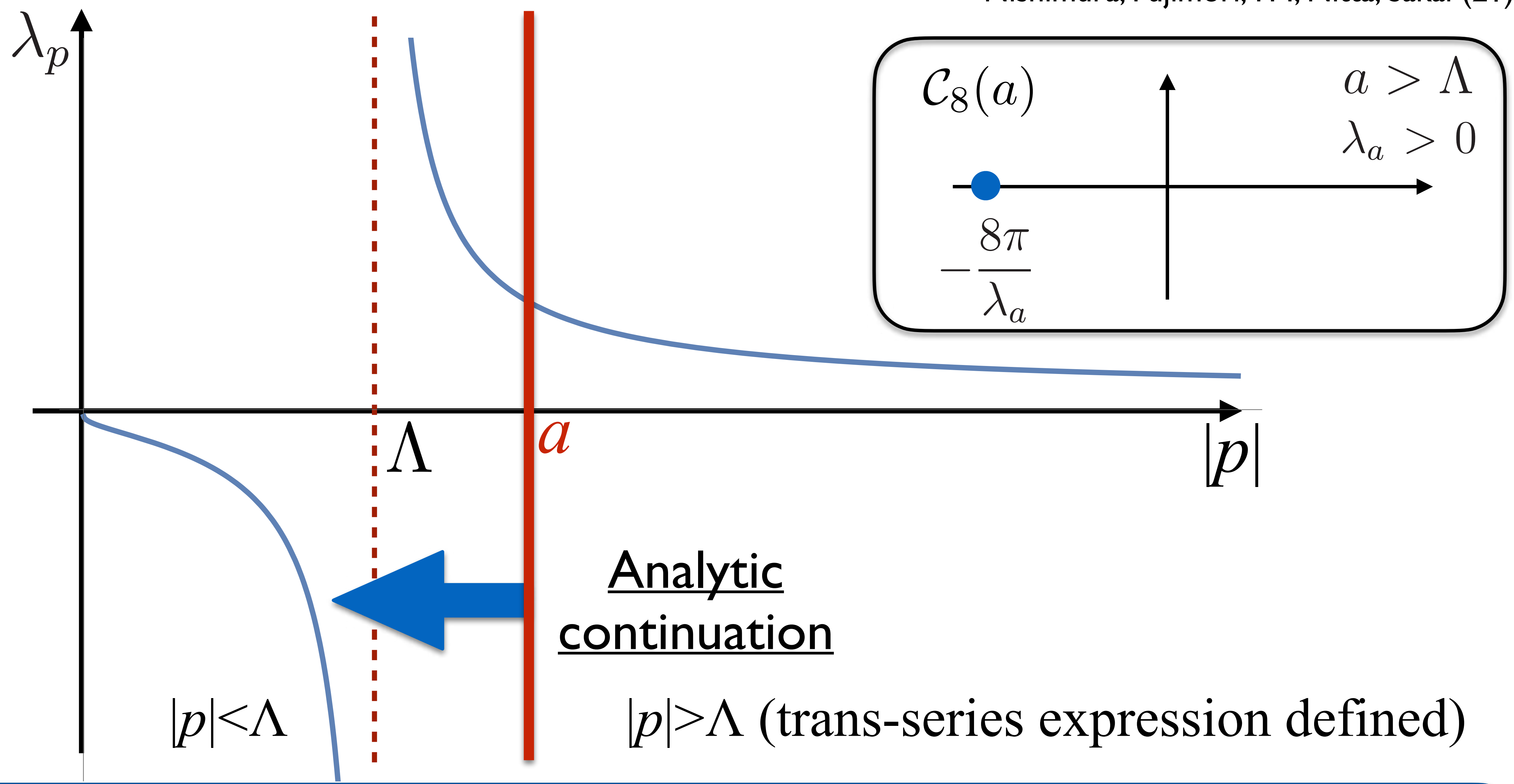


Analytic continuation from $a>\Lambda$ to $a<\Lambda$ is responsible for
(1) existence of ambiguities,
(2) cancellation of ambiguities (Borel singularity of $\Lambda^8$ order moves from negative to positive real axis.)

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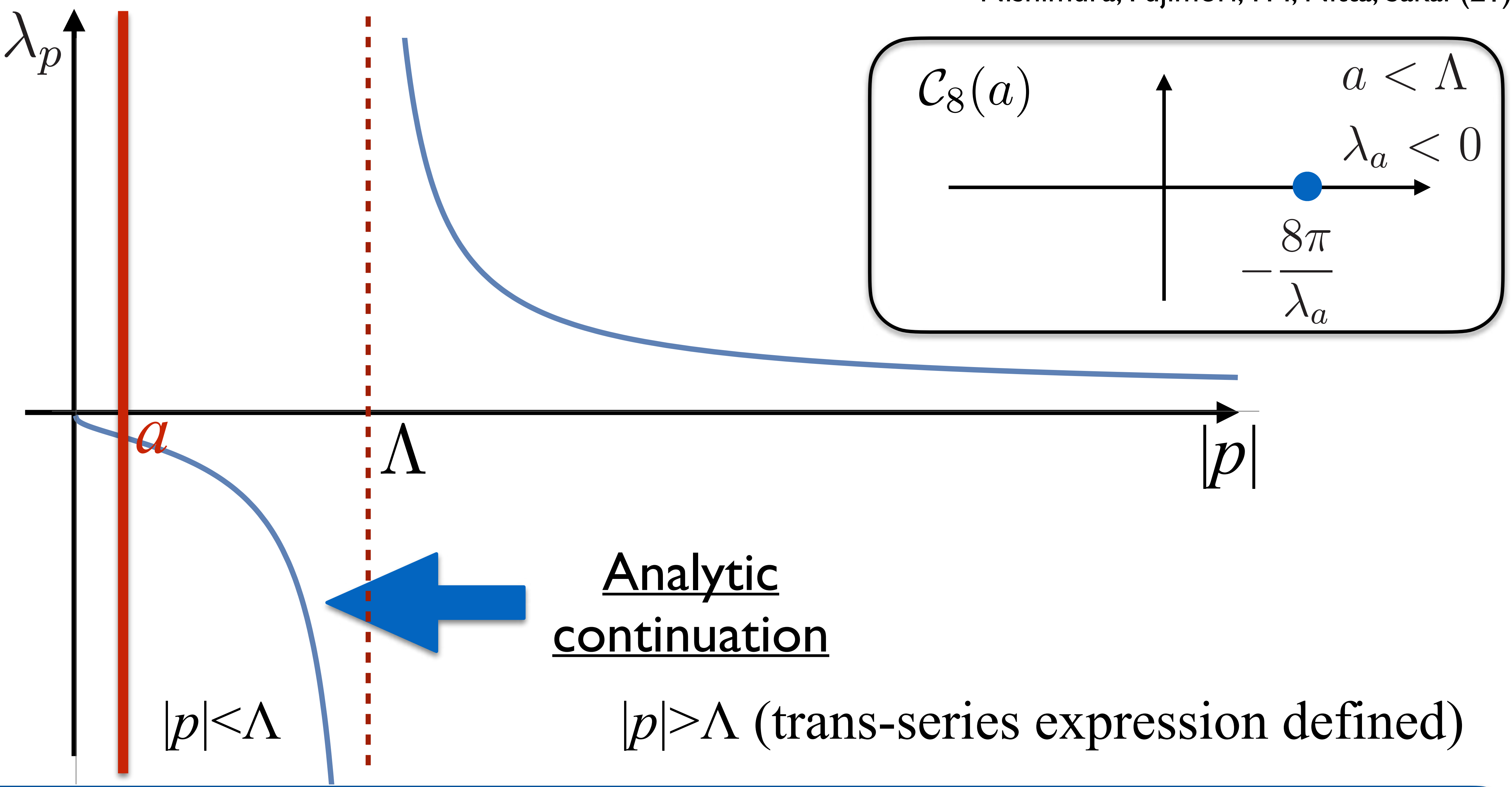


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## ◆2D O(N) and $CP^{N-1}$ sigma models in large N

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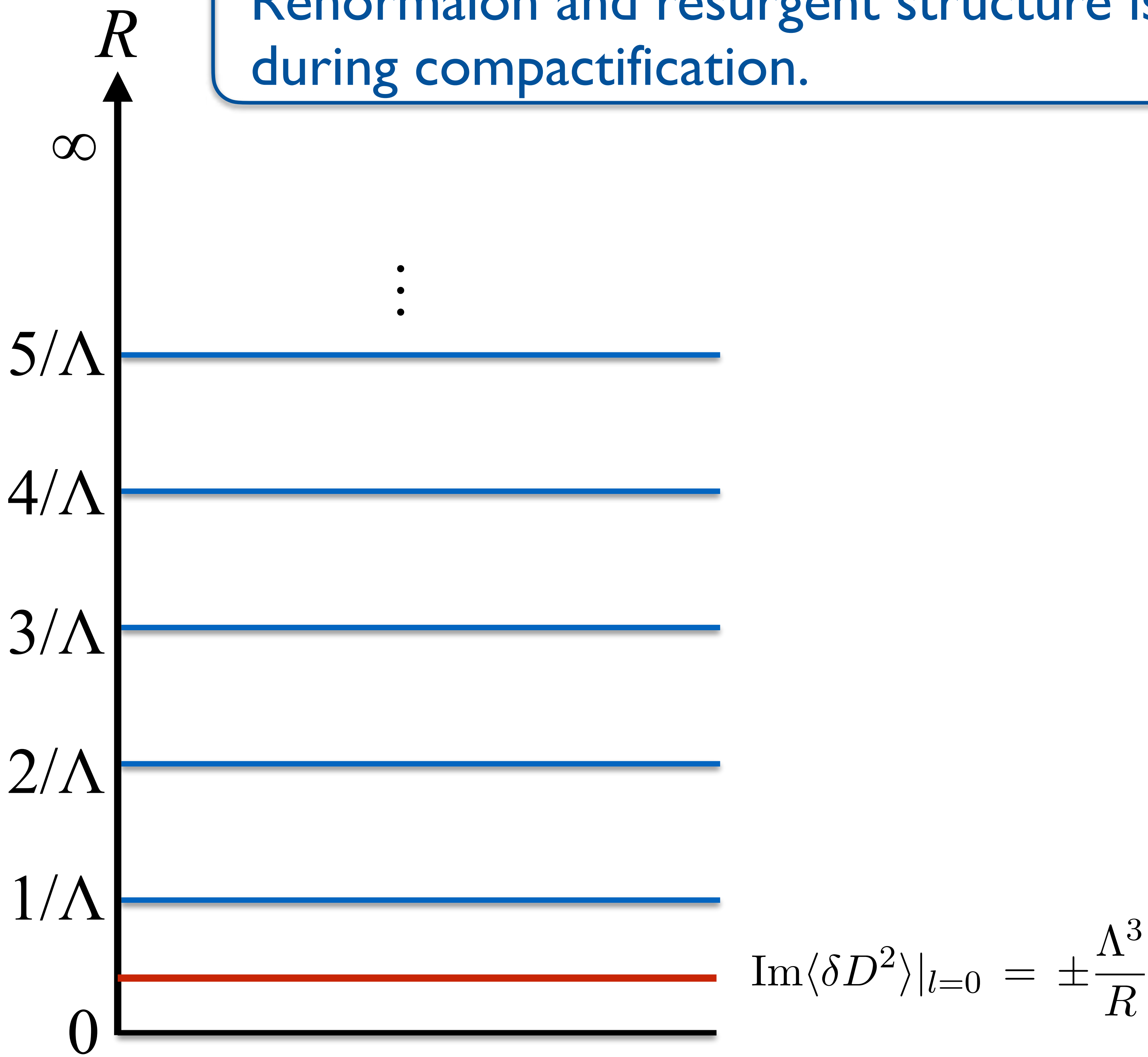
Renormalon and resurgent structure is drastically changed during compactification.



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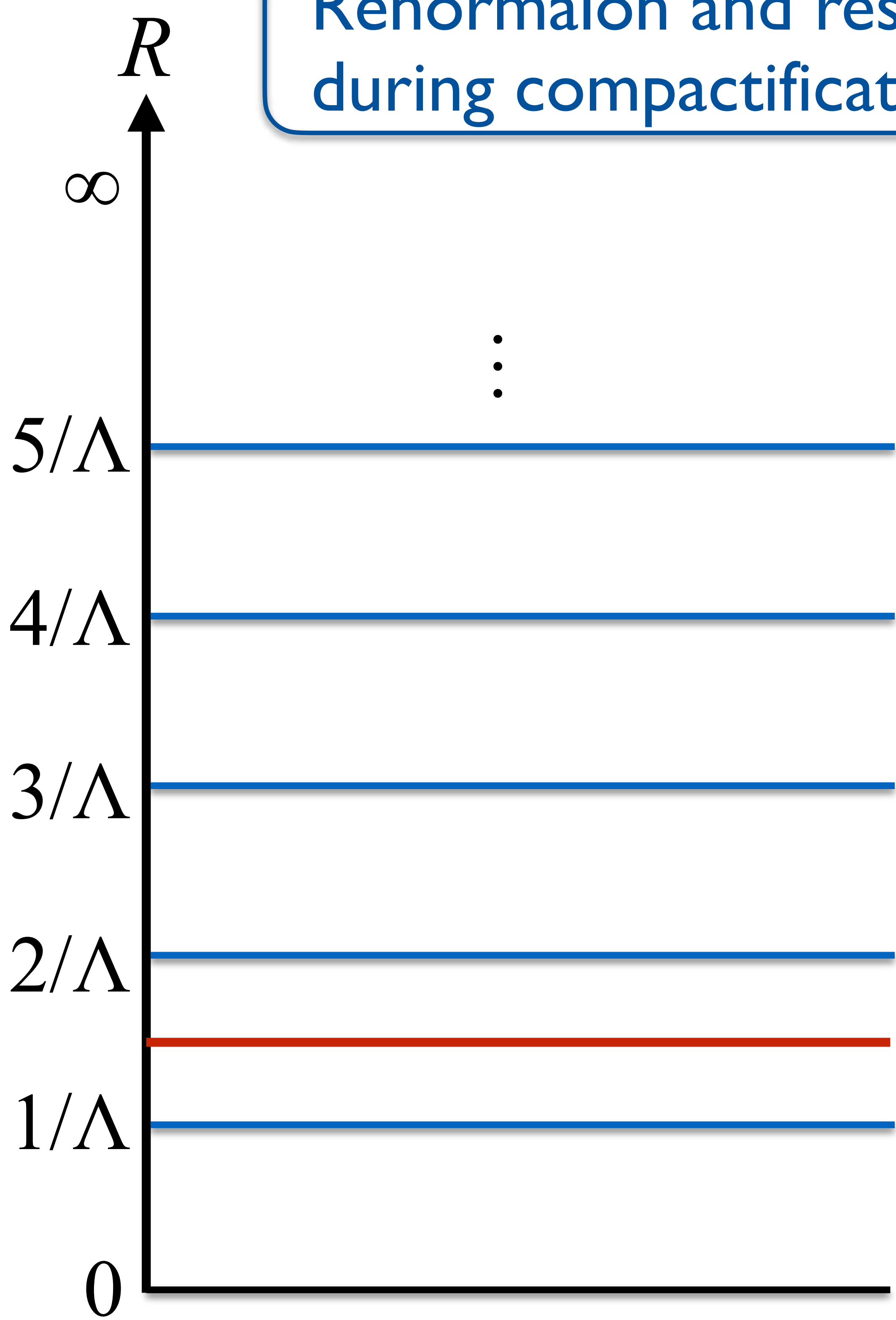


$$\mathrm{Im}\langle \delta D^2 \rangle|_{l=0} = \pm \frac{\Lambda^3}{R}\left(1 + \frac{2}{\sqrt{1-\frac{1}{(R\Lambda)^2}}}\right)$$

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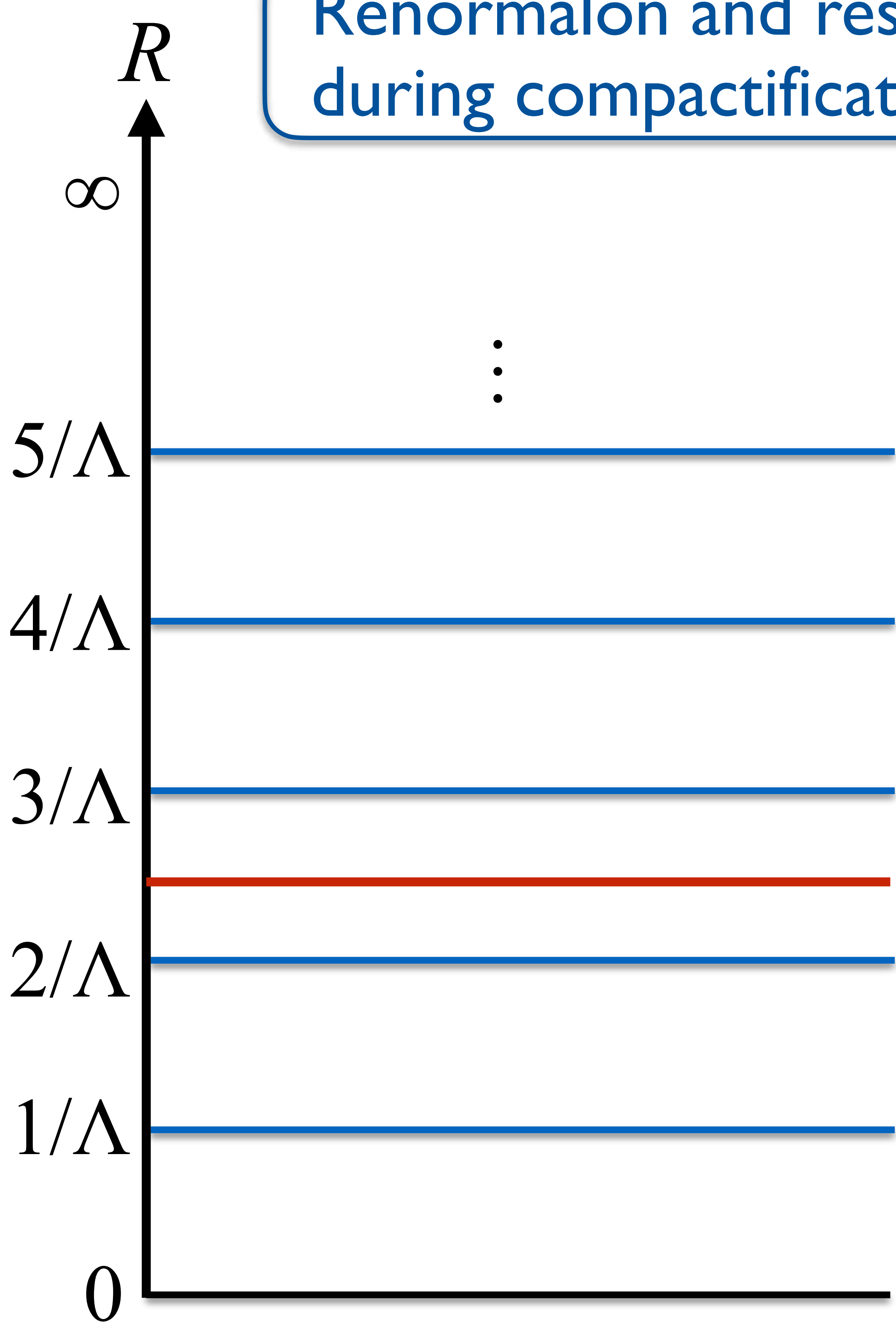


$$\mathrm{Im}\langle \delta D^2 \rangle|_{l=0} = \pm \frac{\Lambda^3}{R}\left(1+\frac{2}{\sqrt{1-\frac{1}{(R\Lambda)^2}}}+\frac{2}{\sqrt{1-\frac{4}{(R\Lambda)^2}}}\right)$$

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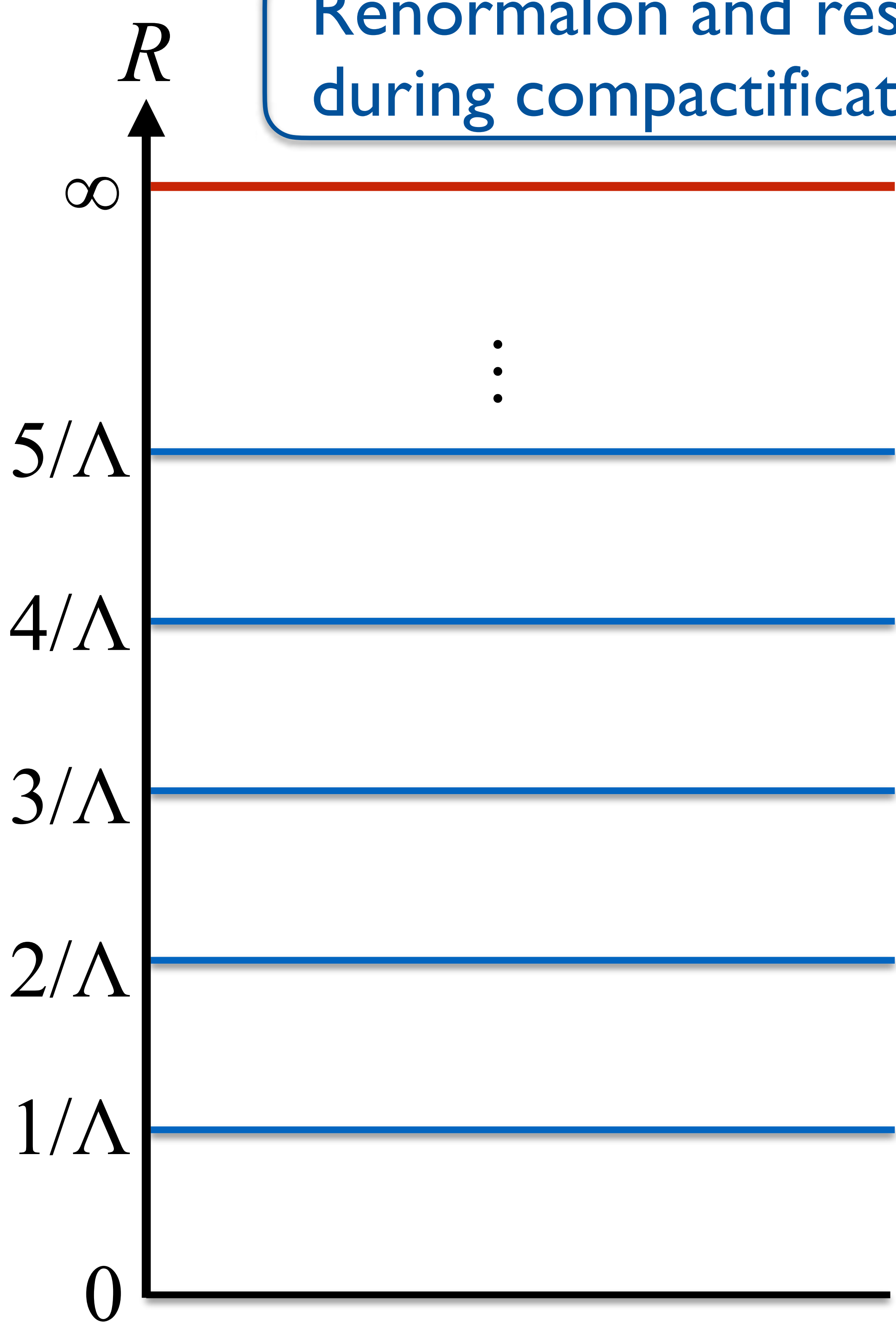



$$\mathrm{Im}\langle \delta D^2 \rangle|_{l=0} = \pm\Lambda^4$$

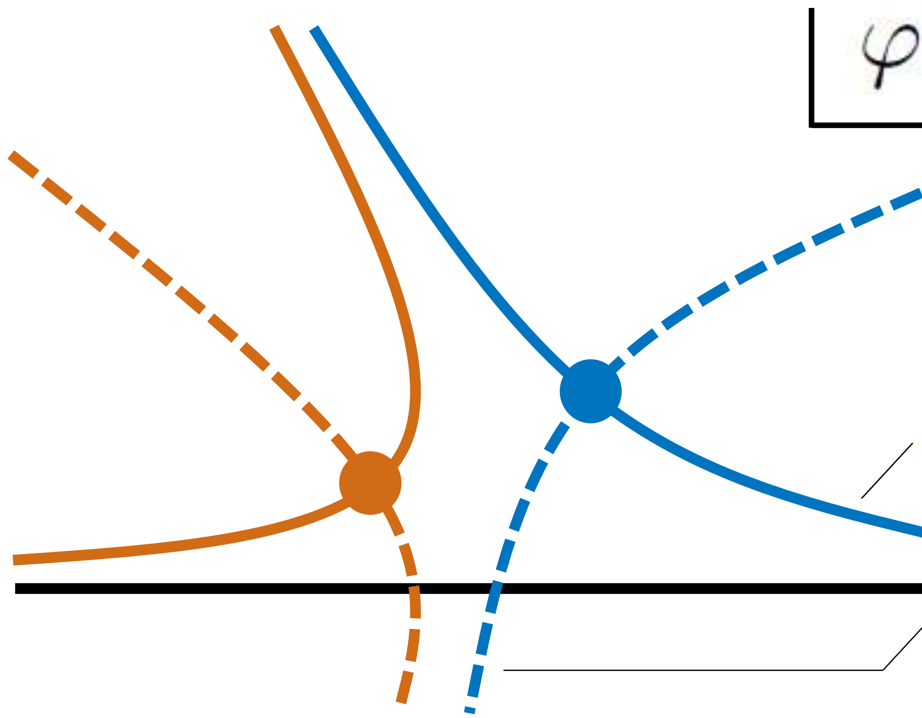
Resurgent structure in asymptotically-free QFT is being figured out.

# 2. Resurgence and phase transition

Can we explore phase transition by resurgence?

## ◆ Resurgence and phase transition

Yoda, Fujimori, Honda, Kamata, TM, Sakai (21)




$$\mathcal{J}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = +\overline{\frac{\delta S[\varphi]}{\delta \varphi}}, \quad \varphi(-\infty) = \varphi_i$$

$$\mathcal{K}_i : \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} = -\overline{\frac{\delta S[\varphi]}{\delta \varphi}}, \quad \varphi(-\infty) = \varphi_i$$

$$\left.\frac{\delta S[\varphi]}{\delta \varphi}\right|_{\varphi=\varphi_i} = 0$$

・Stokes phenomenon : Change of intersection numbers

$$\mathrm{Im}[S[\varphi_i]] = \mathrm{Im}[S[\varphi_j]]$$

Resurgent structure

・Anti-Stokes phenomenon : Change of dominant saddles

$$\mathrm{Re}[S[\varphi_i]] = \mathrm{Re}[S[\varphi_j]]$$

1st order phase transition

Kanazawa, Tanizaki (15), Dunne, et.al. (16)(17)(18)

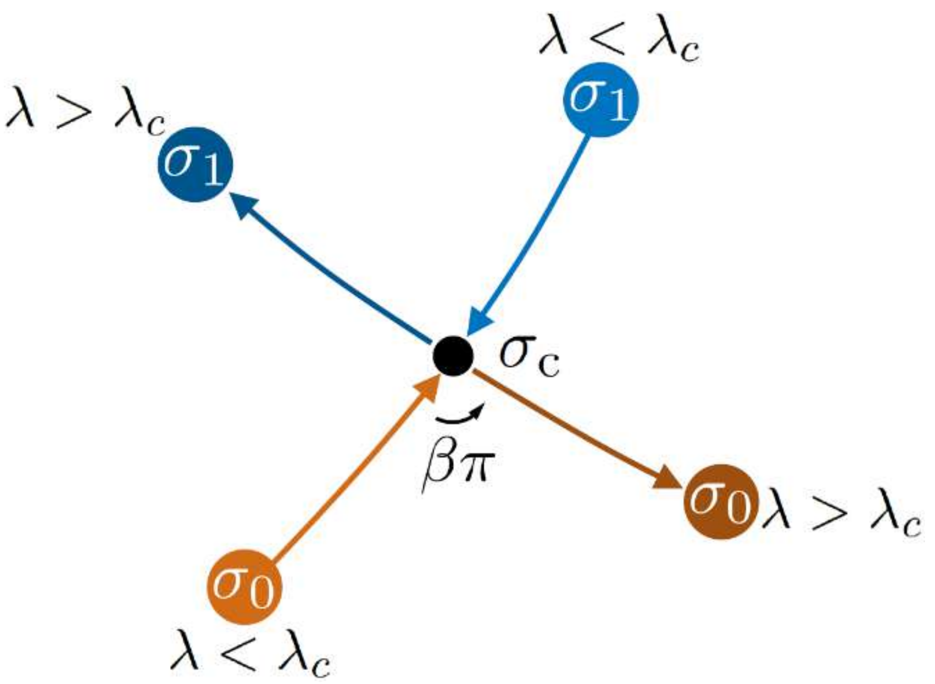
## ◆ Resurgence and phase transition

Yoda, Fujimori, Honda, Kamata, TM, Sakai (21)

**Theorem**

Assume partition function in expression as

$$e^{-NF(\lambda)} = \int \mathrm{d}\sigma \; e^{-N\tilde{S}(\lambda;\sigma)}$$




When $n$ saddles collide with angle $\beta\pi$,
phase transition of order $\lceil (n+1)\beta \rceil$ occurs, where Stokes and anti-Stokes phenomena simultaneously occur.

ceiling function, cf.) $\lceil (2+1)(1/2) \rceil = 2$

- Higher-order phase transitions are understood as collision of saddles, characterized by simultaneous Stokes and anti-Stokes phenomena.
- It means phase transition is detected from collision of Borel singularities!

# Generic argument on phase transition

Fujimori, Honda, Kamata, TM, Sakai, Yoda (21)

## ex.) Airy integral

$$\tilde{S}(\lambda;\sigma) = \frac{i\sigma^3}{3} - i\lambda\sigma. \quad \Rightarrow \quad \tilde{S}_\pm = \mp\frac{2i}{3}(\delta\lambda)^{3/2}.$$

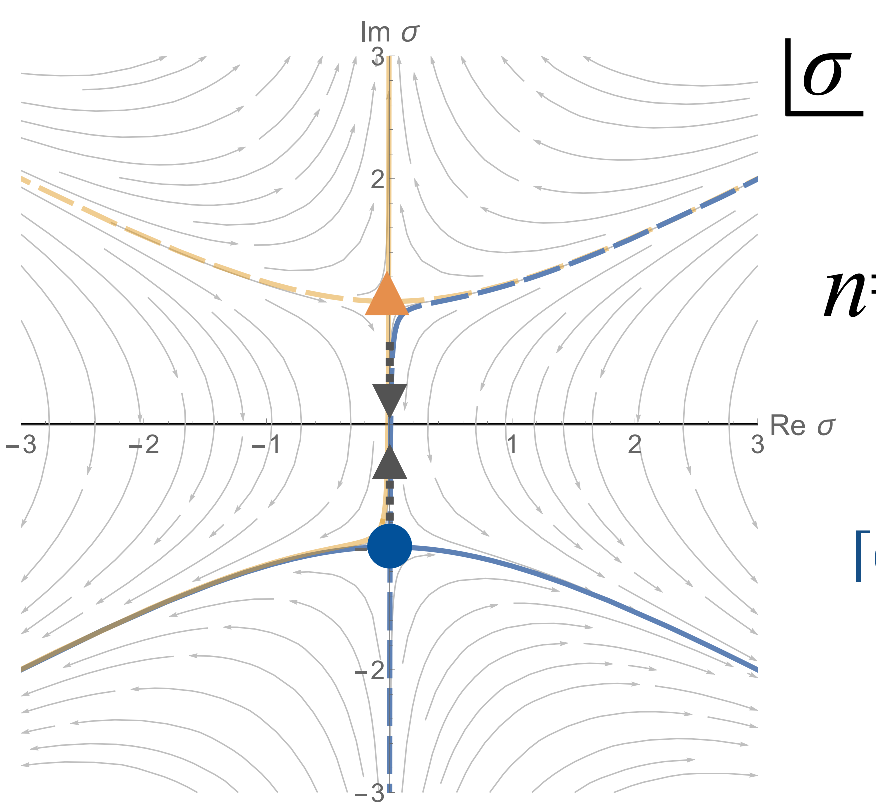
$$\sigma_c = 0, \quad \lambda_c = 0, \quad \delta\sigma_\pm = \pm\delta\lambda^{1/2} \quad \sigma_\pm = \pm\lambda^{1/2}$$

### Free energy

$$F \simeq \begin{cases} \tilde{S}_+ & = \frac{2}{3}(-\delta\lambda)^{3/2} \quad \text{for } \delta\lambda < 0 \\ \tilde{S}_+ + \tilde{S}_- & = 0 \quad \text{for } \delta\lambda > 0 \end{cases}$$

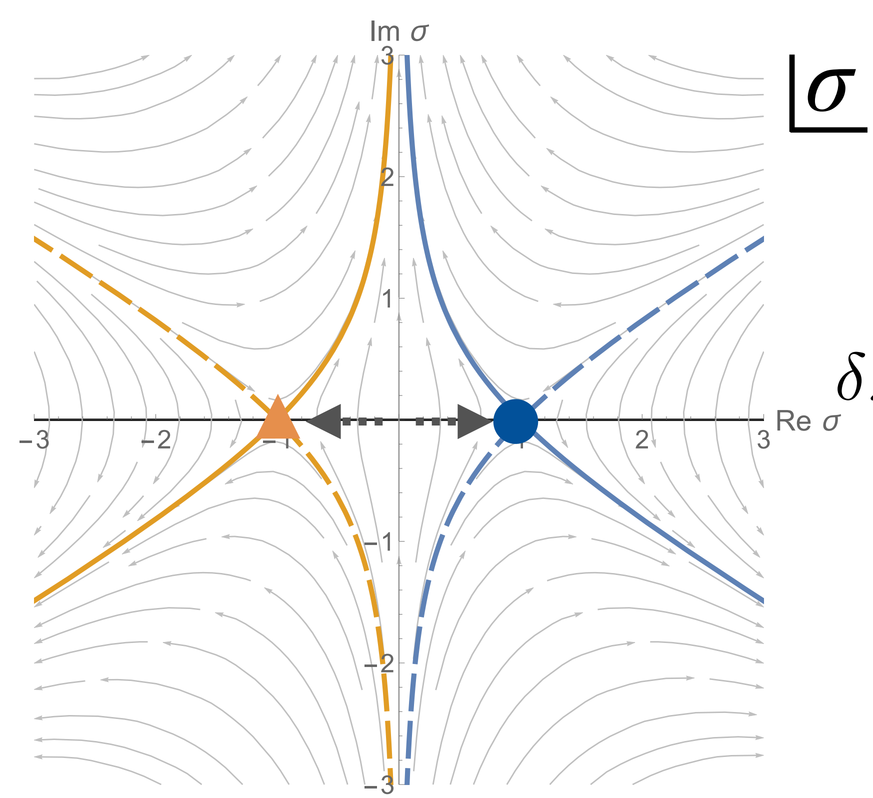
**2nd order phase transition**

$\delta\lambda < 0$




$n$=2, $\beta$=1/2

[(2+1)(1/2)] = 2




$\delta\lambda > 0$

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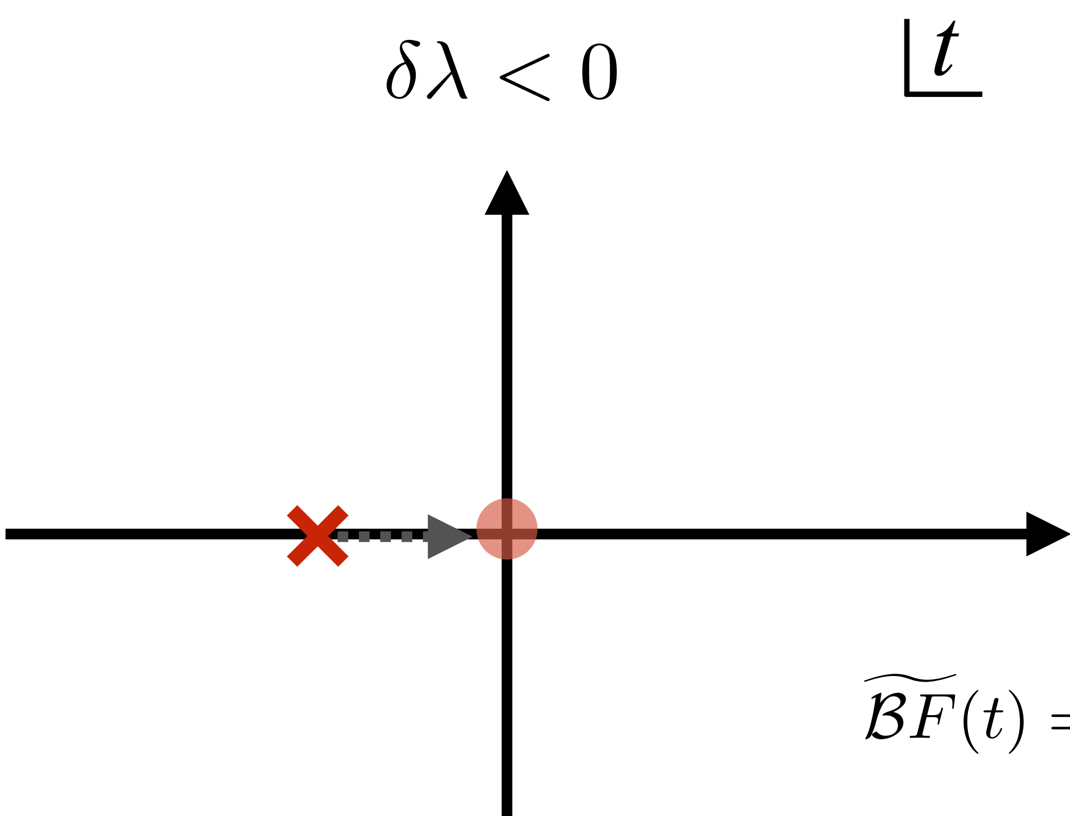
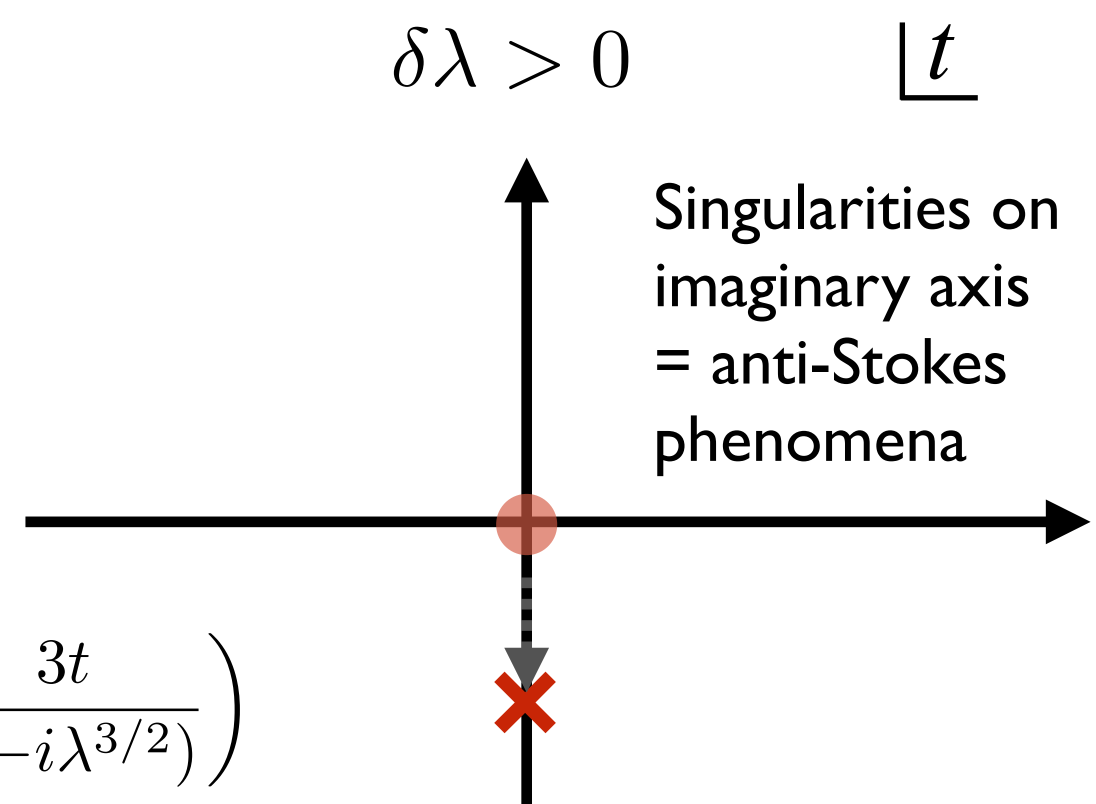
2nd order phase transition








$$\widetilde{\mathcal{B}F}(t) = {}_2F_1\left(\frac{1}{6}, \frac{5}{6}, 1; \frac{3t}{4(-i\lambda^{3/2})}\right)$$

Phase transition and its order is studied by resurgence theory.

# 3. Exact-WKB and its application

What is a generic framework to grasp resurgent structure?

# Exact-WKB

Voros(83) Aoki, Kawai, Takei(91)

## 1. Borel-resum of WKB wave function

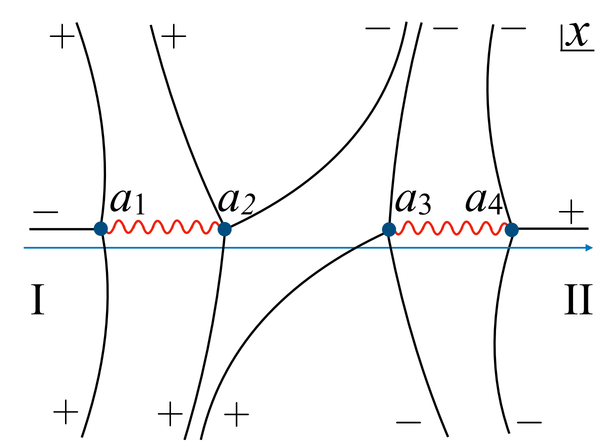
$$\psi_a^{\pm}(x) = e^{\pm\frac{1}{\hbar}\int_a^x \sqrt{Q(x)}dx} \sum_{n=0}^{\infty} \psi_{a,n}^{\pm}(x)\hbar^{n+\frac{1}{2}} \qquad Q(x) = 2(V(x) - E)$$

$$\Rightarrow \quad \mathcal{S}[\psi_a^{\pm}](\hbar) = \int_{\mp z_0}^{\infty e^{i\theta}} e^{-\frac{z}{\hbar}} \mathfrak{B}[\psi_a^{\pm}(x)](z)dz, \qquad \theta = \mathrm{Arg}(\hbar)\,,$$

## 2. Non-Borel-summable points on complex $x$ plane : Stokes curve

Monodromy matrix connects wave functions in two regions separated by Stokes curve

ex.)Stokes curves for double-well



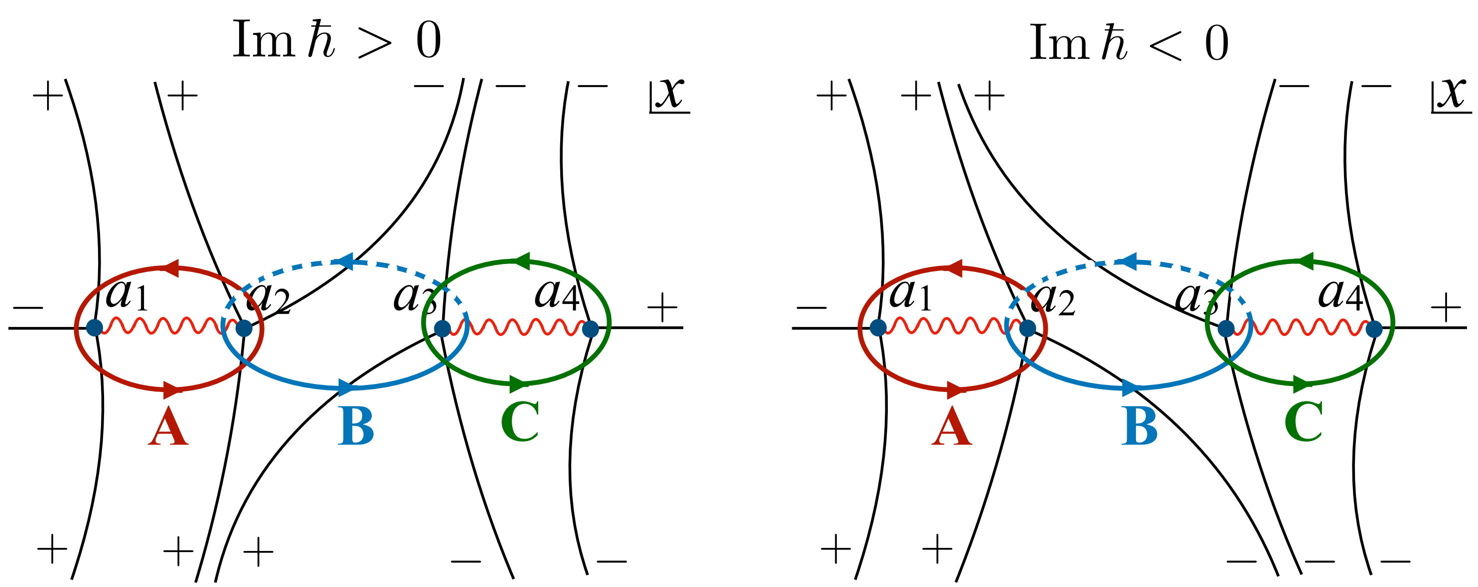

$\mathrm{Im}\,\hbar > 0$

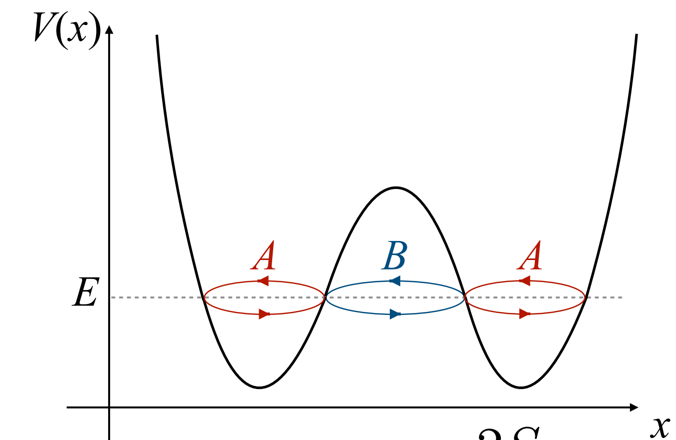
## 3. Matrices for several Stokes curve → connection formula

$$\begin{pmatrix} \psi_{a_1,\mathrm{I}}^{+}(x) \\ \psi_{a_1,\mathrm{I}}^{-}(x) \end{pmatrix} = M_+ N_{a_1a_2} M_+ N_{a_2a_3} M_+ M_- N_{a_3a_4} M_- N_{a_4a_3} N_{a_3a_2} N_{a_2a_1} \begin{pmatrix} \psi_{a_1,\mathrm{II}}^{+}(x) \\ \psi_{a_1,\mathrm{II}}^{-}(x) \end{pmatrix} = \begin{pmatrix} 1 & iD \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_{a_1,\mathrm{II}}^{+}(x) \\ \psi_{a_1,\mathrm{II}}^{-}(x) \end{pmatrix}$$

# Exact-WKB and resurgent structure

Sueishi, Kamata, TM, Unsal (20)(21)




$B \propto e^{-2S_I}$

$$A = e^{\oint_A S_{\rm odd}}, \qquad B = e^{\oint_B S_{\rm odd}}, \qquad C = e^{\oint_C S_{\rm odd}} = 1/A$$

- Relation of $A$,$B$ cycles from normalization : exact quantization condition

$$D = (1+A)(1+C) + AB = 0 \quad \Rightarrow \quad \frac{i\sqrt{2\pi}}{\Gamma(-E+\frac{1}{2})} = \mp \exp\left(-\frac{1}{6g^2} + i\pi E\right)\left(\frac{\hbar}{2}\right)^{-E}$$

- Cancelations of Im. ambiguities of $A$(pert.), $B$(non-pert.) : resurgent structure

$$Z = \underbrace{(\mathrm{Re}[Z_{\rm p}] \pm i\mathrm{Im}[Z_{\rm p}])}_{A} + \underbrace{(\mathrm{Re}[Z_{\rm np}] \pm i\mathrm{Im}[Z_{\rm np}])}_{B}$$

cf.) Yang-san's talk on 30th

# Applications to SG-eq. with time-dependent E

Taya, Fujimori, TM, Nitta, Sakai (21)

- Klein-Gordon eq. with time-dependent gauge field

$$0 = \left[\hbar^2 \partial_t^2 + m^2 + (\boldsymbol{p} - e\boldsymbol{A}(t))^2\right] \phi(t;\boldsymbol{p})$$

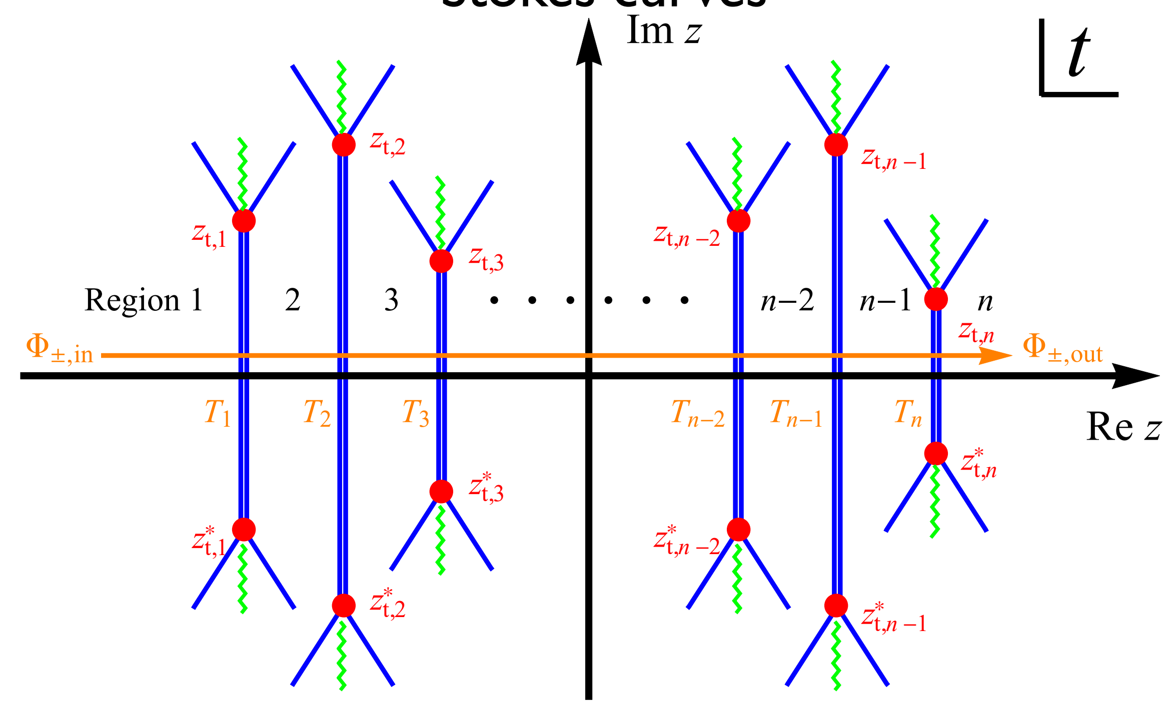
Exact-WKB analysis

$$\begin{pmatrix} \Phi_{+,\mathrm{out}} \\ \Phi_{-,\mathrm{out}} \end{pmatrix} = T_n T_{n-1} \cdots T_2 T_1 \begin{pmatrix} \Phi_{+,\mathrm{in}} \\ \Phi_{-,\mathrm{in}} \end{pmatrix}$$

Pair production density

$$\frac{\mathrm{d}^6 N_{e^-}}{\mathrm{d}\boldsymbol{x}^3 \mathrm{d}\boldsymbol{p}^3} = \frac{\mathrm{d}^6 N_{e^+}}{\mathrm{d}\boldsymbol{x}^3 \mathrm{d}\boldsymbol{p}^3} = \frac{1}{(2\pi\hbar)^3} \left| \sum_{i=1}^{n} \mathrm{e}^{-\mathrm{i}\,\mathrm{Im}\,\sigma_{z_{\mathrm{t},i}}/\hbar} \mathrm{e}^{-S_{z_{\mathrm{t},i}}/\hbar} \right|^2 \left(1 + \mathcal{O}(\mathrm{e}^{-S_{z_{\mathrm{t},i}}/\hbar})\right)$$

Stokes curves




Schwinger mechanism can be understood in terms of Exact-WKB.

cf.) Two-state Floquet systems, Taya, Sueishi, Fujimori, Kamata, TM, Nitta (23) in progress

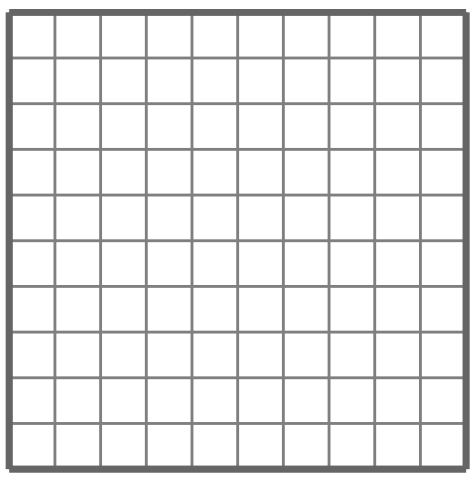
Exact-WKB is useful not only for resurgence
but also for study of dynamics.

# II. Graph theory and lattice

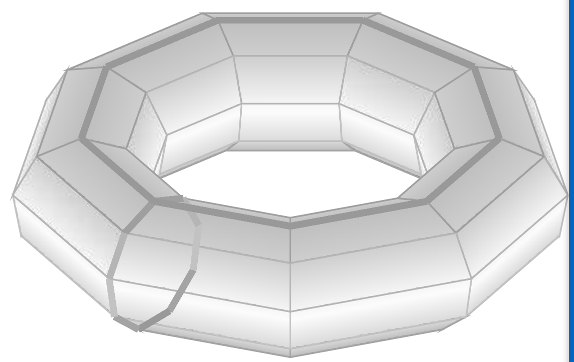
# Lattice and graph

Yumoto, TM (21)
cf.) Ohta, Sakai(20), Ohta, Matsuura(21)
See also Ohta-san's talk on 30th

## Lattice QFT



$\approx$



- Action

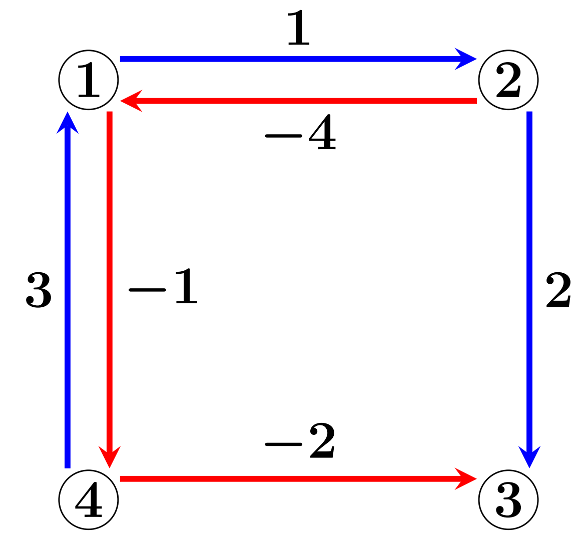
$$S_{T^D} = \sum_n \sum_{\mu=1}^{D} \bar{\psi}_n \gamma_\mu D_\mu^{(\mathrm{PBC})} \psi_n \equiv \bar{\boldsymbol{\psi}} \mathcal{D}^{(\mathrm{PBC})} \boldsymbol{\psi}$$

- Operator

$$\begin{aligned}\mathcal{D} = & \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \gamma_1 \\ & + \mathbf{1}_N \otimes \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \gamma_2 \\ & + \mathbf{1}_N \otimes P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_3 \\ & + P_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \mathbf{1}_N \otimes \gamma_4\end{aligned}$$

## Graph theory

$G = (V, E)$ : Vertex set $V$ and Edge set $E$




- Graph matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -2 & 0 \end{pmatrix}$$

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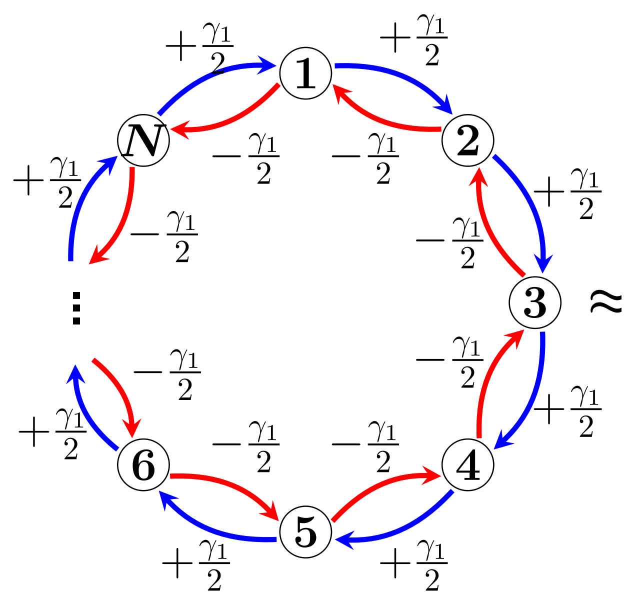
Yumoto, TM (21)
cf.) Ohta, Sakai(20), Ohta, Matsuura(21)

## Lattice QFT

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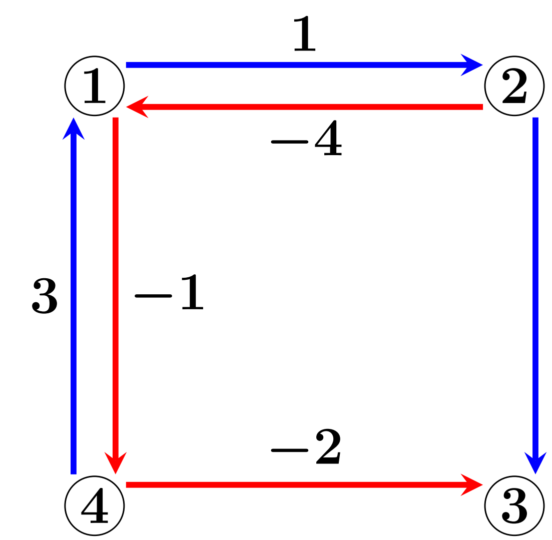


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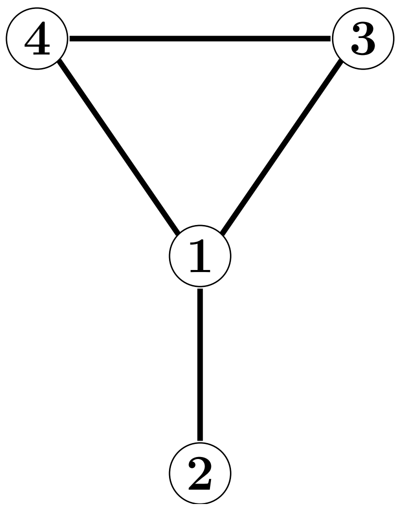
# Definitions of graph

**Definition 1.** ***A graph G is a pair G = (V, E).*** *V is a set of vertices and E is a set of edges.*

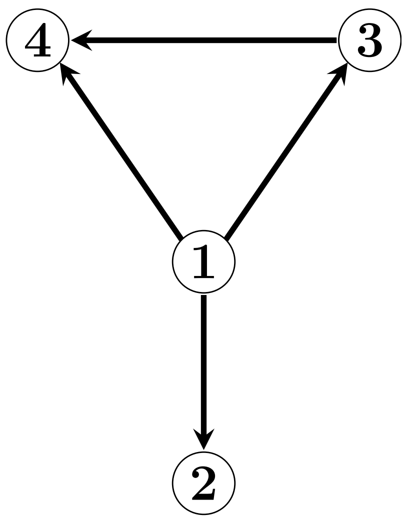
**Definition 2.** ***A directed graph is a pair (V, E) of sets of vertices and edges together with two maps* init *: E → V and* ter *: E → V.*** *The two maps are assigned to every edge $e_{ij}$ with an initial vertex* $\mathrm{init}(e_{ij}) = v_i \in V$ *and a terminal vertex* $\mathrm{ter}(e_{ij}) = v_j \in V$. *If* $\mathrm{init}(e_{ij}) = \mathrm{ter}(e_{ij})$, *the edge $e_{ij}$ is called a loop.*

**Definition 3.** ***A weighted graph has a value (weight) for each edge in a graph.***

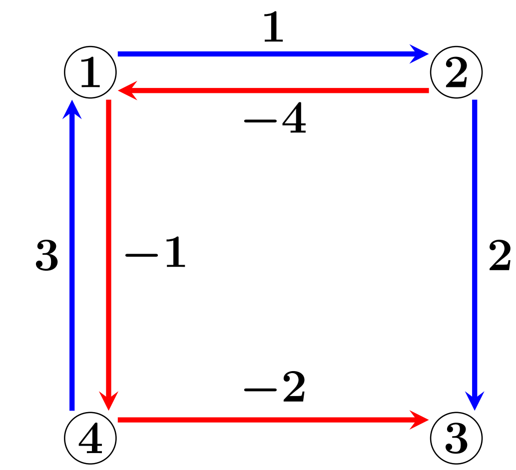
ex.)simple four-vertex graphs




undirected, unweighted




directed, unweighted




directed, weighted

# Spectral graph theory

- Adjacency matrix

$$A_{ij} = \begin{cases} 1 & if\ there\ is\ a\ edge\ between\ i\ and\ j \\ 0 & otherwise \end{cases}$$

- Degree matrix

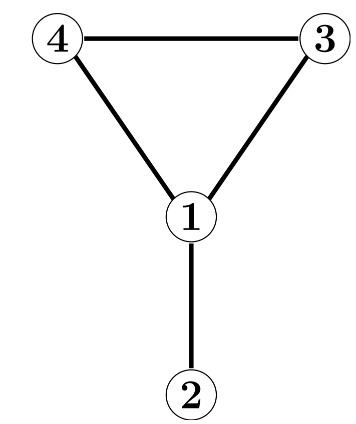
$$D_{ij} = \begin{cases} \deg(v_i) & i = j \\ 0 & otherwise \end{cases}$$

- Laplacian matrix

$$L = D - A$$

cf.)Undirected & unweighted

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$




$$D = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

# Spectral graph theory

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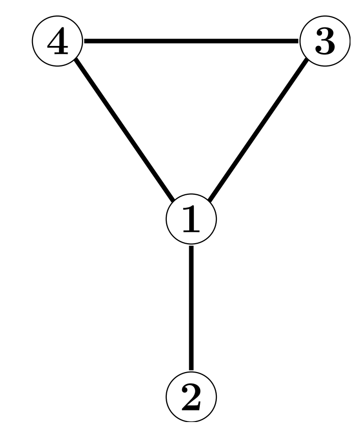


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- Laplacian matrix

related to topology!

$$L = D - A$$

$$|V| - \text{rank}(L) = \beta_0$$

$$|E| - \text{rank}(L) = \beta_1$$

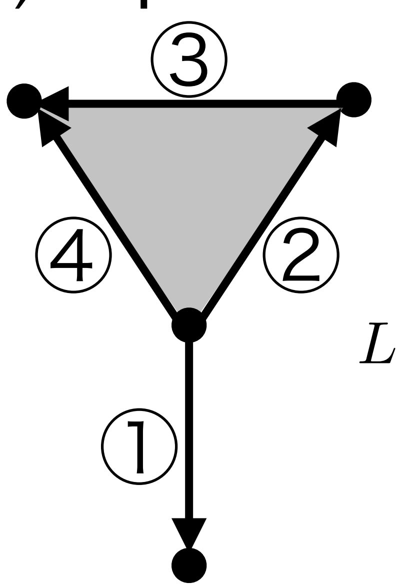
$$L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

# Spectral graph theory

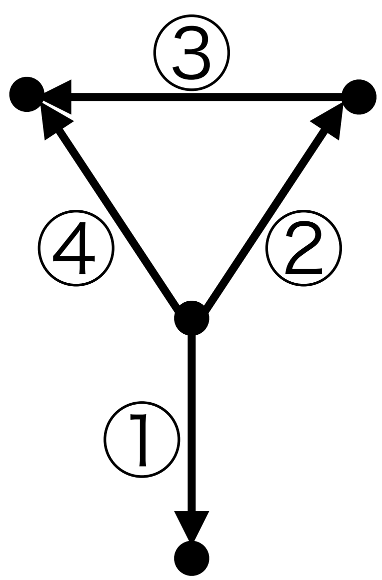
- combinatorial Laplacian matrix

$$(L_1)_{lk} = \begin{cases} 2+\deg(F_l) & l=k \qquad \deg(F_l) \textit{ is the number of faces touching the edge } l \\ 1 & \textit{if both } l \textit{ and } k \textit{ go in or out of the same vertex} \\ -1 & \textit{if one goes in and the other goes out of the same vertex} \\ 0 & \textit{otherwise} \end{cases}$$

ex.)simple four-vertex graphs




$$L_1 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

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$$|E| - \mathrm{rank}(L_1) = \beta_1$$

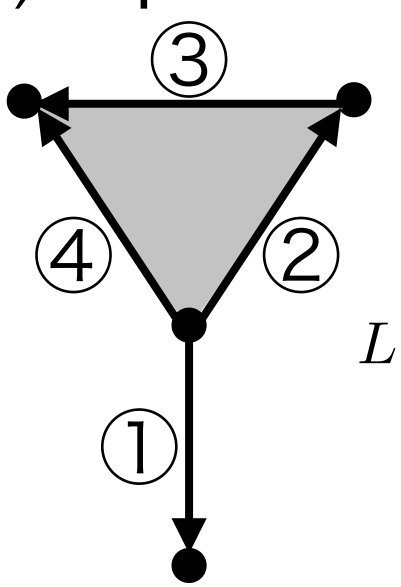
related to topology!

# Spectral graph theory

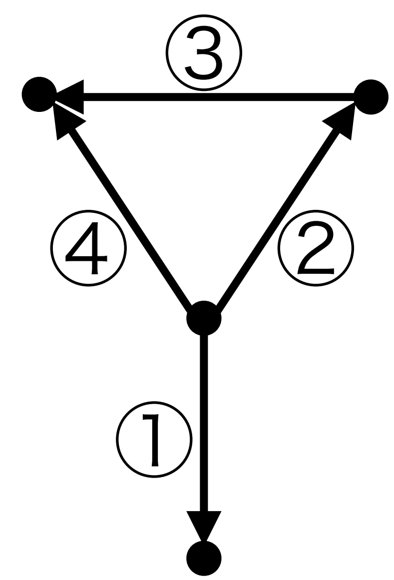
- combinatorial Laplacian matrix

$$(L_1)_{lk} = \begin{cases} 2+\deg(F_l) & l = k \qquad \deg(F_l) \ \textit{is the number of faces touching the edge } l \\ 1 & \textit{if both } l \textit{ and } k \textit{ go in or out of the same vertex} \\ -1 & \textit{if one goes in and the other goes out of the same vertex} \\ 0 & \textit{otherwise} \end{cases}$$

ex.)simple four-vertex graphs




$$L_1 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 3 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$




$$L_1 = \begin{pmatrix} 2 & 1 & 0 & 1 \\ 1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$\beta_q = \dim\left(\mathcal{L}_q(K)\right) - \operatorname{rank}\left(\mathcal{L}_q(K)\right) = \operatorname{nullity}\left(\mathcal{L}_q(K)\right) = \#\text{of zero eigenvalues of } \mathcal{L}_q(K).$$

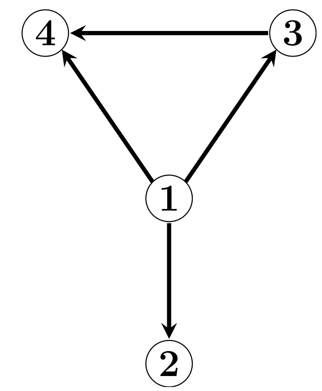
# Spectral graph theory

Yumoto, TM (23)

- Anti-symmetrized adjacency matrix

$$(A_{\text{as}})_{ij} = \begin{cases} 1 & \textit{an edge leaves } i \textit{ and enters } j \\ -1 & \textit{an edge leaves } j \textit{ and enters } i \ , \\ 0 & \textit{otherwise} \end{cases}$$

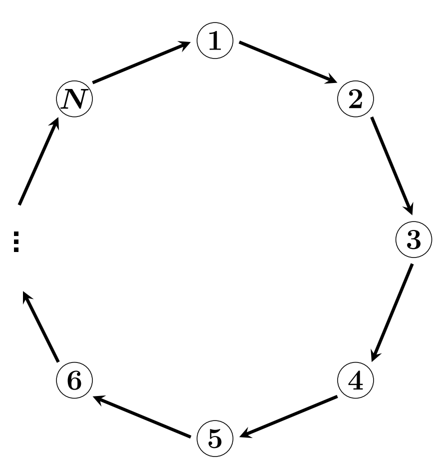
cf.)directed & unweighted




$$A_{\text{as}} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

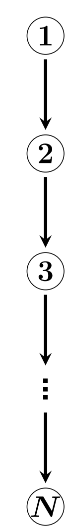
**Anti-symmetrized adjacency matrix =1d free fermion Dirac operator**

ex.)cycle (1d torus)




$$A_{\text{as}}(D_{\text{c}}) = \begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & -1 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 1 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

ex.)path (1d disk)




$$A_{\text{as}}(D_{\text{p}}) = \begin{pmatrix} 0 & 1 & 0 & & 0 & 0 & 0 \\ -1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 0 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 0 & 1 \\ 0 & 0 & 0 & & 0 & -1 & 0 \end{pmatrix}$$

# Spectral graph theory

Yumoto, TM (23)

- Anti-symmetrized adjacency matrix

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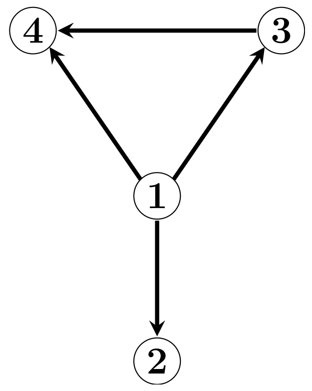



$$A_{\rm as} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{pmatrix}$$

Anti-symmetrized adjacency matrix =1d free fermion Dirac operator

$$\dim\left(\ker A_{\rm as}(G)\right) \leq \beta_0(G) + \beta_1(G) = 2 - |V| + |E|$$

# of exact zeromodes is bounded by Betti numbers

# Spectral graph theory

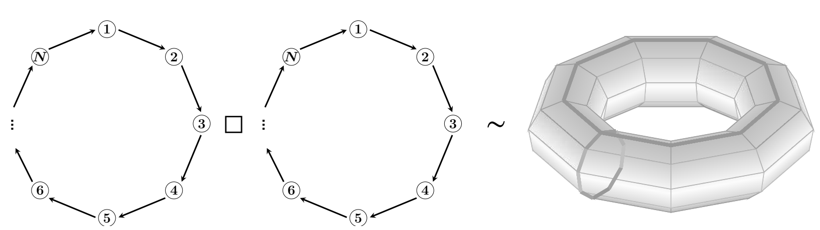
Yumoto, TM (23)

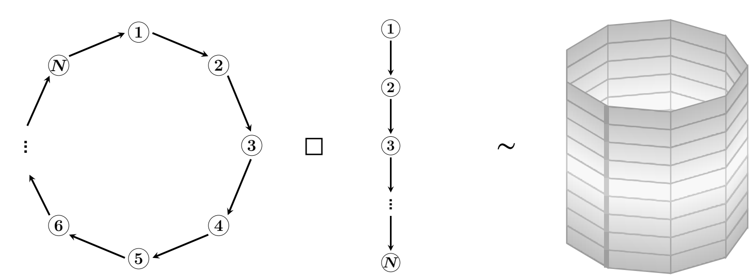
- Anti-symmetrized adjacency matrix in D-dim

$$\mathcal{D}(G) \equiv \sum_{\mu=1}^{D} (A_{\mathrm{as}})_{\mu} \otimes \gamma_{\mu} \qquad G \equiv G_1 \square G_2 \square \cdots \square G_D$$

**Anti-symmetrized adjacency matrix in D-dim**
**= D-dim free fermion Dirac operator**

For the graph G as cartesian product of T1 and B1





$$\frac{\dim\left(\ker \mathcal{D}(G)\right)}{\mathrm{rank}\, \gamma} \leq \sum_{r=0}^{D} \beta_r(M)$$

# of exact zeromodes is bounded by cumulative sum of Betti numbers

# Implications for Lattice theory

Yumoto, TM (23)

- Lattice scalar action

$$S_{\mathrm{b}} = -\sum_{n,\mu} \frac{1}{2}\phi_n(2\phi_n - \phi_{n-\hat{\mu}} - \phi_{n+\hat{\mu}}) = -\frac{1}{2}\phi L \phi$$

→ graph laplacian!

# of zeromodes of lattice free and massless scalar operator

=

0-th Betti number of the graph (lattice)

$$\dim(\mathrm{Ker}\mathcal{B}) = \dim(\mathrm{Ker}L) = \beta_0$$

For any simply connected graph (lattice), the number of zeromodes of free and massless lattice scalar is one.

# Implications for Lattice theory

Yumoto, TM (23)

- Lattice naive fermion

$$S = \sum_{n,\mu} \bar{\psi}_n \gamma_\mu (\psi_{n-\hat{\mu}} - \psi_{n+\hat{\mu}} + m\psi_n) = \bar{\psi}\gamma_\mu A^\mu_{\rm as}\psi$$

# of zero modes of Lattice free naive fermion

<=

sum of all Betti numbers of the graph (lattice)

$$\dim(\mathrm{Ker}\mathcal{D})/\mathrm{rank}\gamma \leq \sum_{n=0}^{D} \beta_n$$

It is consistent with the known result on the number of species in D-dimensional naive fermion.

# Observations

Yumoto, TM (22)

| | sum of $\beta_n(M)$ | max # of free Dirac zeromodes |
|---|---|---|
| 1D torus | 1+1 | 2 |
| 2D torus | 1+2+1 | 4 |
| 3D torus | 1+3+3+1 | 8 |
| 4D torus | 1+4+6+4+1 | 16 |
| $T^D$ | $(1+1)^D$ | $2^D$ |
| Hyperball | 1+0+0+…. | 1 |
| Sphere | 1+0+0+…+1 | 2 |
| $T^D \times R^d$ | $2^D + 0$ | $2^D$ |

The theorem may be generalized to generic lattices.

# Implications for Lattice theory

Yumoto, TM (23)

- Lattice Wilson fermion term

$$S_{\mathrm{W}} = \sum_{n,\mu} \bar\psi_n \gamma_\mu (\psi_{n-\hat\mu} - \psi_{n+\hat\mu}) + \frac{1}{2}\sum_{n,\mu} \bar\psi_n (2\psi_n - \psi_{n-\hat\mu} - \phi_{n+\hat\mu})$$

$$= \bar\psi[\mathcal{D} + \mathcal{W}]\psi$$

$$= \bar\psi \left[ (A_{\mathrm{as}})_\mu \otimes \gamma_\mu + \frac{L}{2} \otimes \mathbf{1} \right] \psi \, .$$

graph laplacian!

$$[\mathcal{D}, \mathcal{W}] = \left[ A_{\mathrm{as}}, \frac{L}{2} \right] = 0$$

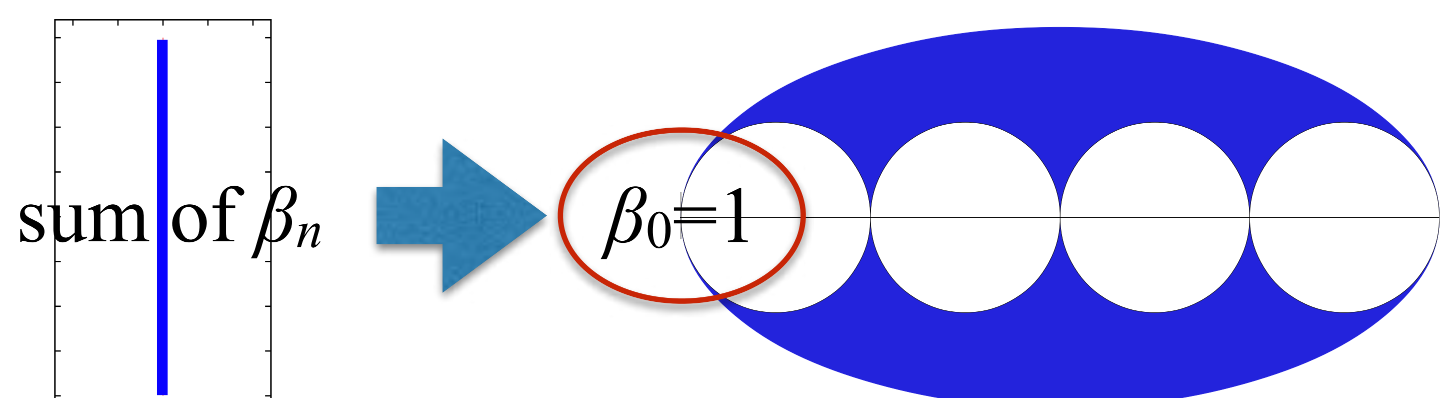
# of zero modes of Lattice Wilson term

=

1st Betti number of the graph (lattice)

Wilson term:

filtration equipment to extract zeromodes corresponding to $\beta_0$



Introduction of link variables is expected to lead to more non-trivial theorems.

# Summary

- Symmetry is a powerful tool to explore QFT, while quantum gravity seems to forbid global internal symmetry.
- We are at the stage to humbly consider a way of examining QFT without the aid of symmetry.
- Resurgence theory suits QFT with small symmetry since resurgent structure tends to emerge more vividly for smaller symmetry.
- Lattice theory is established setup to study QFT without the aid of symmetry. Equivalence to "graph theory" helps study lattice theory.