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Color confinement due to topological defects -restoration of residual gauge symmetries-

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based on N. Fukushima and K.-I. Kondo, 2023, [arXiv:2304.14008 [hep-th]]

§ Introduction

- Quark confinement is well understood based on the dual super conductor picture where condensation of magnetic monopoles and antimonopoles occurs.

- Gluon confinement is less understood with the dual superconductor picture.

→ We recall the color confinement criterion derived by Kugo and Ojima (1979).

- If the Kugo and Ojima (KO) criterion is satisfied, all colored objects cannot be observed.

- The KO criterion was derived only in the Lorenz gauge $\partial^{\mu}\mathscr{A}_{\mu} = 0$, and clearly gauge dependent.

→ The KO criterion is not directly applied to the other gauge fixing conditions.

§ Introduction

- The residual local gauge symmetry is the local gauge symmetry remaining even after imposing the gauge fixing condition.
- This symmetry is "spontaneously broken" in the perturbative vacuum, but we can derive the condition of the disappearance of the massless NG pole associated this breaking.

→ The residual gauge symmetry is restored in the true confining vacuum of QCD.

- In the Lorenz gauge, KO criterion is derived as the condition of the restoration of the residual gauge symmetry which gauge transformation function $\omega(x)$ is linear in the coordinate x. (Hata(1982))(ω should satisfy $\partial_\mu \omega(x) = 0$ from $\delta_\omega A_\mu(x) = \partial_\mu \omega(x)$)
- We can generalize the condition of the restoration of the residual gauge symmetry into more general gauge transformation including topological configurations. (Kondo, Fukushima(2022))
- In the Maximal Abelian (MA) gauge, the restoration can be also achieved, and we can understand the confinement.

→ In this talk, we consider MA gauge from the viewpoint of the momentum space and examine the effect of the topological configuration.

§ Plan

- The restoration condition of the residual gauge symmetry in the Maximal Abelian gauge
- Topological configuration which contribute to the path integral
- General discussion of the restoration condition
- Actual calculation of the restoration condition
- Summary

- In the manifestly Lorentz covariant operator formalism on the indefinite metric state space $\mathcal{V}$, we suppose that the (nilpotent) BRST symmetry exists.

- Let $\mathcal{V}_{\mathrm{phys}}$ be the physical state space with a semi-positive definite metric $\langle \mathrm{phys}|\mathrm{phys}\rangle \geq 0$. Using BRST charge Q_B,

$$\mathcal{V}_{\mathrm{phys}} = \{|\mathrm{phys}\rangle \in \mathcal{V}; Q_B|\mathrm{phys}\rangle = 0\} \subset \mathcal{V}. \qquad (1)$$

- We decompose the Lie-algebra valued quantity to the diagonal Cartan part and remaining off-diagonal part.

- The gauge field $\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)T_A$ with the generators T_A $(A = 1, \cdots, N^2 - 1)$ of the Lie algebra $su(N)$ has the decomposition:

$$\mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)T_A = a_\mu^j(x)H_j + A_\mu^a(x)T_a, \qquad (2)$$

where H_j are the Cartan generators and T_a are the remaining generators of the Lie algebra.

- We choose the Maximal Abelian (MA) gauge

$$\left(\mathscr{D}^\mu[a]A_\mu(x)\right)^a := \partial^\mu A_\mu^a(x) + g f^{ajb} a^{\mu j}(x) A_\mu^b(x) = 0. \tag{3}$$

- Since the MA gauge does not fix the diagonal components, we further impose the Lorenz gauge for the diagonal components

$$\partial^\mu a_\mu^j = 0. \tag{4}$$

- The Lagrangian in the MA gauge is given as below

$$\mathscr{L} = \mathscr{L}_{\rm inv} + \mathscr{L}_{\rm GF+FP},$$

$$\mathscr{L}_{\rm inv} = -\frac{1}{4}\mathscr{F}_{\mu\nu}^A \mathscr{F}^{\mu\nu A} + \mathscr{L}_{\rm matter}(\psi, D_\mu\psi),$$

$$\mathscr{L}_{\rm GF+FP} = -i\boldsymbol{\delta}_B\left\{\bar{C}^a\left(\mathscr{D}^\mu[a]A_\mu + \frac{\alpha}{2}B\right)^a\right\} - i\boldsymbol{\delta}_B\left\{\bar{c}^j\left(\partial^\mu a_\mu + \frac{\beta}{2}b\right)^j\right\}. \tag{5}$$

- We consider the local gauge transformation with the Lie algebra valued transformation function $\omega(x)$

$$\delta^{\omega}\mathscr{A}_{\mu}(x) = \mathscr{D}_{\mu}\omega(x) \coloneqq \partial_{\mu}\omega(x) + g\mathscr{A}_{\mu}(x) \times \omega(x),$$
$$\delta^{\omega}\varphi(x) = ig\omega(x)\varphi(x), \qquad \delta^{\omega}\mathscr{B}(x) = g\mathscr{B}(x) \times \omega(x),$$
$$\delta^{\omega}\mathscr{C}(x) = g\mathscr{C}(x) \times \omega(x), \ \delta^{\omega}\bar{\mathscr{C}}(x) = g\bar{\mathscr{C}}(x) \times \omega(x). \tag{6}$$

Since this is residual symmetry, $\omega^{j}(x)$ should satisfy the Laplace equation.

- We can calculate the Noether current $\mathscr{J}^{\mu}_{\omega}$ associated with (6) and the divergence of this current is equal to the transformation of the Lagrangian density under the transformation (6)

$$\partial_{\mu}\mathscr{J}^{\mu}_{\omega} = \delta^{\omega}\mathscr{L} = \delta^{\omega}\mathscr{L}_{\mathrm{GF+F}}$$
$$= i\boldsymbol{\delta}_{B}\partial_{\mu}\bar{c}^{j}\partial^{\mu}\omega^{j} + i\boldsymbol{\delta}_{B}\partial^{\mu}\left(\mathscr{D}_{\mu}[\mathscr{A}]\bar{\mathscr{C}}\right)^{a}\omega^{a} + i\boldsymbol{\delta}_{B}\left(\mathscr{D}_{\mu}[\mathscr{A}]\bar{\mathscr{C}}\right)^{a}\partial^{\mu}\omega^{a}. \tag{7}$$

→ The local gauge current $\mathscr{J}^{\mu}_{\omega}$ is conserved in the physical state space.
((7) is BRST transformation δ_B exact)(Kondo, Fukushima(2022))

- Now, we focus on Abelian (diagonal) part.
- The restoration condition of the residual symmetry is written as the condition of the disappearance of the massless pole. In a sector of the field $\Phi^{\ell}(y) = \Phi_1^{\ell_1}(y_1)\cdots\Phi_n^{\ell_n}(y_n)$, we focus on the following Ward-Takahashi (WT) identity

$$0 = \lim_{p\to 0} i\int d^D x e^{ipx}\partial_\mu^x\langle J_\omega^\mu(x)\Phi^\ell(y)\rangle$$
$$= \lim_{p\to 0}\sum_{a=1}^{n} e^{ipy_a}\left\langle\Phi_1^{\ell_1}(y_1)\cdots\delta^\omega\Phi_a^{\ell_a}(y_a)\cdots\Phi_n^{\ell_n}(y_n)\right\rangle + i\int d^D x e^{ipx}\langle\partial_\mu^x J_\omega^\mu(x)\Phi^\ell(y)\rangle . \tag{8}$$

- Using $\langle\delta_B F\rangle = 0$ and Schwinger-Dyson equation $\int d\mu \frac{\delta}{\delta\Phi(x)} e^{iS}F = 0$, we get from (7):

$$i\langle\partial_\mu^x J_\omega^\mu(x)\Phi^\ell(y)\rangle = -\partial^\mu\omega^j(x)\left\langle\delta_B\left(\partial_\mu\bar{c}^j(x)\right)\Phi^\ell(y)\right\rangle = -\partial^\mu\omega^j(x)\langle\partial_\mu\bar{c}^j(x)\delta_B\Phi^\ell(y)\rangle$$
$$= -\partial^\mu\omega^j(x)\left[\frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\langle\partial^\nu\bar{c}^j(x)\delta_B\Phi^\ell(y)\rangle + \left(g_{\mu\nu} - \frac{\partial_\mu^x\partial_\nu^x}{\partial_x^2}\right)\langle\partial^\nu\bar{c}^j(x)\delta_B\Phi^\ell(y)\rangle\right]$$
$$= \partial^\mu\omega^j(x)\frac{\partial_\mu^x}{\partial_x^2}\left\langle\frac{\delta}{\delta c^j(x)}\delta_B\Phi^\ell(y)\right\rangle. \tag{9}$$

→
$$\boxed{\lim_{p\to 0}\sum_{a=1}^{n} e^{ip\ \ a}\left\langle\Phi_1^{\ell_1}(y_1)\cdots\delta^\omega\Phi_a^{\ell_a}(y_a)\cdots\Phi_n^{\ell_n}(y_n)\right\rangle + i\int d^D x e^{ipx}\partial^\mu\omega^j(x)\frac{\partial_\mu^x}{\partial_x^2}\left\langle\frac{\delta}{\delta c^j(x)}\delta_B\Phi^\ell(y)\right\rangle} \tag{10}$$

- n gauge field sector: $\Phi(y) = a^{\ell_1}_{\rho_1}(y_1)\cdots a^{\ell_n}_{\rho_n}(y_n)$

$$0 = \lim_{p\to 0}\sum_{a=1}^{n}\int d^D x\, e^{ipx}\partial^\mu \omega^{\ell_a}(x)\left(g_{\mu\rho_a} - \frac{\partial^x_\mu \partial^x_{\rho_a}}{\partial^2_x}\right)\delta^D(x-y_a)$$
$$\times\left\langle a^{\ell_1}_{\rho_1}(y_1)\cdots a^{\ell_{a-1}}_{\rho_{a-1}}(y_{a-1})a^{\ell_{a+1}}_{\rho_{a+1}}(y_{a+1})\cdots a^{\ell_n}_{\rho_n}(y_n)\right\rangle \quad (11)$$

- n quark-antiquark pair: $\Phi(y) = \psi(y_1)\cdots\psi(y_n)\bar{\psi}(z_1)\cdots\bar{\psi}(z_n)$

$$0 = \lim_{p\to 0}\sum_{a=1}^{n}\int d^D x\, e^{ipx}\left(\omega^j(x) + \partial^\mu \omega^j(x)\frac{\partial^x_\mu}{\partial^2_x}\right)\left(\delta^D(x-y_a) - \delta^D(x-z_a)\right)\langle\Phi\rangle \quad (12)$$

The momentum representation of the topological configuration

- In order to examine the effect of the topological configuration to the restoration of the residual symmetry, in $SU(2)$, we consider following topological configuration with one defect, where Q corresponds to topological charge.

$$\partial_\mu \omega^j(x) = Q \frac{\eta^j_{\mu\nu} x^\nu}{x^2} h(x^2) \quad (Q \propto g^{-1},\ j = 3,\ \mu, \nu = 1,2,\cdots,D) \tag{13}$$

- If $h(x^2) \equiv 1$, (13) include
  $D = 2$:Abrikosov-Nielsen-Olesen vortex,
  $D = 3$:Wu-Yang monopole,
  $D = 4$:Alfaro-Fubini-Furlan meron.

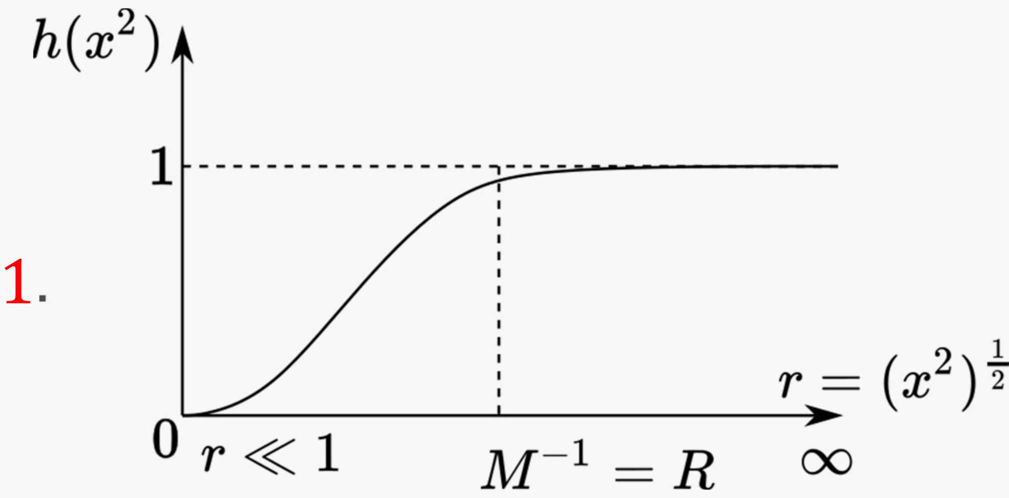
The divergence of the action(UV divergence)

- When $h(x^2) \equiv 1$, topological configuration (13) results in a divergent Euclid action S_E as $x \to 0$.

→ In order for such configurations to contribute to the path integral, $h(x^2) \to 0$ moderately as $x \to 0$

$$e^{-S_E} > 0 \Leftarrow S_E < \infty \tag{14}$$

- To obtain the finite S_E, $h(x^2)$ must approaches 0 of order $r^{2\delta} (\delta > 1/4)$ in $r \ll 1$ and it must approach 1 rapidly in $r \gg 1$. ($D = 2$: Kondo(2018), $D = 3$: Nishino et al. (2018))



§ General discussion of the restoration condition

- For simplicity, in $SU(2)$, we focus on one gauge field sector $\Phi(y) = a^j_\lambda(y)$

$$I_\lambda \coloneqq \lim_{p\to 0} \int d^D x \ e^{ip(x-y)} \partial^\mu \omega^j(x) \left(g_{\mu\lambda} - \frac{\partial_\mu \partial_\lambda}{\partial^2} \right) \delta^D(x-y). \tag{15}$$

- In our previous paper(Kondo, Fukushima(2022)), we put $h(x^2) \equiv 1$ and demonstrated the restoration of the residual symmetry is satisfied $I_\lambda = 0$ in the MA gauge.
- Using Fourier transformation

$$\widetilde{\partial_\mu \omega}^j(q) \coloneqq \int d^D x \, e^{iqx} \partial_\mu \omega^j(x) \,,$$

we obtain

$$I_\lambda = \lim_{p\to 0} \int \frac{d^D k}{(2\pi)^D} \ e^{-i(p-k)y} \widetilde{\partial_\mu \omega}^j(p-k) \left(g_{\mu\rho} - \frac{k_\mu k_\lambda}{k^2} \right), \tag{16}$$

- From translationally invariance, we can put $y = 0$

$$I_\lambda = \lim_{p\to 0} \int \frac{d^D k}{(2\pi)^D} \ \widetilde{\partial_\mu \omega}^j(p-k) \left(g_{\mu\rho} - \frac{k_\mu k_\lambda}{k^2} \right). \tag{17}$$

Two region

- We put $q \coloneqq p - k$

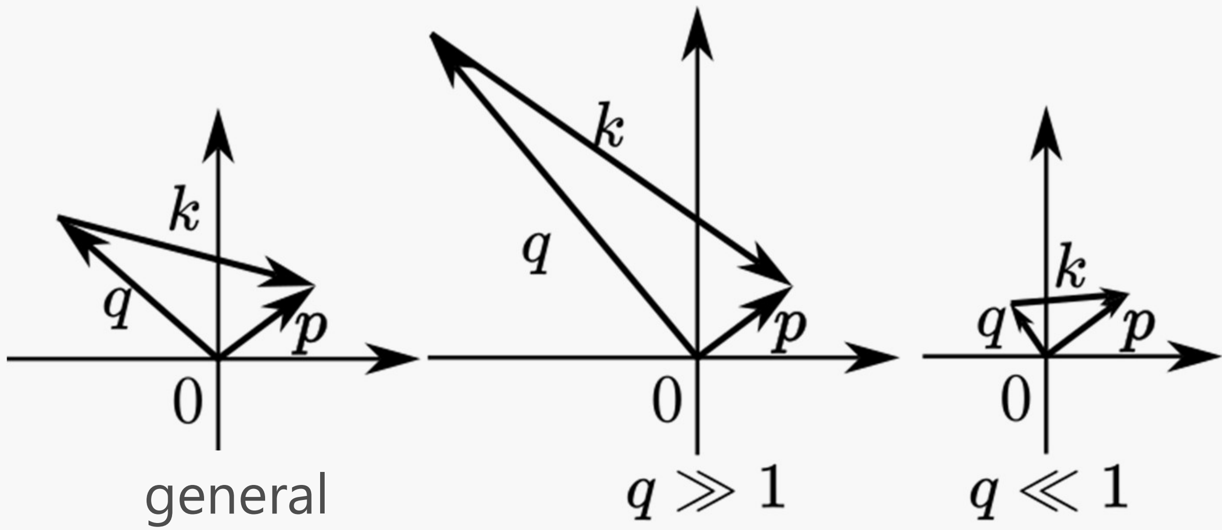
$$\widetilde{\partial_\mu \omega^j}(q) \coloneqq \int d^D x\, e^{iqx} \partial_\mu \omega^j(x)\,, I_\lambda = \lim_{p\to 0} \int \frac{d^D k}{(2\pi)^D}\, \widetilde{\partial_\mu \omega^j}(q) \left(g_{\mu\lambda} - \frac{k_\mu k_\lambda}{k^2} \right). \tag{17'}$$

- We fix $p \ll 1$.
- $r = (x^2)^{\frac{1}{2}} \ll 1 \Rightarrow q \gg 1 \Rightarrow k \gg 1$

→ UV region (of k)

- $r \gg 1 \Rightarrow q \ll 1 \Rightarrow k \simeq p \to 0$

→ IR region (of k)



§ General discussion of the restoration condition

- When $h(x^2) \equiv 1$, Fourier transformation of the topological configuration (13) is given by

$$\widetilde{\partial_\mu \omega}^j(q) \coloneqq iQC_D \eta^{j\mu\nu} \frac{q_\nu}{|q|^D}, \qquad C_D \coloneqq 2^{D-1}\pi^{\frac{D}{2}}\Gamma\left(\frac{D}{2}\right). \tag{18}$$

- In $r \ll 1$, we multipled $r^{2\delta}(\delta > 1/4)$.

→ In $q \gg 1$, it corresponds to dividing $\widetilde{\partial_\mu \omega}(q)$ of (18) by $q^{2\delta}$

$$\widetilde{\partial_\mu \omega}^j(q) \sim iQC_D \eta^{j\mu\nu} \frac{q_\nu}{|q|^{D+2\delta}} \quad (\delta > 1/4) \tag{18'}$$

- We discuss the restoration of the residual symmetry by using this $\widetilde{\partial_\mu \omega}$.

$$I_\lambda = -iQ\left(\frac{D}{8} + \frac{\delta}{4}\right) \frac{\Gamma(\delta)\Gamma(1-\delta)\Gamma\left(\frac{D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma\left(\frac{D}{2}+\delta+1\right)\Gamma\left(\frac{D}{2}-\delta+1\right)} \lim_{p\to 0} \eta^j_{\lambda\nu} p^\nu (p^2)^{-\delta} \tag{19}$$

- In this case, we can take the limit $p \to 0$ in the integrand of the k when we estimate the contribution from the UV region $k \gg k_R$ i.e. $p \ll k$

$$I_\lambda \to \boxed{-i \frac{QC_D}{2\delta(4\pi)^{\frac{D}{2}}\Gamma\left(\frac{D}{2}+1\right)} \lim_{p\to 0} \eta^j_{\lambda\nu} p^\nu \frac{1}{k_R^{2\delta}}} \tag{19'}$$

→ δ play the role of avoiding the UV divergence $(\delta > 1/4)$.

§ Actual calculation of the restoration condition

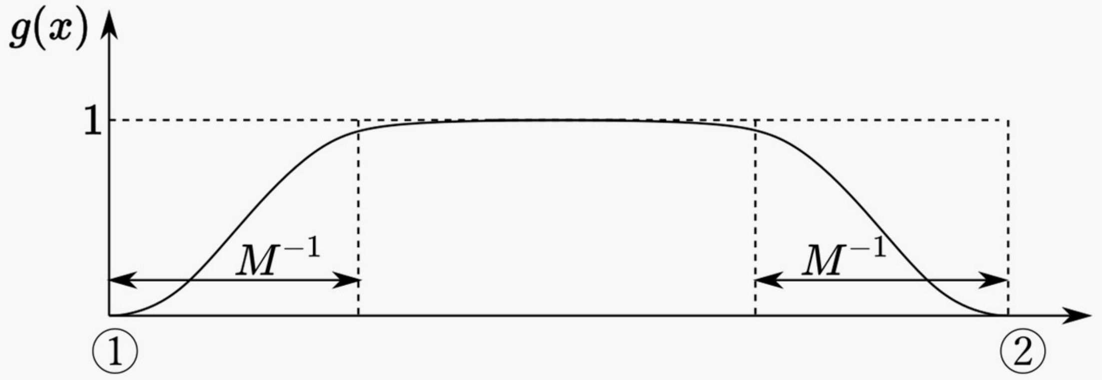
- Since $q \ll 1$ corresponds to $r \gg 1$, $\delta \to 0$.
- We cannot avoid IR divergence when the configuration has only one defect.

→ We introduce the topological configuration with the multiple pairs of defect.

$$\partial_\mu \omega^j(x) = \sum_s^n Q_s \frac{\eta^j_{\mu\nu}(x-x_s)^\nu}{|x-x_s|^2} g\big((x-x_s)^2\big) \quad (Q_s \propto g^{-1},\ \ j=3,\ \ \mu,\nu \tag{20}$$

- When topological defects are sufficiently separated, the IR region of one defect① corresponds to the UV region of another defect②.

→ IR divergence can be eliminated.



§ Actual calculation of the restoration condition

- In $D = 3$, we consider the topological configuration consisted of the multiple monopole.

→ Magnetic monopole plasma cause the Debye screening. (Polyakov(1977))

→ The gauge field becomes massive.

- This corresponds to the origin of the appearance of the dimensional scale $M^{-1} = R$ of $h(x^2)$. ($D = 4$:Callan-Dashen-Gross(1978))

- If we introduce a topological configuration with multiple defects, these results are consistent with the confinement picture where the vacuum condensation of topological defects lead to confinement.

§ Summary

- We performed a detailed calculation of the residual symmetry restoration conditions for the one gauge field sector from viewpoint of momentum space.
- The topological configurations we have dealt with so far yield the divergence of the restoration condition which is independent of dimensions and momentum.
- The condition of the residual symmetry restoration has the IR divergence and the UV divergence.
- The UV divergence corresponds to the divergence of the Euclid action and It was speculated that this could be avoided by correcting the topological configuration to contribute to path integrals.
- When multiple pairs of topological defects are introduced (sufficiently separated), the IR region of one defect corresponds to the UV region of another defect and we can avoid the IR divergence.
- These are consistent with the idea that confinement occurs due to vacuum condensation.

Future perspective

- To give the complete discussion for confinement, it is necessary to consider interactions among various fields, including different kinds of fields such as the ghost and gauge field.
- A physical picture of symmetry restoration due to topological configurations contributing to the path integral is provided. Therefore, based on this picture, it is necessary to discuss the condensation of topological defects in arbitrary dimensions.