Three ways of calculating composite-particle spectra of gauge theories in the Hamiltonian formalism

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Mass spectrum of hadrons

- Non-perturbative calc. by **Monte Carlo simulation**

  lattice gauge theory in the Lagrangian formalism

- **Tensor network** and **Quantum computation**

  complementary approach in the Hamiltonian formalism

  👍 without sign problem 👍 treat excited states directly

  → enable us to study finite density QCD, θ term, real-time evolution, ⋯

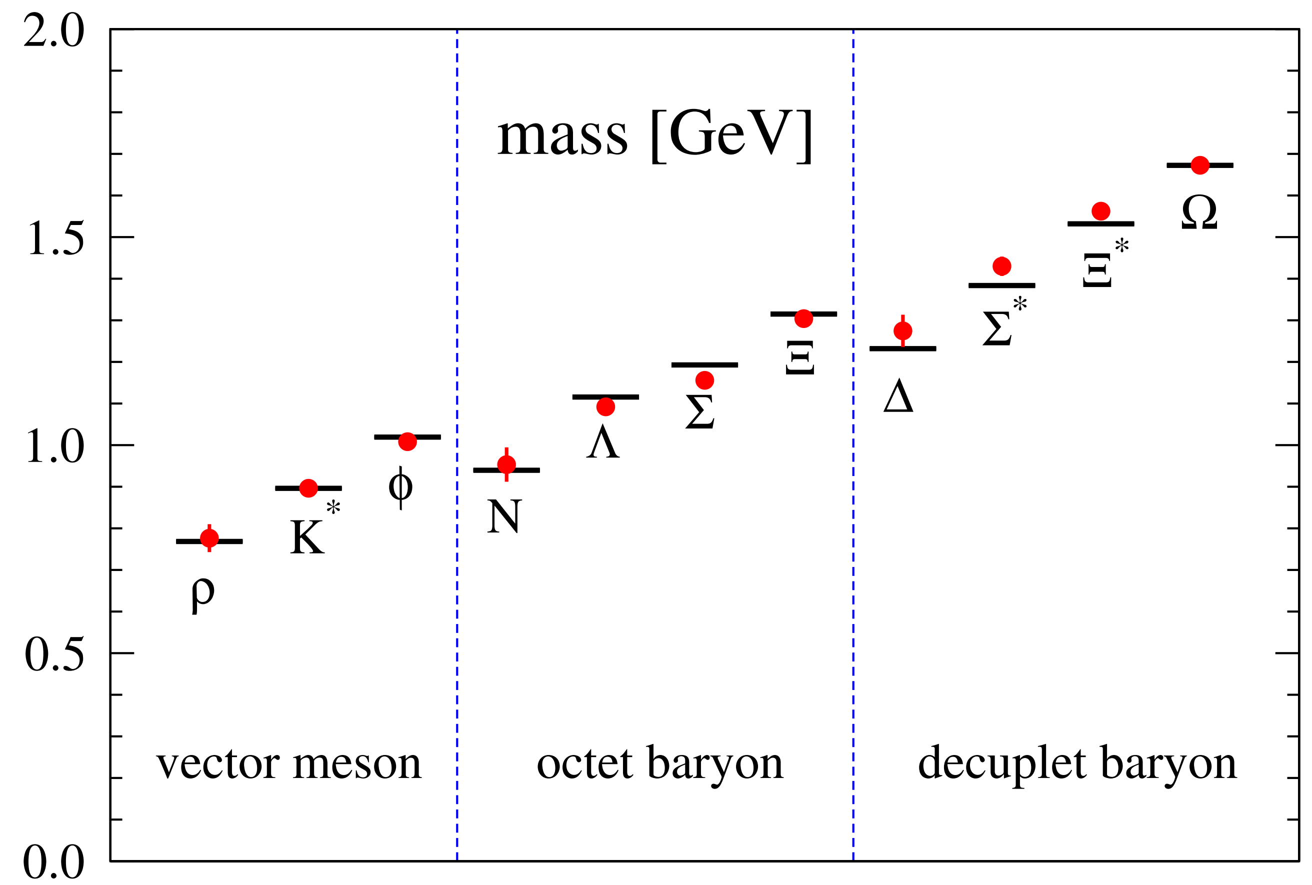

FIG. 24 (color online). Light hadron spectrum extrapolated to the physical point using m_π, m_K and m_Ω as input. Horizontal bars denote the experimental values.

[PACS-CS collab. (2009)]

<u>aim of this work</u>

compute the hadron mass spectrum in the Hamiltonian formalism

"Hadrons" in 2-flavor Schwinger model

Schwinger model = quantum electrodynamics in 1+1d

- the simplest nontrivial gauge theory sharing some features with QCD

$$\mathscr{L} = -\frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_f}\left[i\bar{\psi}_f\gamma^\mu\left(\partial_\mu + iA_\mu\right)\psi_f - m\bar{\psi}_f\psi_f\right]$$

quantum numbers

- isospin J : SU(2) acting on the flavor doublet
- parity P
- G-parity $G = Ce^{i\pi J_y}$: generalization of C

"mesons" (a type of hadron)

$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$: $J^{PG} = 1^{-+}$

$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right)$: $J^{PG} = 0^{--}$

$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$: $J^{PG} = 0^{++}$

Short summary

- three distinct methods for computing the mass spectrum

  (1) correlation-function scheme — conventional method in lattice QCD

  (2) one-point-function scheme — make good use of the boundary effect

  (3) dispersion-relation scheme — obtain the excited states directly

- demonstration in the 2-flavor Schwinger model using tensor network (DMRG)

- results of the three methods are consistent with each other

Calculation strategy

- Hamiltonian on the lattice (staggered fermion + open boundary)

$$H = \frac{g^2 a}{2} \sum_{n=0}^{N-2} \left(L_n + \frac{\theta}{2\pi} \right)^2 + \sum_{f=1}^{N_f} \left[\frac{-i}{2a} \sum_{n=0}^{N-2} \left(\chi_{f,n}^\dagger U_n \chi_{f,n+1} - \chi_{f,n+1}^\dagger U_n^\dagger \chi_{f,n} \right) + m_{\mathrm{lat}} \sum_{n=0}^{N-1} (-1)^n \chi_{f,n}^\dagger \chi_{f,n} \right]$$

- solving Gauss law condition to remove L_n

[Kogut & Susskind (1975)]

[Dempsey et al. (2022)]

- gauge fixing to $U_n = 1$

- Jordan-Wigner transformation for Nf=2

$$\chi_{1,n} = \sigma_{1,n}^- \prod_{j=0}^{n-1} (-\sigma_{2,j}^z \sigma_{1,j}^z), \quad \chi_{2,n} = \sigma_{2,n}^- (-i\sigma_{1,n}^z) \prod_{j=0}^{n-1} (-\sigma_{2,j}^z \sigma_{1,j}^z)$$

—> spin Hamiltonian with a finite-dimensional Hilbert space

Density-matrix renormalization group (DMRG)

[White (1992)] [Schollwock (2005)]

variational method to find eigenstates of H using MPS ansatz

- cost function: energy $E = \langle \Psi | H | \Psi \rangle$
- update $A_i(s_i)$ to decrease E
- introduce a cutoff ε to control the accuracy
  singular values smaller than ε are neglected in SVD
  small ε <—> large D_i <—> high accuracy <—> high cost

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0)\, A_1(s_1)\ \cdots\right] |s_0\, s_1\ \cdots\rangle$$

$A_i(s_i)$: $D_{i-1} \times D_i$ matrix

D_i : bond dimension

- ℓ-th excited state $|\Psi_\ell\rangle$ —> cost function: $\langle \Psi_\ell | H | \Psi_\ell \rangle + W \sum_{\ell'=0}^{\ell-1} \left| \langle \Psi_{\ell'} | \Psi_\ell \rangle \right|^2$

The C++ library of ITensor is used in this work. [Fishman et al. (2022)]

Simulation result ($\theta = 0$)

(1) Correlation-function scheme

(2) One-point-function scheme

(3) Dispersion-relation scheme

Simulation result ($\theta = 0$)

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(2) One-point-function scheme

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Result of pion correlation function

- pion correlation: $C_\pi(r) = \langle \pi(x)\pi(y) \rangle \qquad r = |x - y|$
- effective mass: $M_{\pi,\mathrm{eff}}(r) = -\frac{d}{dr}\log C_\pi(r)$

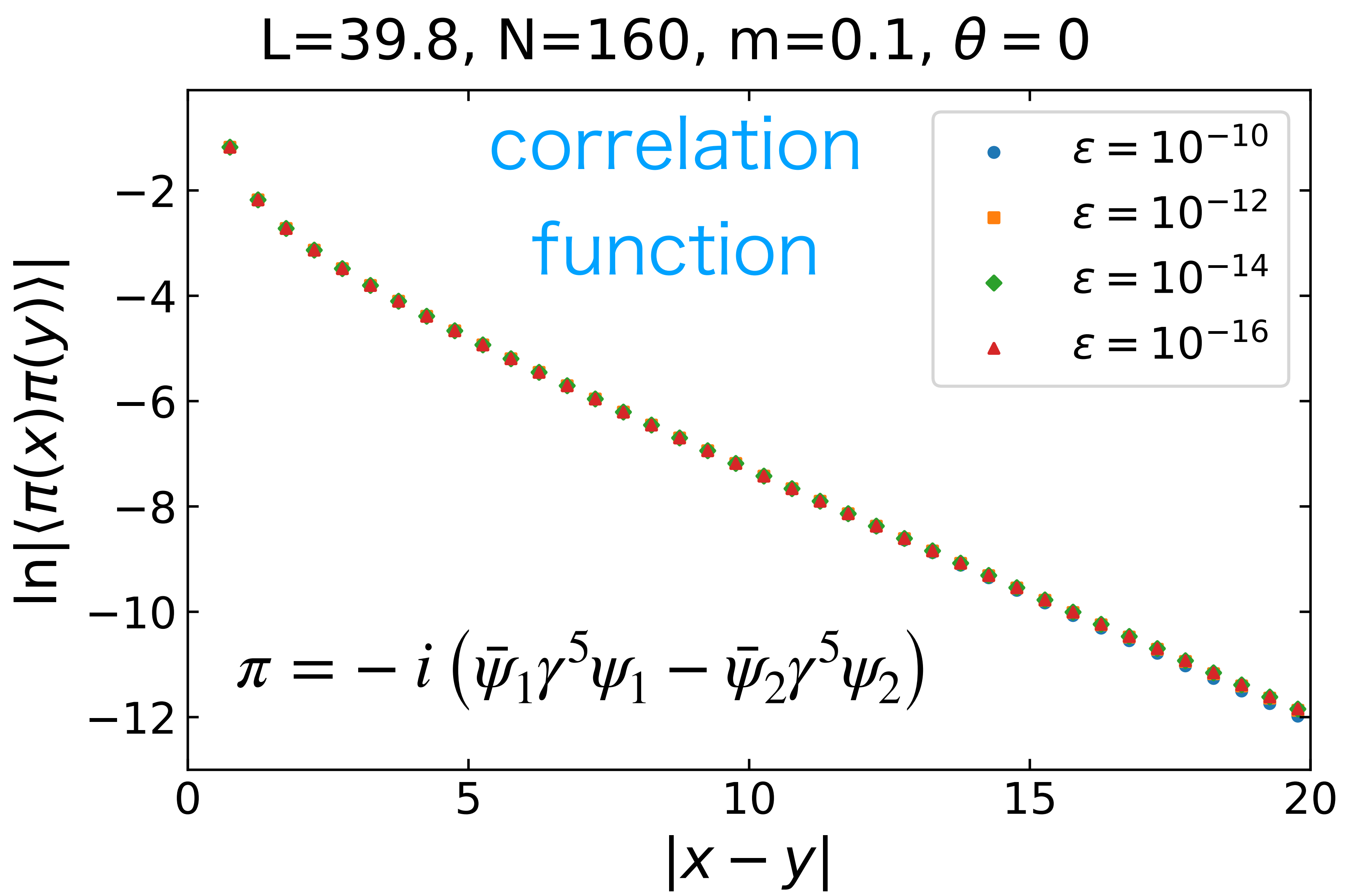
<u>plateau value = pion mass?</u>

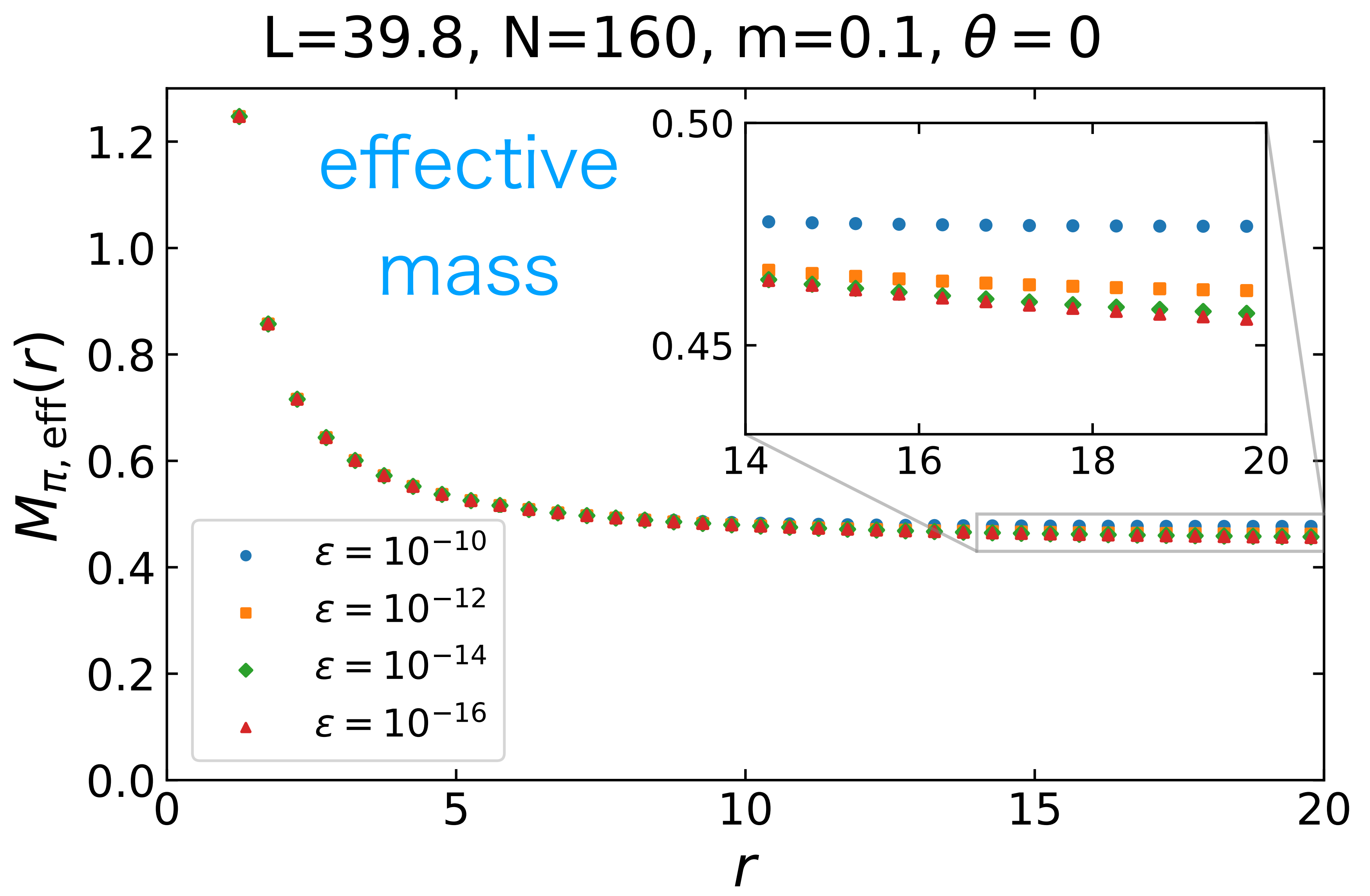
⚠️ plateau behavior gets modified in accurate calc.

$\varepsilon = 10^{-10}$ $(D_i \sim 400)$: $M_{\pi,\mathrm{eff}}(r)$ is almost flat

$\varepsilon = 10^{-16}$ $(D_i \sim 2800)$: $M_{\pi,\mathrm{eff}}(r)$ depends on r

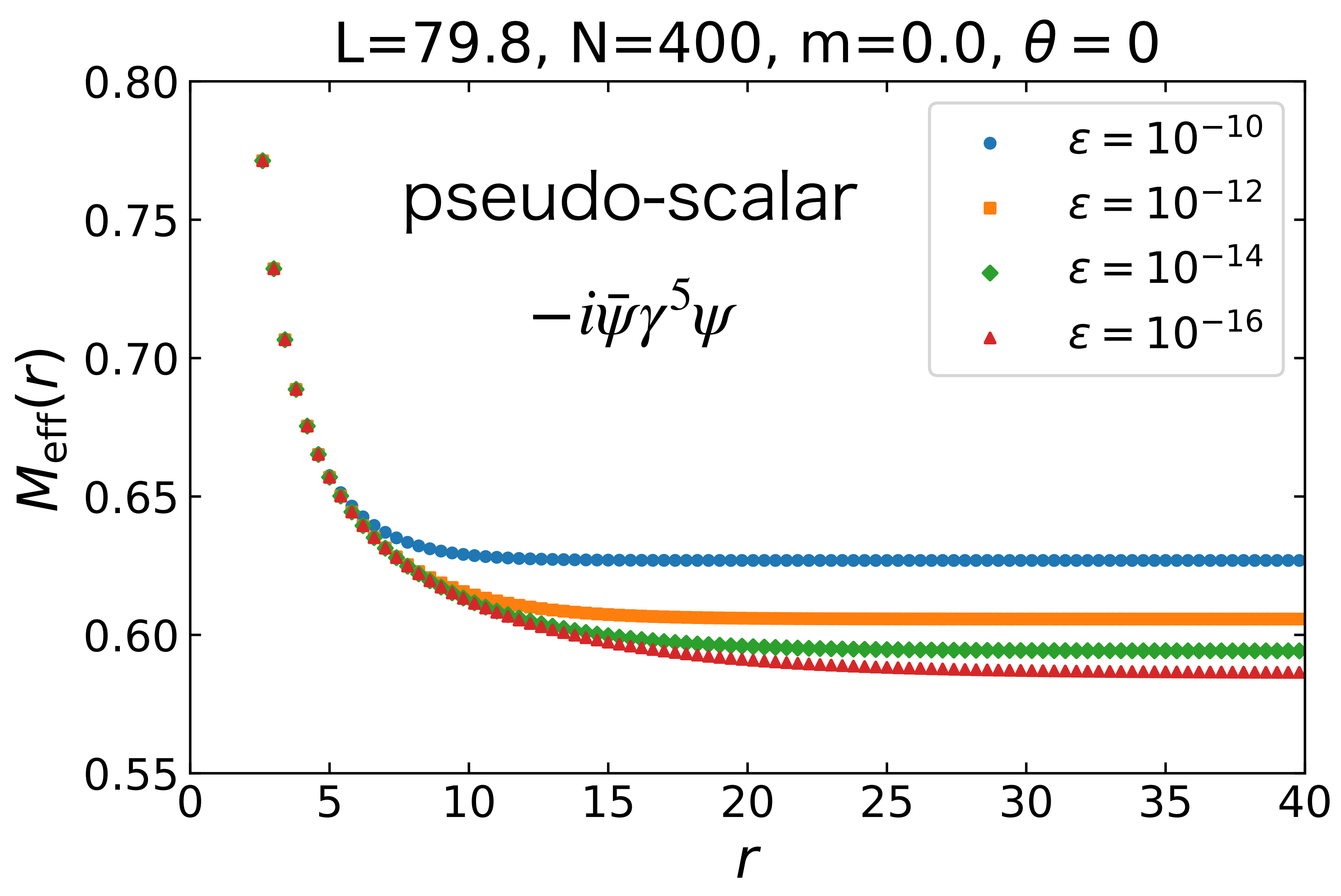
- What's happened?



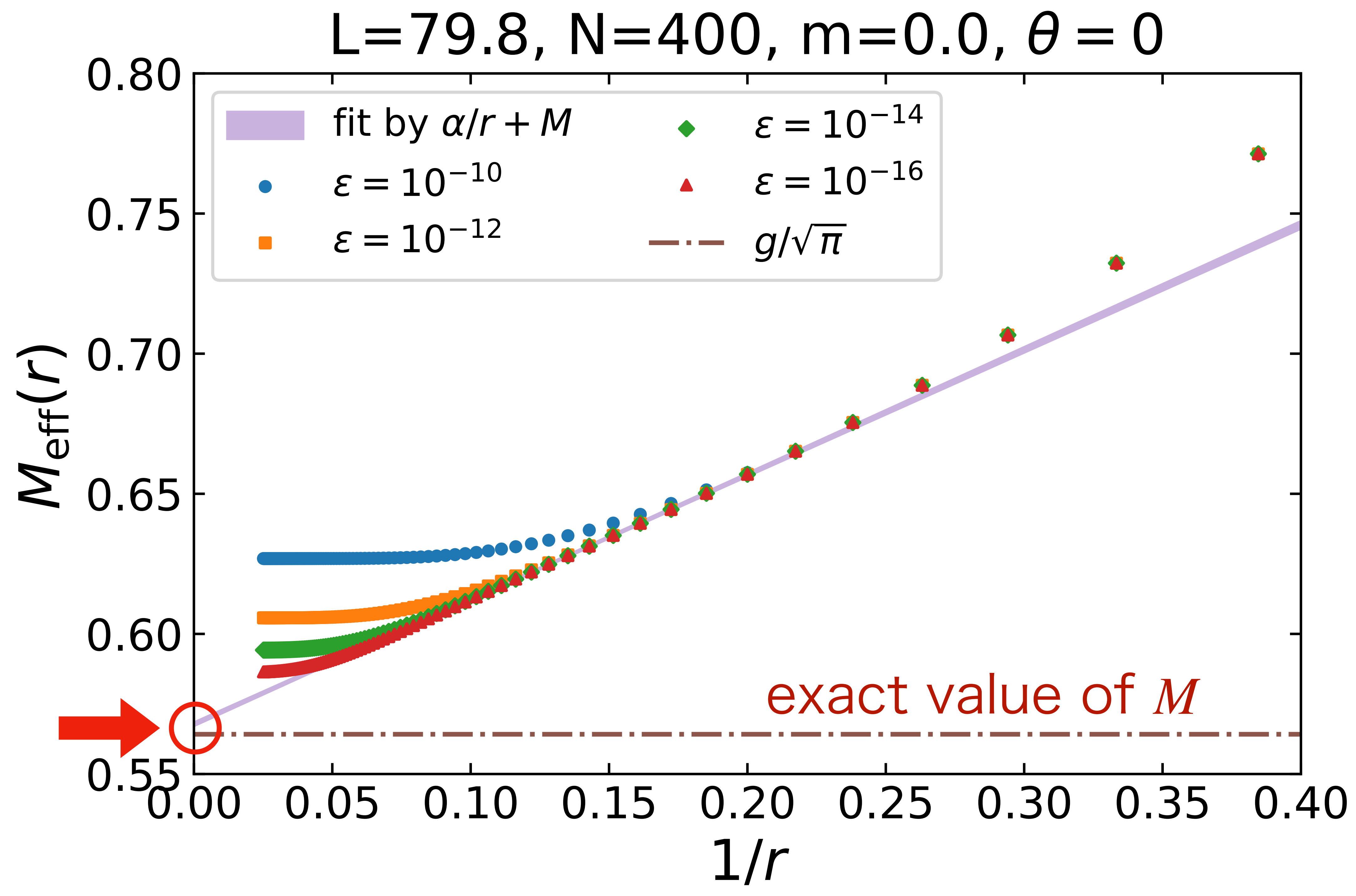


Yukawa-type correlation → 1/r term

- (1+1)d free particle with mass M: $\langle\phi(x,t)\phi(y,t)\rangle \sim \frac{1}{\sqrt{Mr}}e^{-Mr} \;\;\rightarrow\;\; M_{\mathrm{eff}}(r) \sim \frac{\alpha}{r} + M$
- massless Nf=1 Schwinger model (exactly solvable)



plot against $\frac{1}{r}$

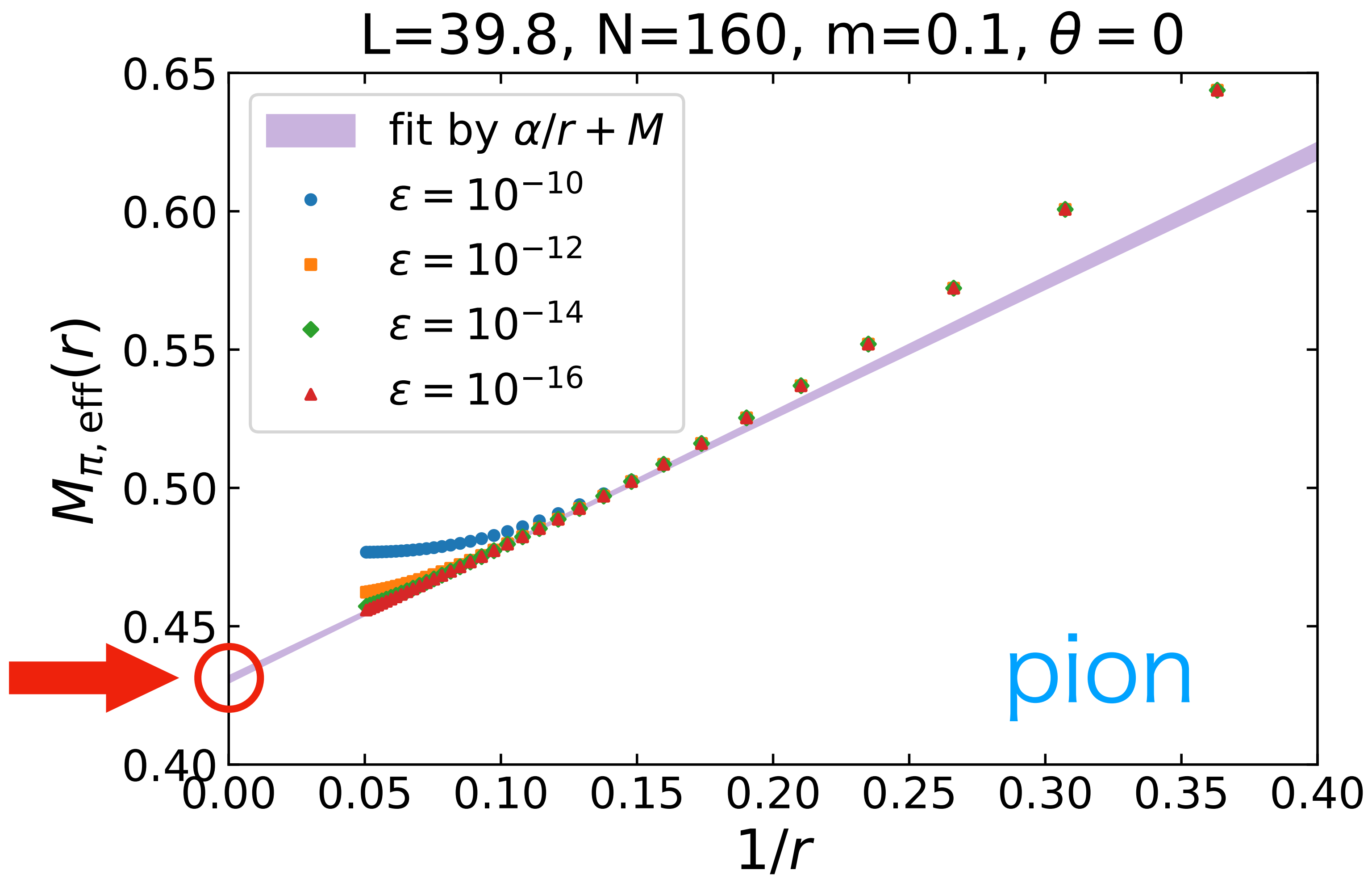


- difficult to reproduce $1/r$ term by MPS
- $r \rightarrow \infty$ extrapolation is required

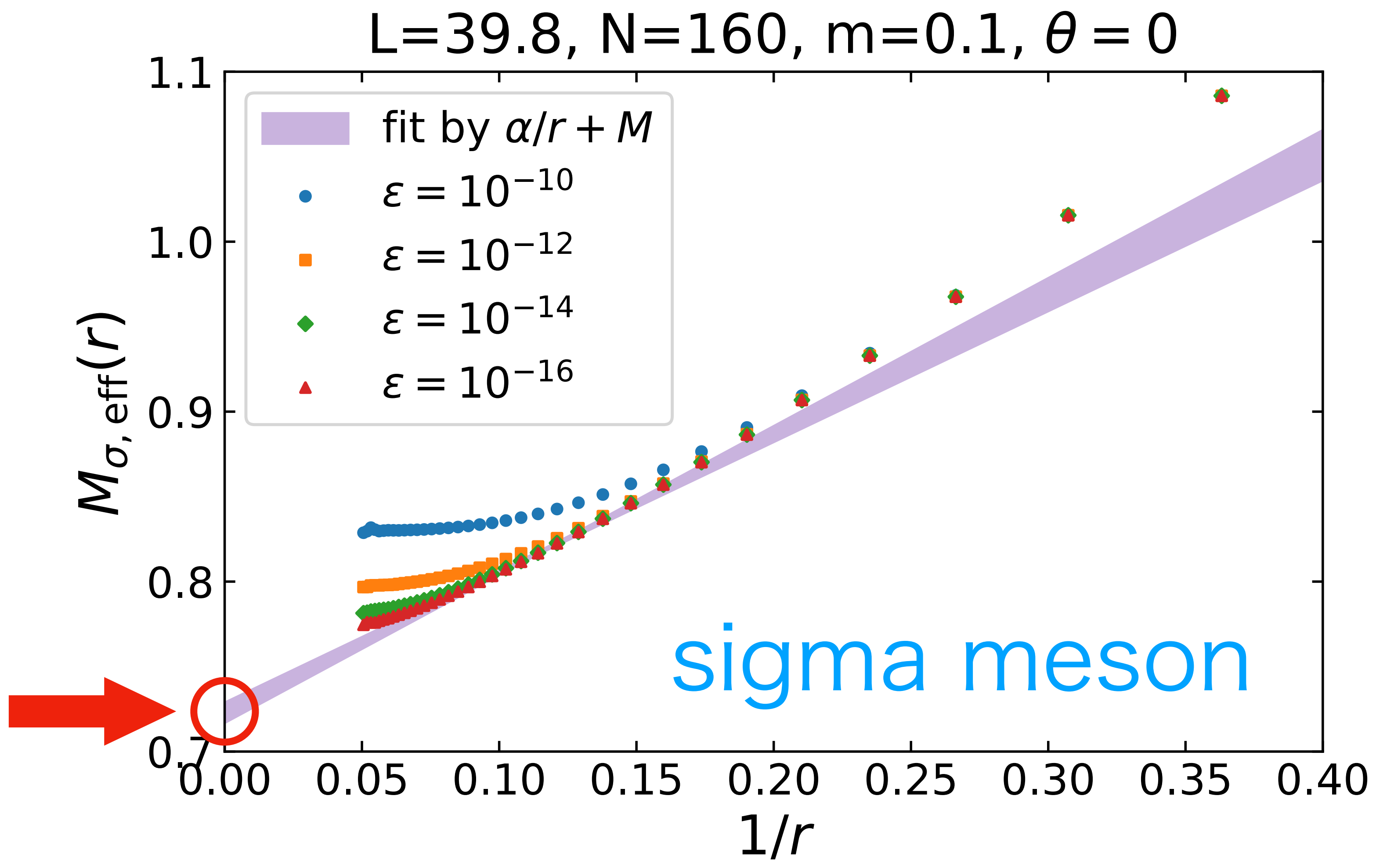
Result of the Nf=2 model

extrapolate the effective mass to $r \to \infty$ using the result for $\varepsilon = 10^{-16}$

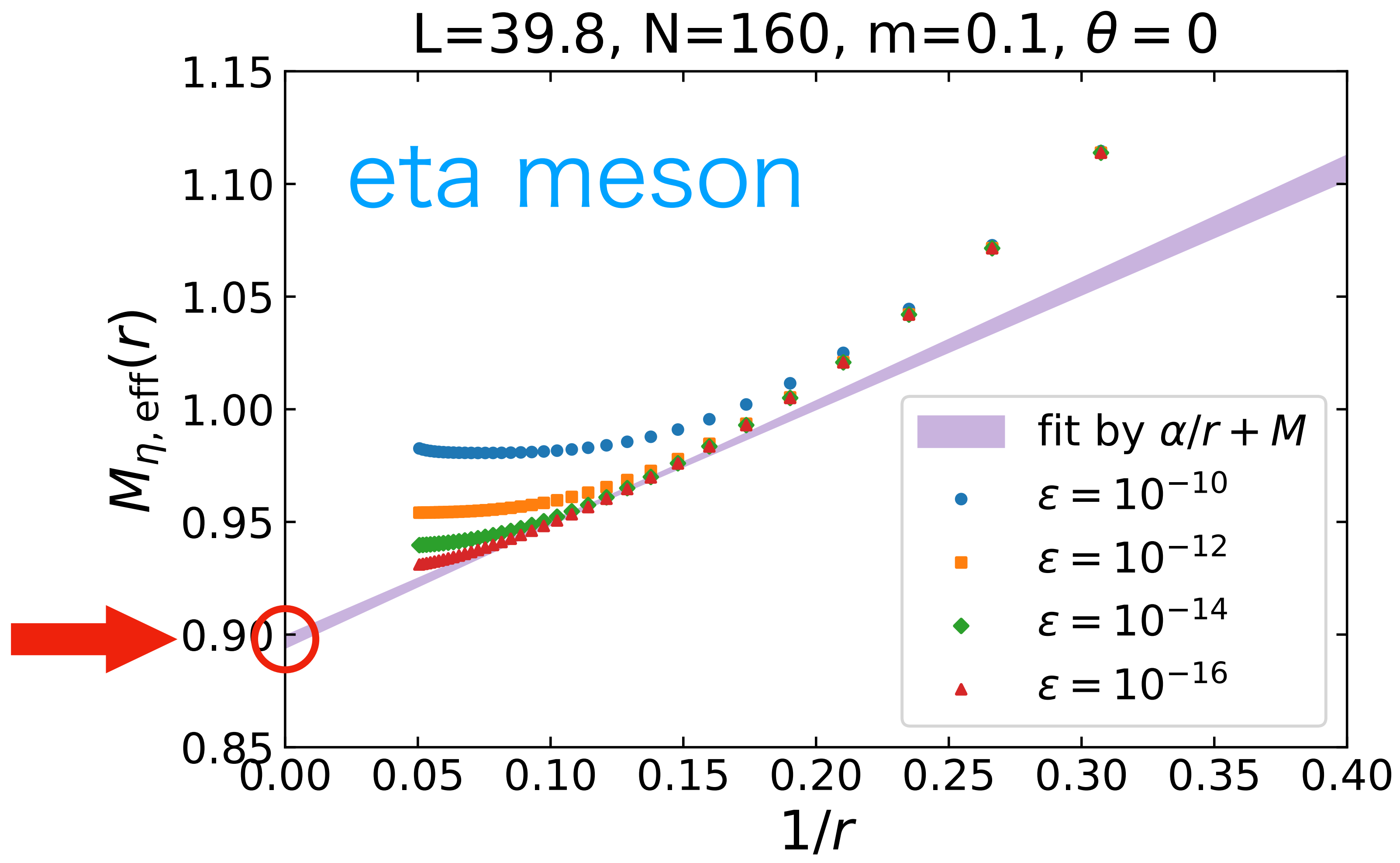
$$\pi = -i\left(\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2\right)$$



$$\sigma = \bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$$



$$\eta = -i\left(\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2\right)$$



	pion	sigma	eta
M	0.431(1)	0.722(6)	0.899(2)
α	0.477(9)	0.83(5)	0.51(2)

Simulation result ($\theta = 0$)

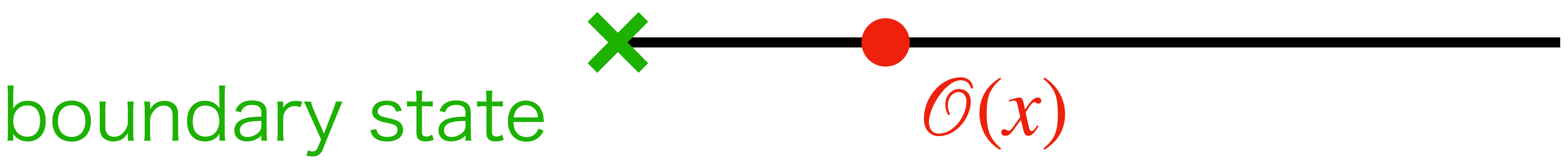
(1) Correlation-function scheme

(2) One-point-function scheme

(3) Dispersion-relation scheme

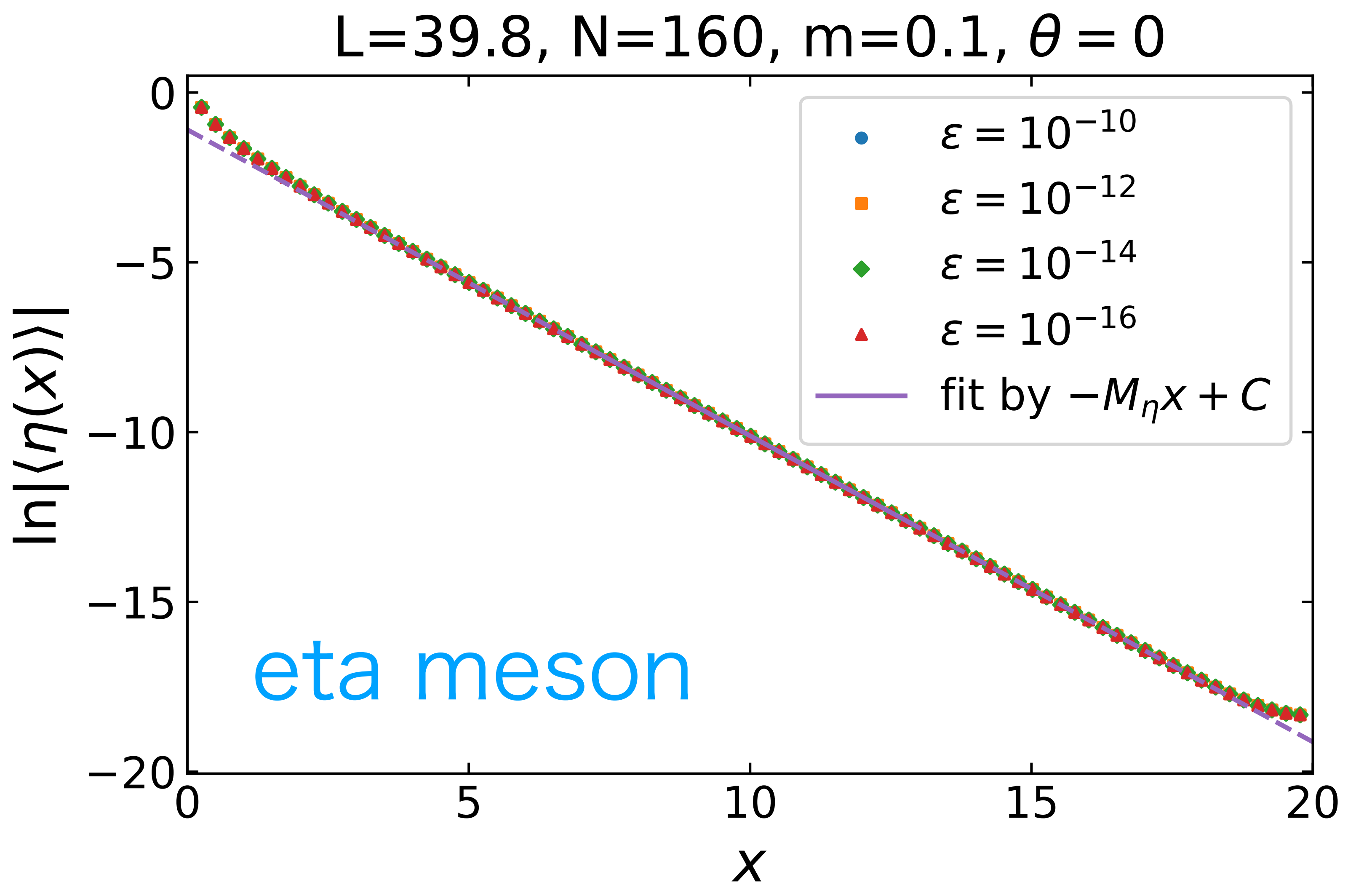
Result of eta and sigma meson

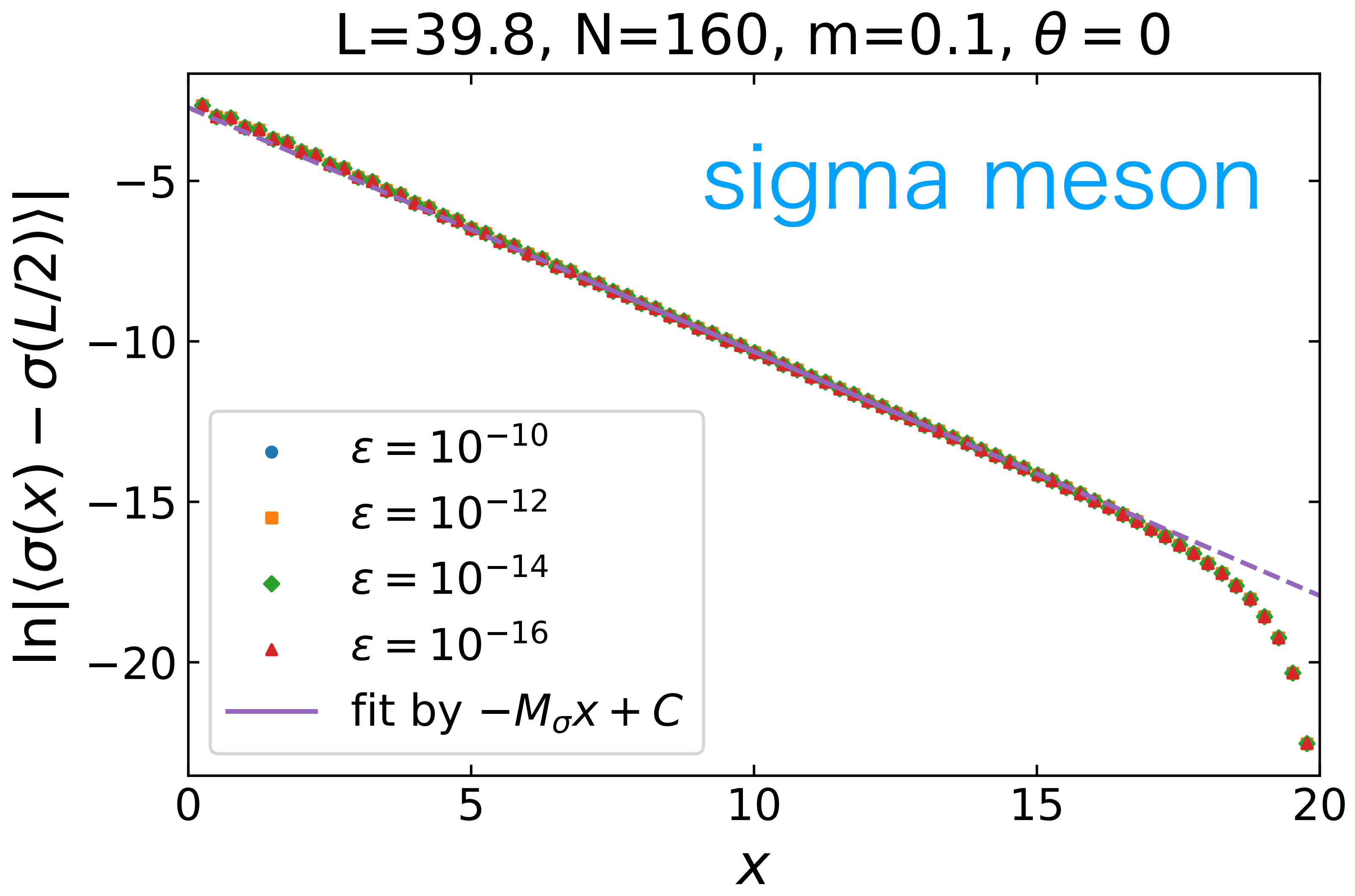
- At $\theta = 0$, the open boundary can be a source of iso-singlet states. (~wall source)
- one-point function: $\langle \mathcal{O}(x) \rangle \sim \exp(-Mx)$



- ε-dependence is NOT observed
  —> systematic error from truncating D_{eff} is sufficiently small

- eta: $M = 0.9014(1),\ C = -1.096(1)$
- sigma: $M = 0.761(2),\ C = -2.71(2)$



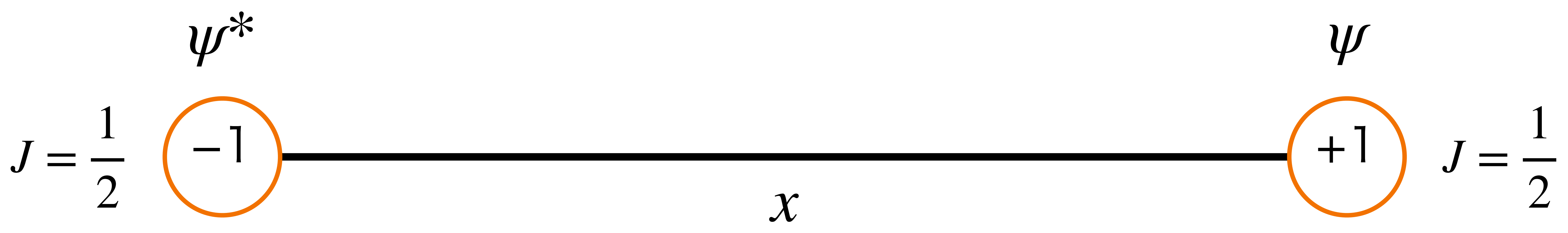


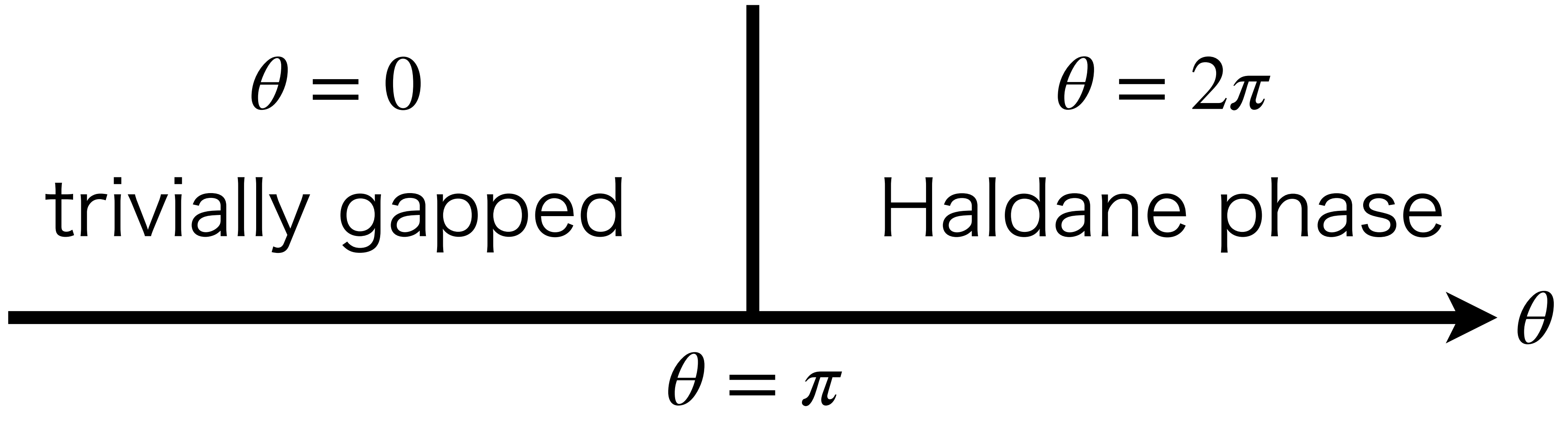
Pion: tricky case

⚠️ triplet cannot be measured $\langle \pi(x) \rangle = 0$ at $\theta = 0$

solution: introduce a background electric field by setting $\theta = 2\pi$

- Dirac fermions with charge ± 1 are induced as edge modes
- isospin 1/2 at the boundary → a source of iso-triplet mesons



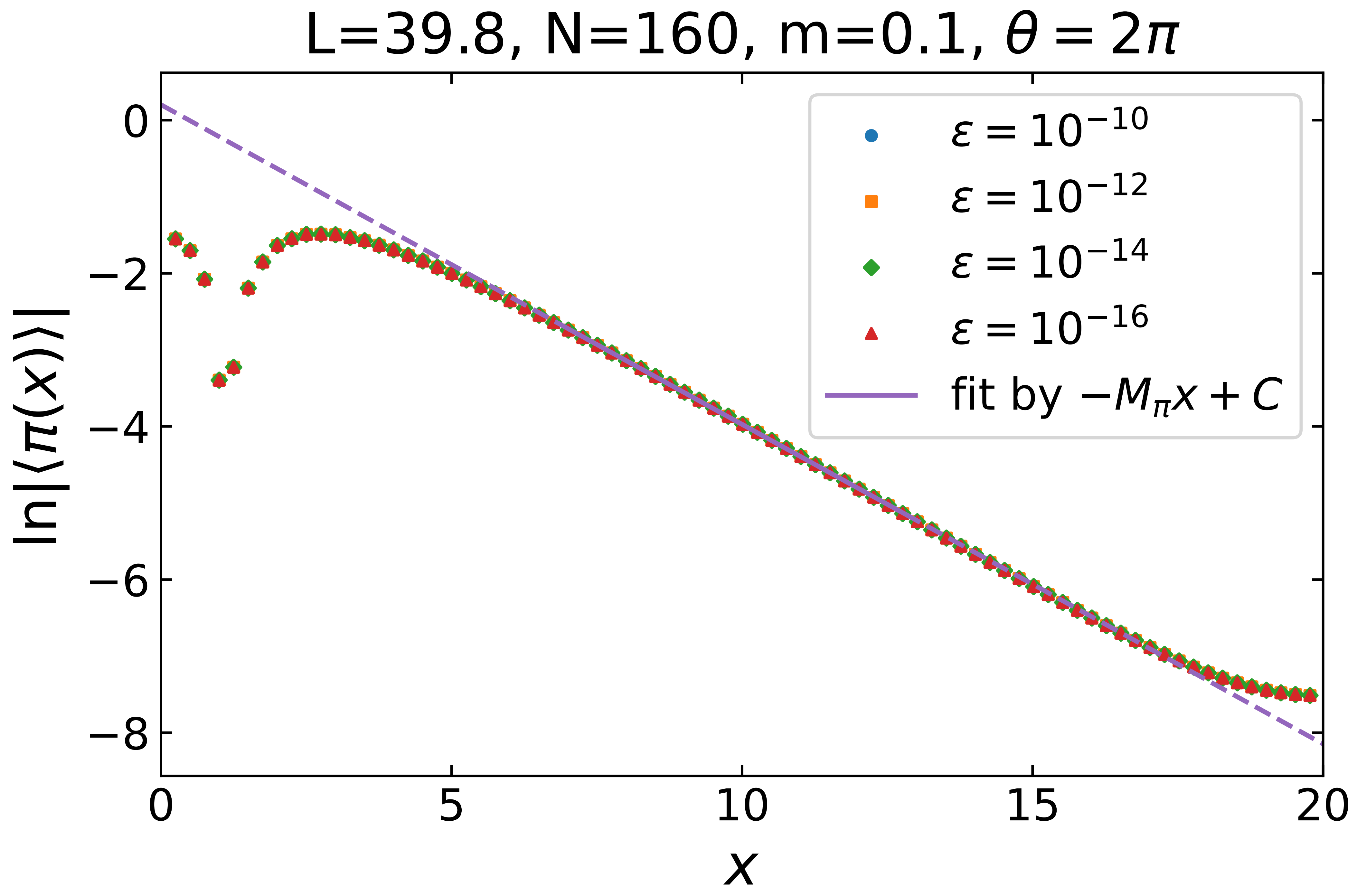


cf.) similar SPT phase
to anti-ferro. Heisenberg chain
[Chen et al. (2011)] [Kapustin (2014)]

Result of pion

- generate the ground state at $\theta = 2\pi$
- one-point function: $\langle \pi(x) \rangle \sim \exp(-Mx)$

- $M = 0.4175(9), \quad C = 0.203(9)$
- ε-dependence is NOT observed



	pion	sigma	eta
M	0.4175(9)	0.761(2)	0.9014(1)

Simulation result ($\theta = 0$)

(1) Correlation-function scheme

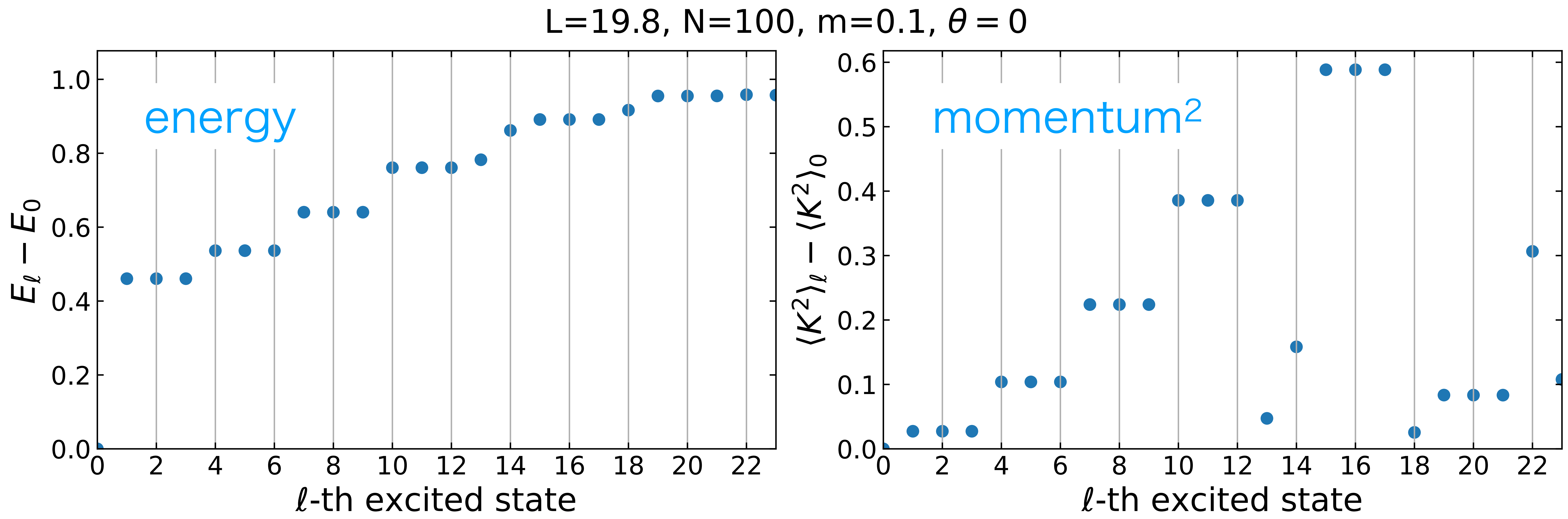
(2) One-point-function scheme

(3) Dispersion-relation scheme

Energy and momentum

- energy gap: $\Delta E_\ell = E_\ell - E_0$ momentum square: $\Delta K_\ell^2 = \langle K^2 \rangle_\ell - \langle K^2 \rangle_0$
- triplets —> pion? singlets —> sigma or eta meson?

identify the states by measuring quantum numbers: $\mathbf{J}^2,\ J_z,\ G = Ce^{i\pi J_y}$



Result of quantum numbers

- triplets: $\mathbf{J}^2 = 2, \ \ J_z = (0, \pm 1), \ \ G > 0$

  —> pion ($J^{PG} = 1^{-+}$)

- singlets: $\mathbf{J}^2 = 0, \ \ J_z = 0,$

  $G > 0$ (ℓ = 13,14,22) —> sigma meson ($J^{PG} = 0^{++}$)

  $G < 0$ (ℓ = 18,23) —> eta meson ($J^{PG} = 0^{--}$)

singlets

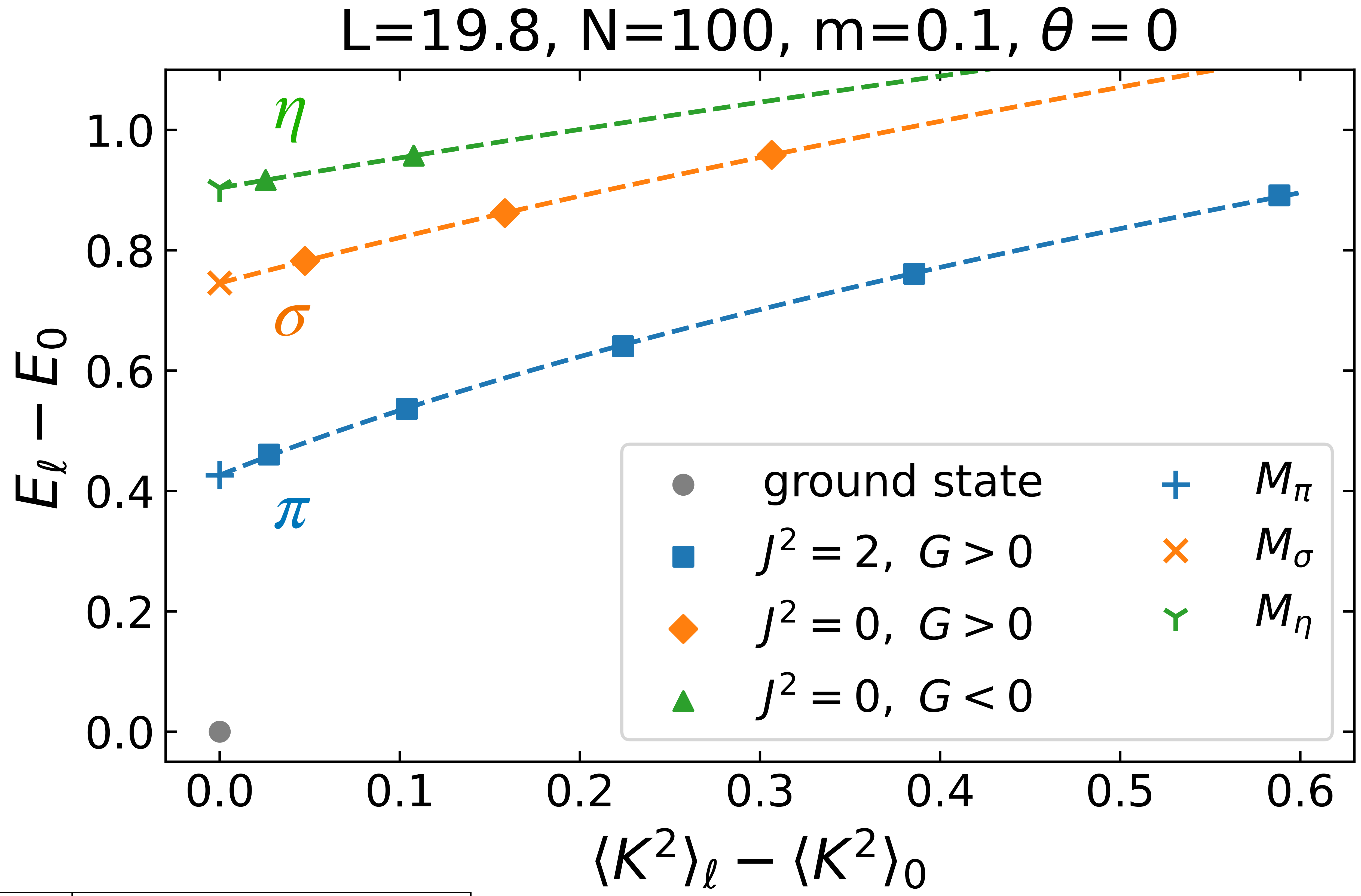
ℓ	$\boldsymbol{J}^2$	J_z	G
0	0.00000003	-0.00000000	0.27984227
13	0.00000003	0.00000000	0.27865844
14	0.00000003	0.00000000	0.27508176
18	0.00000028	0.00000006	-0.27390909
22	0.00001537	0.00000115	0.26678987
23	0.00003607	-0.00000482	-0.27664779

triplets

ℓ	$\boldsymbol{J}^2$	J_z	G
1	2.00000004	0.99999997	0.27872443
2	2.00000012	-0.00000000	0.27872416
3	2.00000004	-0.99999996	0.27872443
4	2.00000007	0.99999999	0.27736066
5	2.00000006	0.00000000	0.27736104
6	2.00000009	-0.99999998	0.27736066
7	2.00000010	1.00000000	0.27536687
8	2.00000002	0.00000000	0.27536702
9	2.00000007	-0.99999998	0.27536687
10	2.00000007	0.99999998	0.27356274
11	2.00000005	0.00000001	0.27356277
12	2.00000007	-0.99999999	0.27356274
15	1.99999942	0.99999966	0.27173470
16	2.00000052	0.00000000	0.27173482
17	2.00000015	-1.00000003	0.27173470
19	2.00009067	1.00004377	0.27717104
20	2.00002578	-0.00000004	0.27717020
21	2.00003465	-1.00001622	0.27717104

Result of dispersion relation

- plot ΔE_ℓ against ΔK_ℓ^2 for each meson
- fit the data points by $\Delta E = \sqrt{b^2 \Delta K^2 + M^2}$



	pion	sigma	eta
M	0.426(2)	0.7456(5)	0.9037
b	1.017(4)	1.087(2)	0.9622

Summary

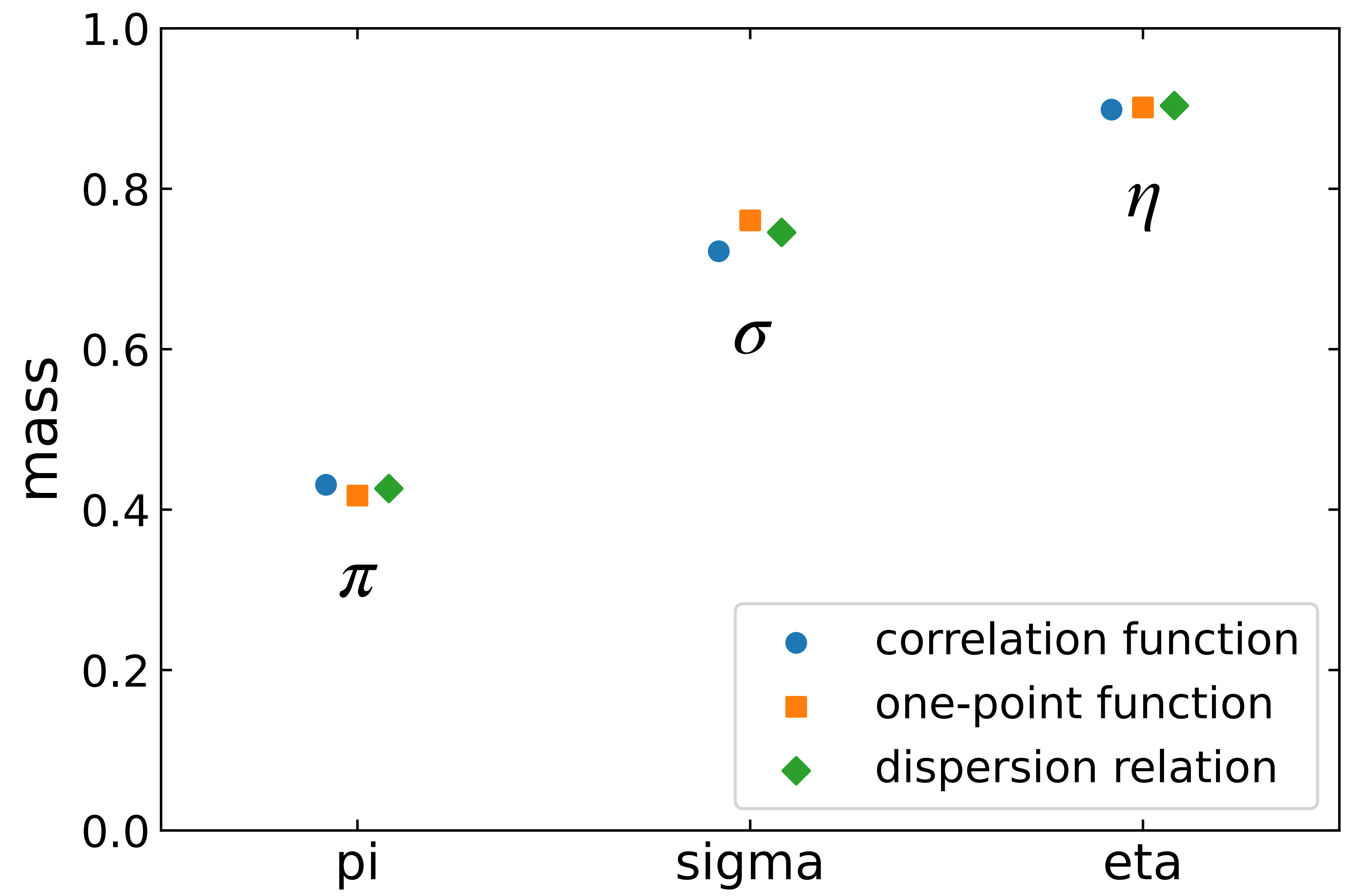
- The three results are consistent with each other and look promising.
- consistent with predictions by bosonization

✔ $M_\pi < M_\sigma < M_\eta$ —> U(1) problem

✔ $M_\eta \sim \mu \quad (\mu = g\sqrt{2/\pi} \sim 0.8)$

✔ $M_\sigma / M_\pi = \sqrt{3}$ within 5% deviation

[Coleman (1976)] [Dashen et al. (1975)]



	correlation func.	one-point func.	dispersion
M_σ/M_π	1.68	1.82	1.75

Discussion

(1)correlation-function scheme

👍 generic method applicable to any case

😥 sensitive to the bond dimension of MPS —> 😊 quantum computation?

(2)one-point-function scheme

👍 needs to increase NEITHER the bond dimension NOR the system size

😥 only the lowest state of the same quantum number as the boundary

(3)dispersion-relation scheme

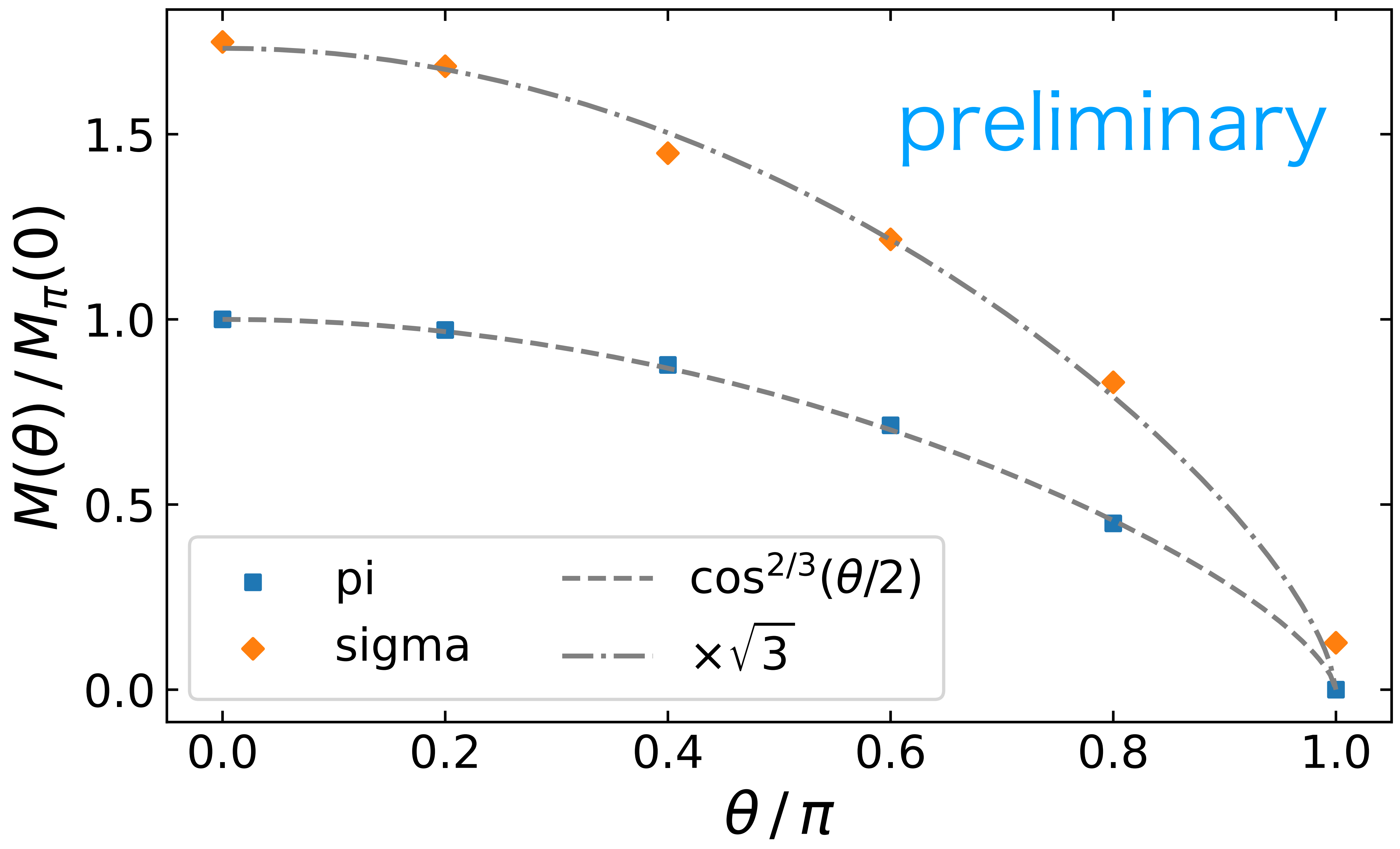
👍 obtain various states heuristically / directly see wave functions

😥 computational cost to generate many excited states

Application to $\theta \neq 0$

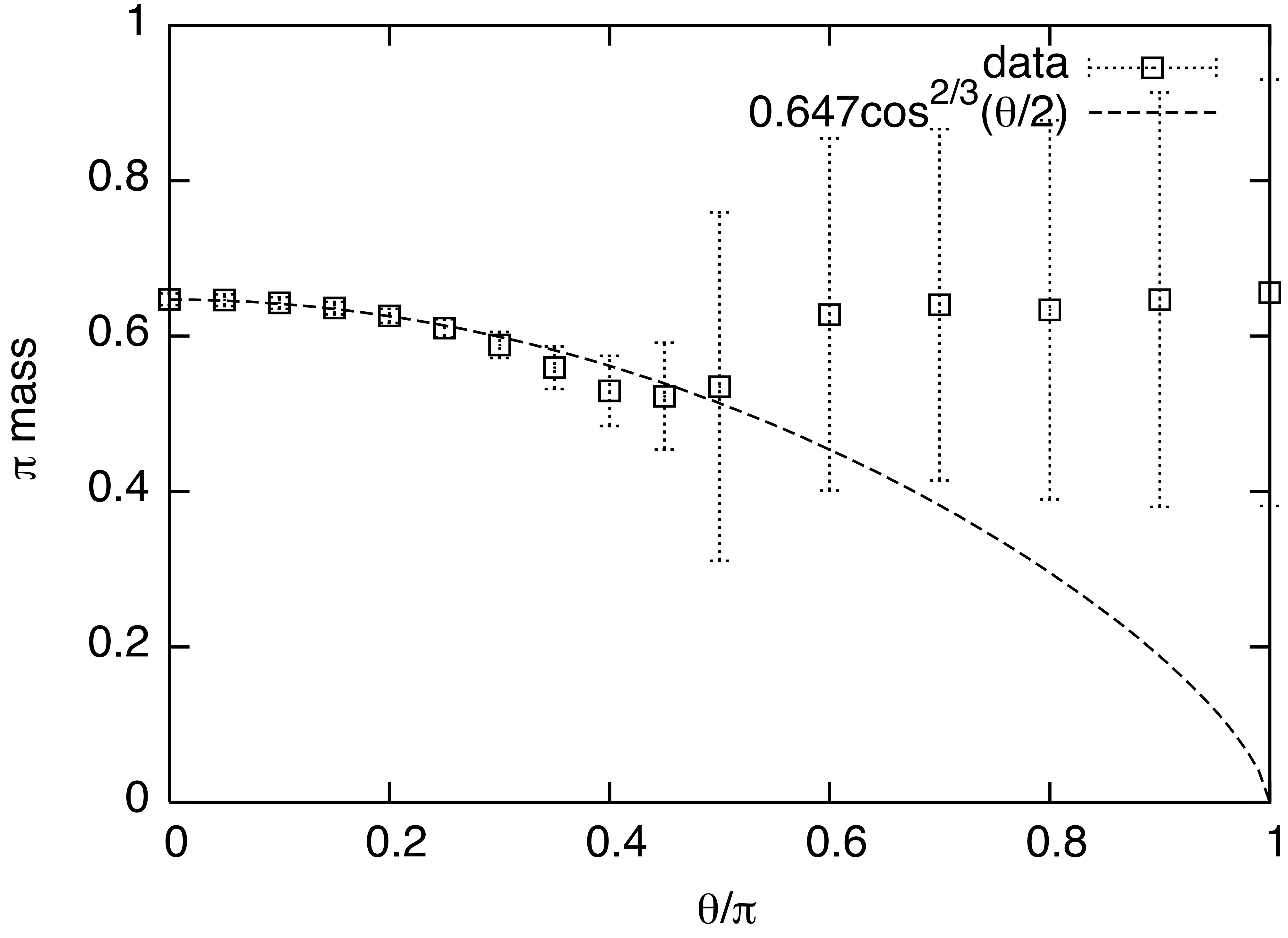
- We focus on $\theta = 0$ so far but application to $\theta \neq 0$ is straightforward!
- result of (3) dispersion-relation scheme

L=19.8, N=100, m=0.1



Monte Carlo result

[Fukaya & Onogi (2003)]



Thank you for listening.

Analytic study by bosonization

- massive 2-flavor Schwinger model is not exactly solvable
- Abelian bosonization can be used in the strong coupling region $g \gg m > 0$

$$H = N_m \left[\frac{1}{2}\Pi_+^2 + \frac{1}{2}(\partial_1 \phi_+)^2 + \frac{\mu^2}{2}\phi_+^2 + \frac{1}{2}\Pi_-^2 + \frac{1}{2}(\partial_1 \phi_-)^2 - 2cm^2 \cos\left(\sqrt{2\pi}\phi_+ - \frac{\theta}{2}\right) \cos\left(\sqrt{2\pi}\phi_-\right) \right]$$

[Coleman (1976)]

- $\phi_+ \to$ eta meson: $M_\eta = \mu + O(m)$ $\qquad \mu^2 = 2g^2/\pi$
- low energy spectrum $E < \mu$ is described by the sine-Gordon model;

  soliton/anti-soliton of ϕ_- and their bound states

  $\to$ pion: $M_\pi \propto |m\mu^{1/2}\cos(\theta/2)|^{2/3}$, $\quad$ sigma meson: $M_\sigma = \sqrt{3}M_\pi$ $\quad \to M_\pi < M_\sigma < M_\eta$

[Dashen et al. (1975)]

Hamiltonian of Schwinger model

- Hamiltonian of the Nf-flavor Schwinger model with a theta term

$$H = \int dx \left\{ \frac{g^2}{2}\left(\Pi - \frac{\theta}{2\pi}\right)^2 + \sum_{f=1}^{N_f}\left[-i\bar{\psi}_f\gamma^1\left(\partial_1 + iA_1\right)\psi_f + m\bar{\psi}_f\psi_f\right]\right\}$$

- lattice Hamiltonian with open boundary condition

$$H = \frac{g^2 a}{2}\sum_{n=0}^{N-2}\left(L_n + \frac{\theta}{2\pi}\right)^2 + \sum_{f=1}^{N_f}\left[\frac{-i}{2a}\sum_{n=0}^{N-2}\left(\chi_{f,n}^\dagger U_n \chi_{f,n+1} - \chi_{f,n+1}^\dagger U_n^\dagger \chi_{f,n}\right) + m_{\mathrm{lat}}\sum_{n=0}^{N-1}(-1)^n \chi_{f,n}^\dagger \chi_{f,n}\right]$$

- U_n : link variable, L_n : conjugate momentum, $\chi_{f,n}$: staggered fermion

[Kogut & Susskind (1975)]

- $m_{\mathrm{lat}} = m - N_f g^2 a/8$ ← taking $O(a)$ correction into account

[Dempsey et al. (2022)]

Gauge field can be removed

- Gauss law condition:

$$\partial_1 \Pi + \sum_{f=1}^{N_f} \psi_f^\dagger \psi_f = 0 \quad \to \quad L_n - L_{n-1} = \sum_{f=1}^{N_f} \left[\chi_{f,n}^\dagger \chi_{f,n} + \frac{(-1)^n - 1}{2} \right] \quad (L_n \sim \text{electric field})$$

Gauss law can be solved under the open boundary condition

$$L_n = \sum_{f=1}^{N_f} \sum_{k=0}^{n} \chi_{f,k}^\dagger \chi_{f,k} + \frac{N_f}{2} \left(\frac{(-1)^n - 1}{2} - n \right) \quad \text{setting} \; L_{-1} = 0$$

- Link variables are absorbed by gauge fixing $U_n = 1$

$\to$ H is written only by fermionic operators (no propagating d.o.f of gauge field)

Map to the spin system

- Jordan-Wigner transformation:
  fermion operator → spin operator

$$\chi_{1,n} = \sigma^-_{1,n} \prod_{j=0}^{n-1} (-\sigma^z_{2,j}\sigma^z_{1,j}) \qquad \chi_{2,n} = \sigma^-_{2,n}(-i\sigma^z_{1,n}) \prod_{j=0}^{n-1} (-\sigma^z_{2,j}\sigma^z_{1,j})$$

$$\sigma^{\pm}_{f,n} = \frac{1}{2}(\sigma^x_{f,n} \pm i\sigma^y_{f,n}) \qquad \left[\sigma^a_{f,n}, \sigma^b_{f',n'}\right] = 2i\,\delta_{ff'}\,\delta_{nn'}\,\epsilon^{abc}\,\sigma^c_{f,n}$$

→ spin Hamiltonian with a finite-dimensional Hilbert space

- convenient to apply the tensor network methods or quantum computation

- In this work, we employ the tensor network method.

Explicit form of the Hamiltonian

- spin Hamiltonian: $H = H_{\mathrm{gauge}} + H_{\mathrm{kin}} + H_{\mathrm{mass}}$

$$H_{\mathrm{gauge}} = \frac{g^2 a}{8} \sum_{n=0}^{N-2} \left[\sum_{f=1}^{N_f} \sum_{k=0}^{n} \sigma^z_{f,k} + N_f \frac{(-1)^n + 1}{2} + \frac{\theta}{\pi} \right]^2$$

$$H_{\mathrm{kin}} = \frac{-i}{2a} \sum_{n=0}^{N-2} \left(\sigma^+_{1,n} \sigma^z_{2,n} \sigma^-_{1,n+1} - \sigma^-_{1,n} \sigma^z_{2,n} \sigma^+_{1,n+1} + \sigma^+_{2,n} \sigma^z_{1,n+1} \sigma^-_{2,n+1} - \sigma^-_{2,n} \sigma^z_{1,n+1} \sigma^+_{2,n+1} \right)$$

$$H_{\mathrm{mass}} = \frac{m_{\mathrm{lat}}}{2} \sum_{f=1}^{N_f} \sum_{n=0}^{N-1} (-1)^n \sigma^z_{f,n} + \frac{m_{\mathrm{lat}}}{2} N_f \frac{1 - (-1)^N}{2}$$

Singular value decomposition (SVD)

- a kind of matrix decomposition

  $M = U\Lambda V$

  M : $m \times n$ complex matrix

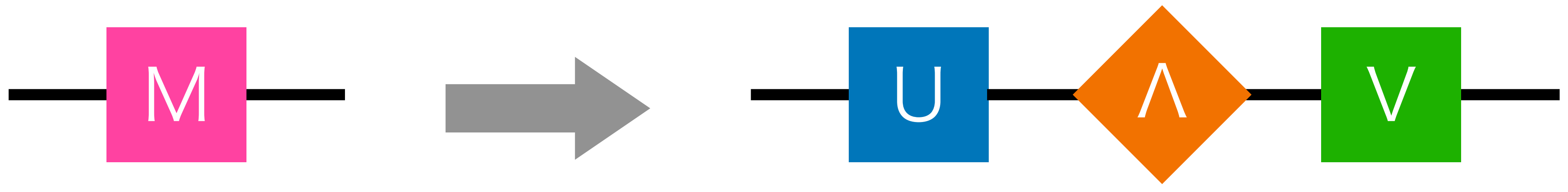
  U : $m \times m$ unitary matrix

  V : $n \times n$ unitary matrix

Λ : $m \times n$ "diagonal" matrix

ex) $\Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & \ddots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \lambda_m & 0 & \cdots \end{pmatrix}$ $(m < n)$

- singular values: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\min(m,n)} \geq 0$

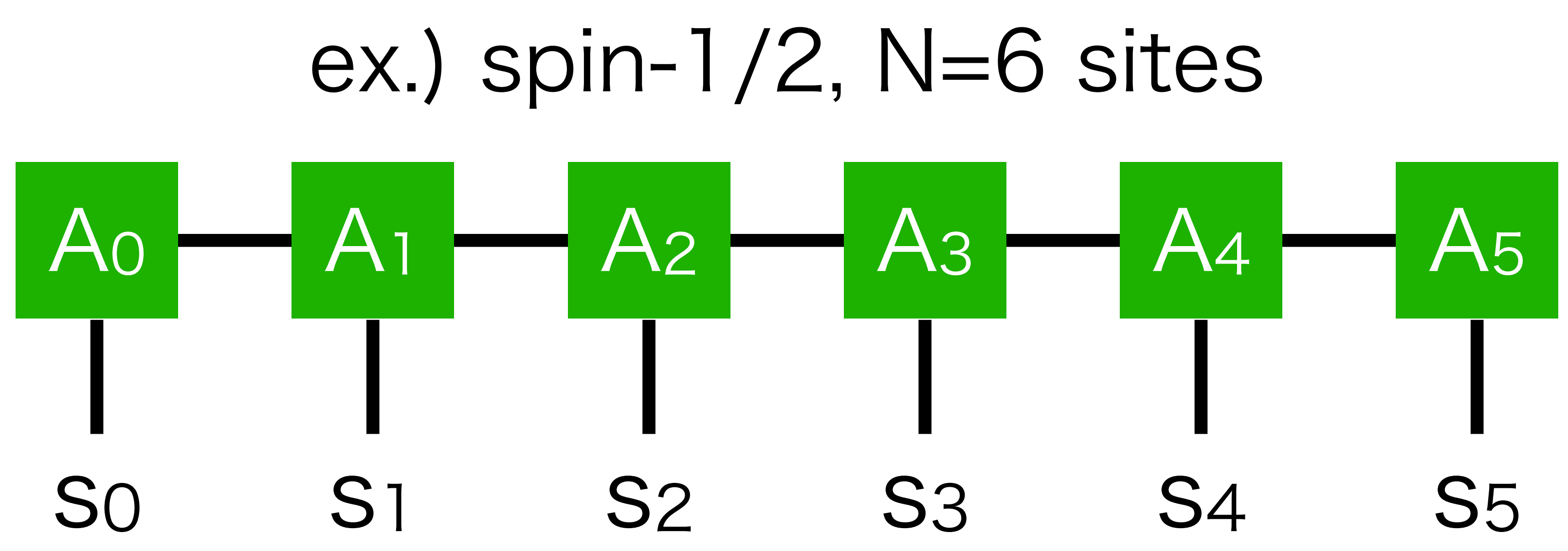


- low-rank approximation by keeping only the largest D singular values

  $$M = U\Lambda V \simeq U\,\mathrm{diag}(\lambda_1, \cdots, \lambda_D, 0, \cdots, 0)\, V =: M_{\mathrm{SVD}}$$

  $\rightarrow$ rank-D matrix which minimizes $\|M - M_{\mathrm{SVD}}\|^2 = \sum_{i=D+1}^{\min(m,n)} \lambda_i^2$

Approximation by MPS

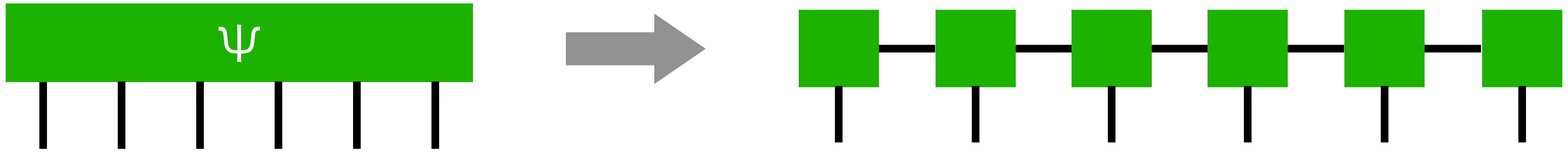


matrix product state (MPS)

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0)\,A_1(s_1)\ \cdots\right] |\,s_0\,s_1\ \cdots\rangle$$

- $A_i(s_i)$: $D_{i-1} \times D_i$ matrix with a spin index $s_i \in \{\uparrow, \downarrow\}$ (D_i : bond dimension)
- Any state can be written as MPS by repeating SVD, but $D_i = O(2^{N/2})$ in general.

$$|\Psi\rangle = \sum_{\{s_i\}} \Psi(s_0, s_1, \cdots)\,|\,s_0\,s_1\ \cdots\rangle$$



- MPS with a cutoff $D_i \leq \mathrm{const}$ approximates the low-energy state efficiently for 1+1d gapped systems of any size N. → numerical cost = $O(ND^3)$

Density-matrix renormalization group (DMRG)

[White (1992)] [Schollwock (2005)]

variational method to find eigenstates of H using MPS ansatz

- cost function: energy $E = \langle \Psi | H | \Psi \rangle$

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}\left[A_0(s_0) A_1(s_1) \cdots\right] | s_0 s_1 \cdots \rangle$$

- update $A_i(s_i)$ to decrease E

  by local optimization and SVD (sweep)

$A_i(s_i)$: $D_{i-1} \times D_i$ matrix

D_i : bond dimension

- We introduce a cutoff ε to control the accuracy.

  D_i is determined so that the truncation error is $\|M - M_{\mathrm{SVD}}\|^2 < \varepsilon$ in SVD.

  small ε <—> large D_i <—> high accuracy <—> high cost

DMRG for excited states

- ℓ-th excited state $|\Psi_\ell\rangle$: the lowest energy eigenstate under the orthogonality condition $\langle\Psi_{\ell'}|\Psi_\ell\rangle = 0$ for $\ell' = 0,1,\cdots,\ell-1$

- obtained by DMRG for the Hamiltonian with the projection term

$$H_\ell = H + W\sum_{\ell'=0}^{\ell-1}|\Psi_{\ell'}\rangle\langle\Psi_{\ell'}| \qquad W > 0$$

<—> cost function: $\langle\Psi_\ell|H|\Psi_\ell\rangle + W\sum_{\ell'=0}^{\ell-1}\left|\langle\Psi_{\ell'}|\Psi_\ell\rangle\right|^2$

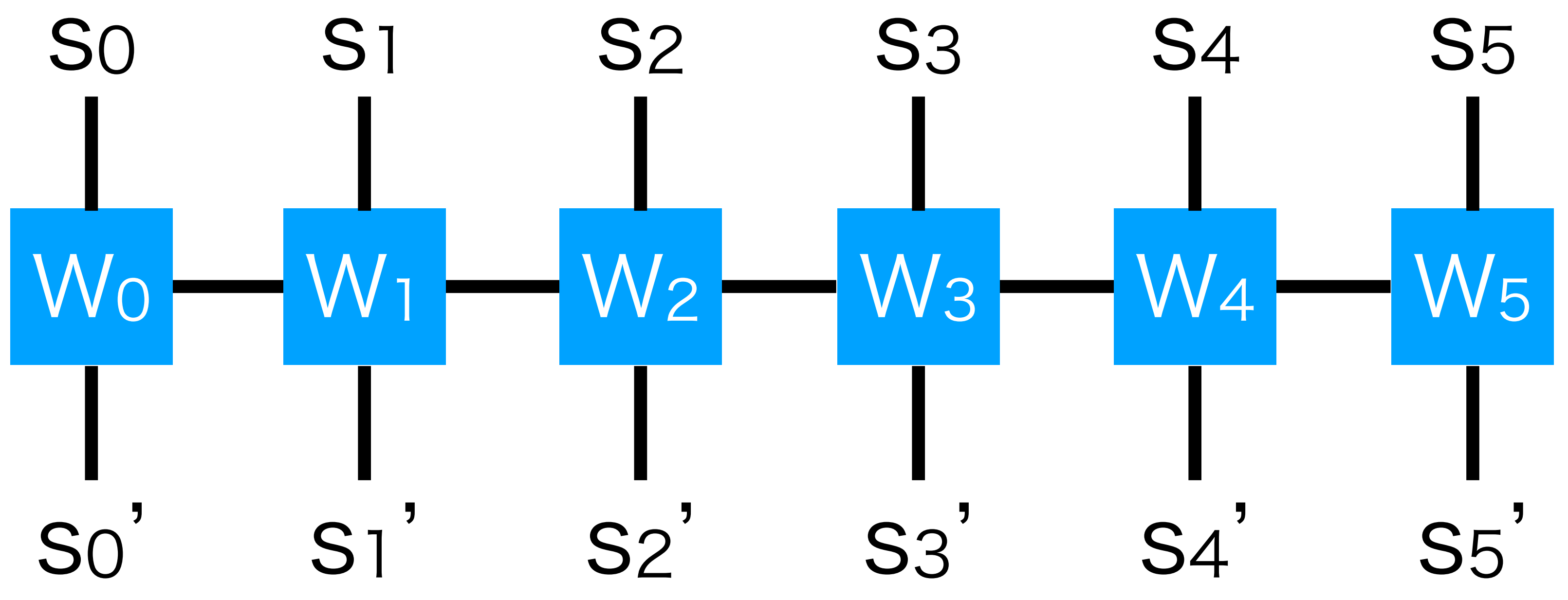
- We can generate the excited state step by step from the bottom.

The C++ library of ITensor is used in this work. [Fishman et al. (2022)]

Calculation of expectation values

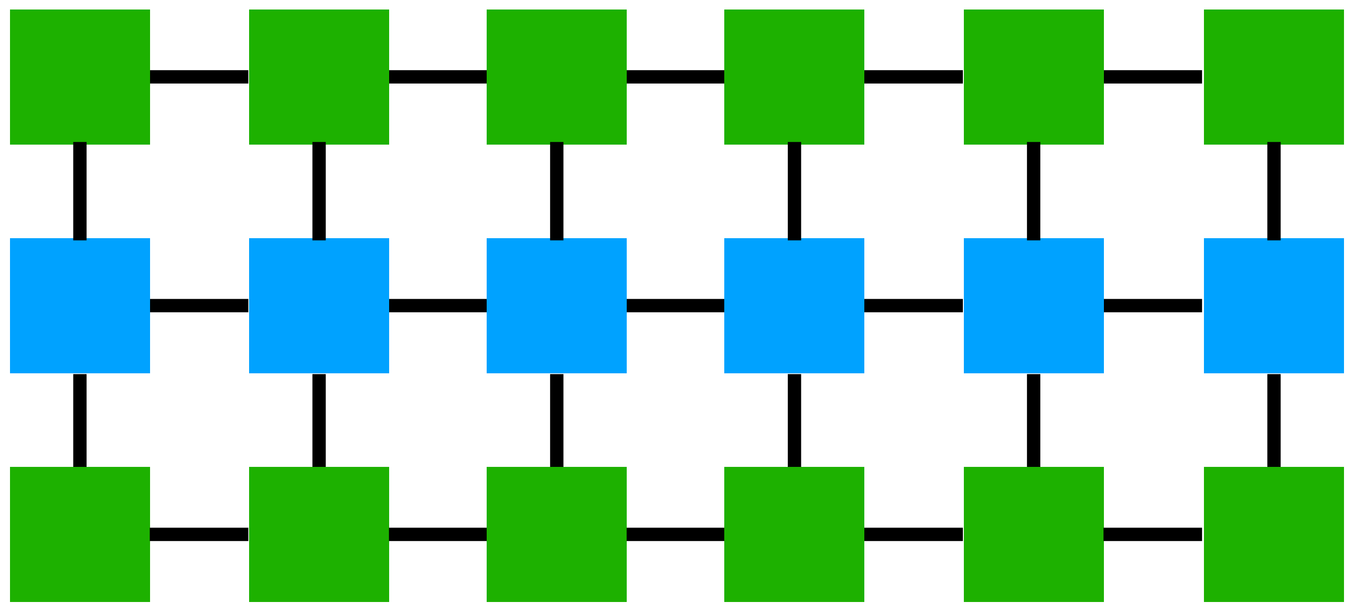
Matrix Product Operator (MPO)

$$\mathcal{O} = \sum_{\{s_i'\}} \sum_{\{s_i\}} \mathrm{Tr}\left[W_0(s_0', s_0) W_1(s_1', s_1) \cdots\right] |s_0' s_1' \cdots\rangle\langle s_0 s_1 \cdots|$$

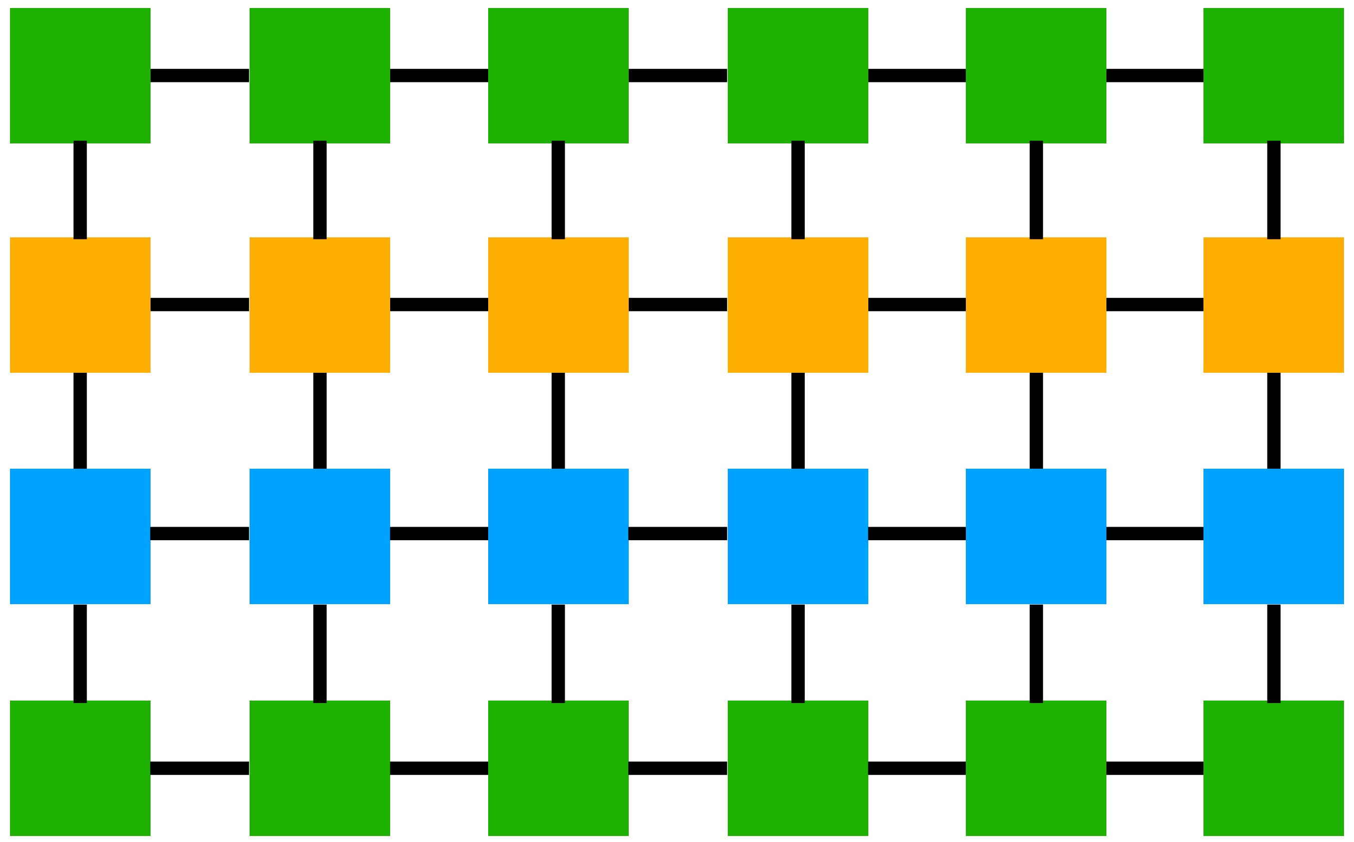


- Expectation values are computed by contracting MPS and MPO

$\langle \Psi | \mathcal{O} | \Psi \rangle$



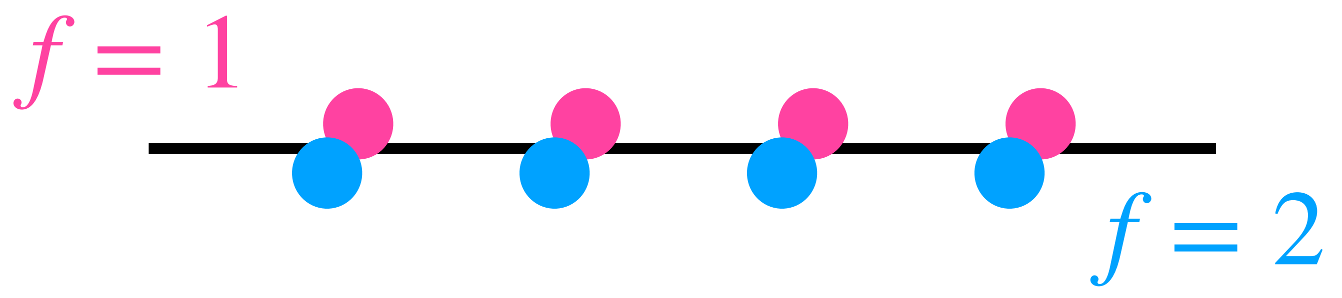
$\langle \Psi | \mathcal{O}_1 \mathcal{O}_2 | \Psi \rangle$



Arrangement of flavors

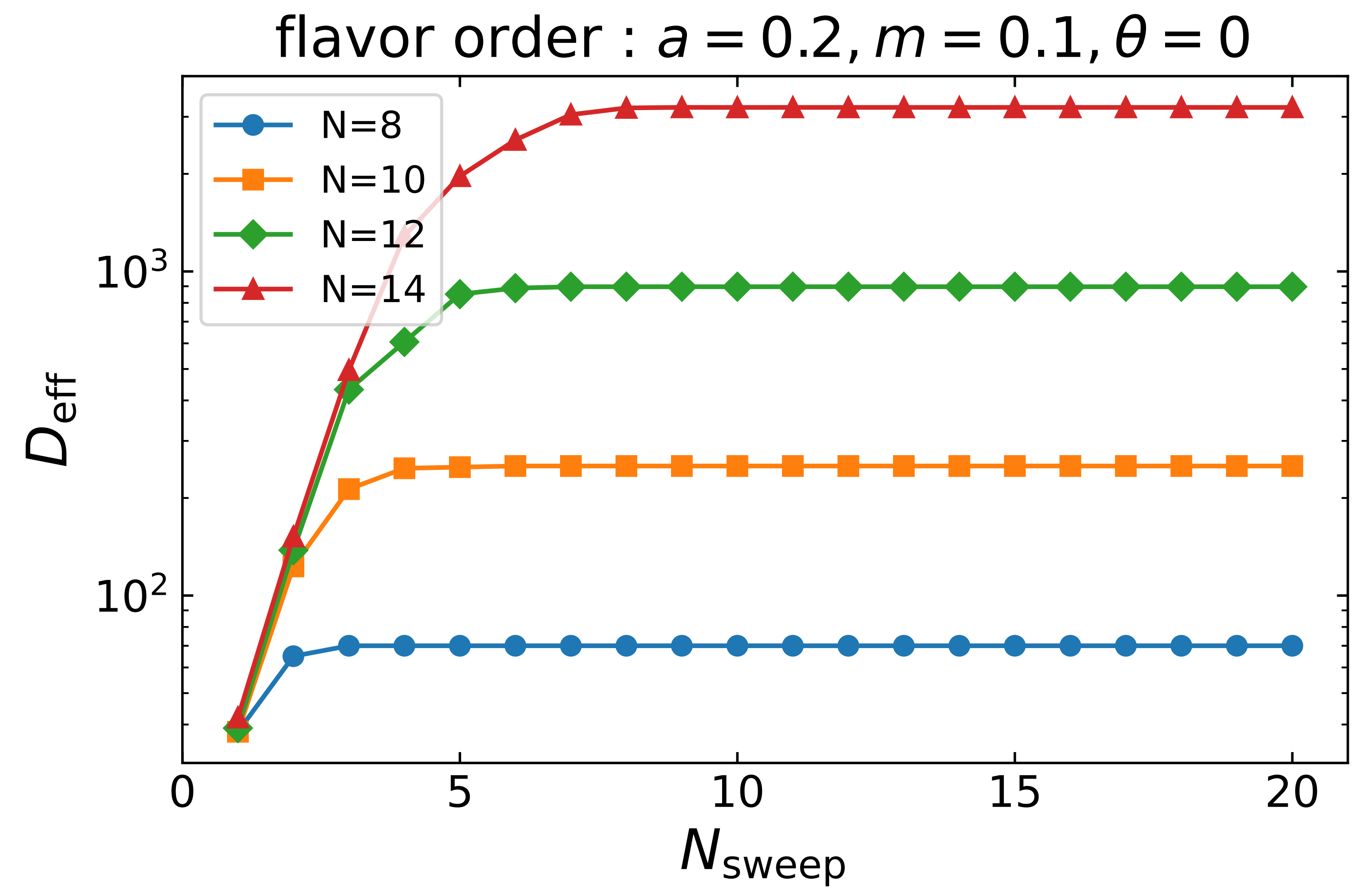
- We need to arrange $N_f \times N$ spins on the 1-dimensional lattice to use MPS.

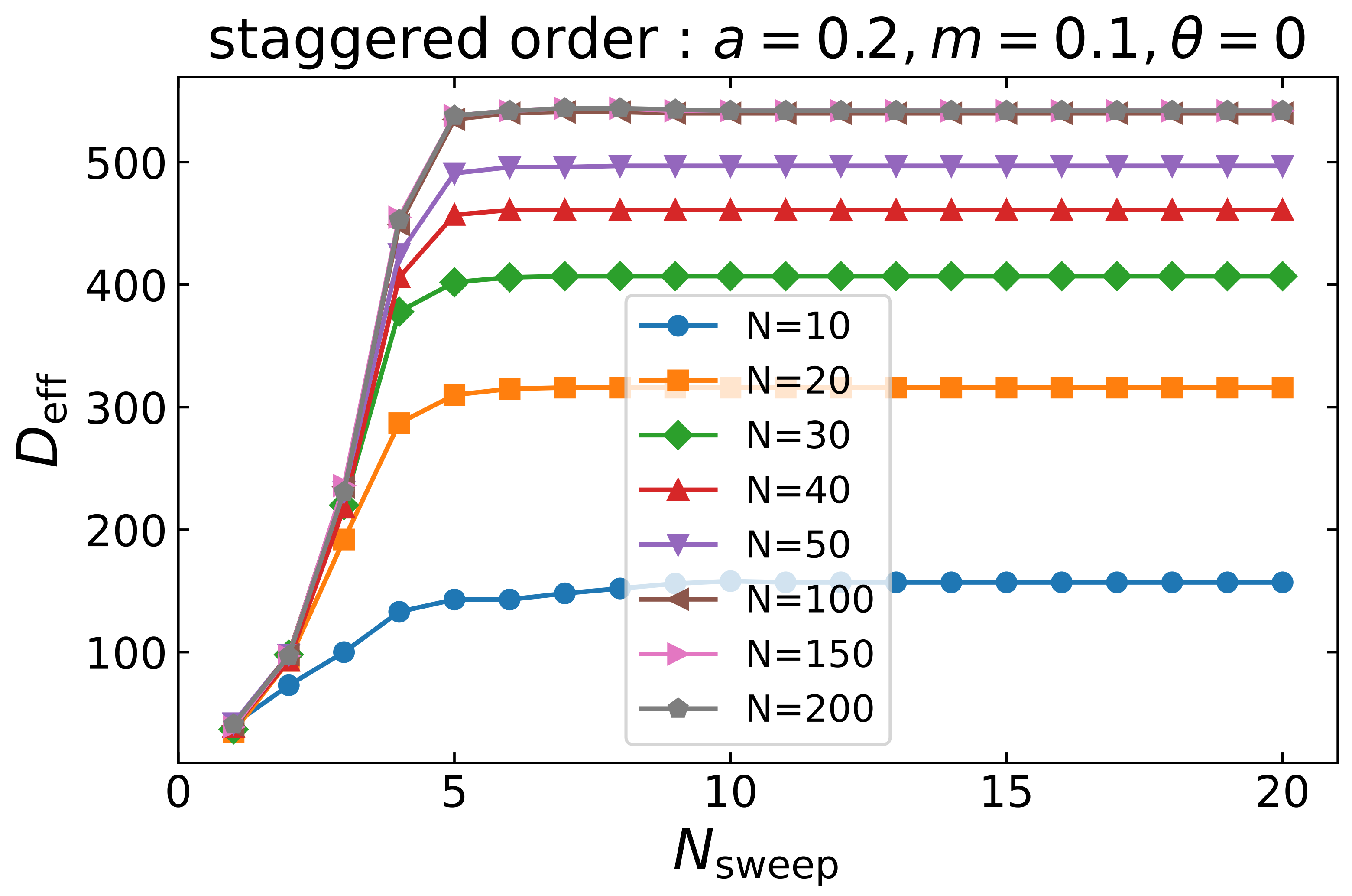
ex.) $N = 4,\ N_f = 2$

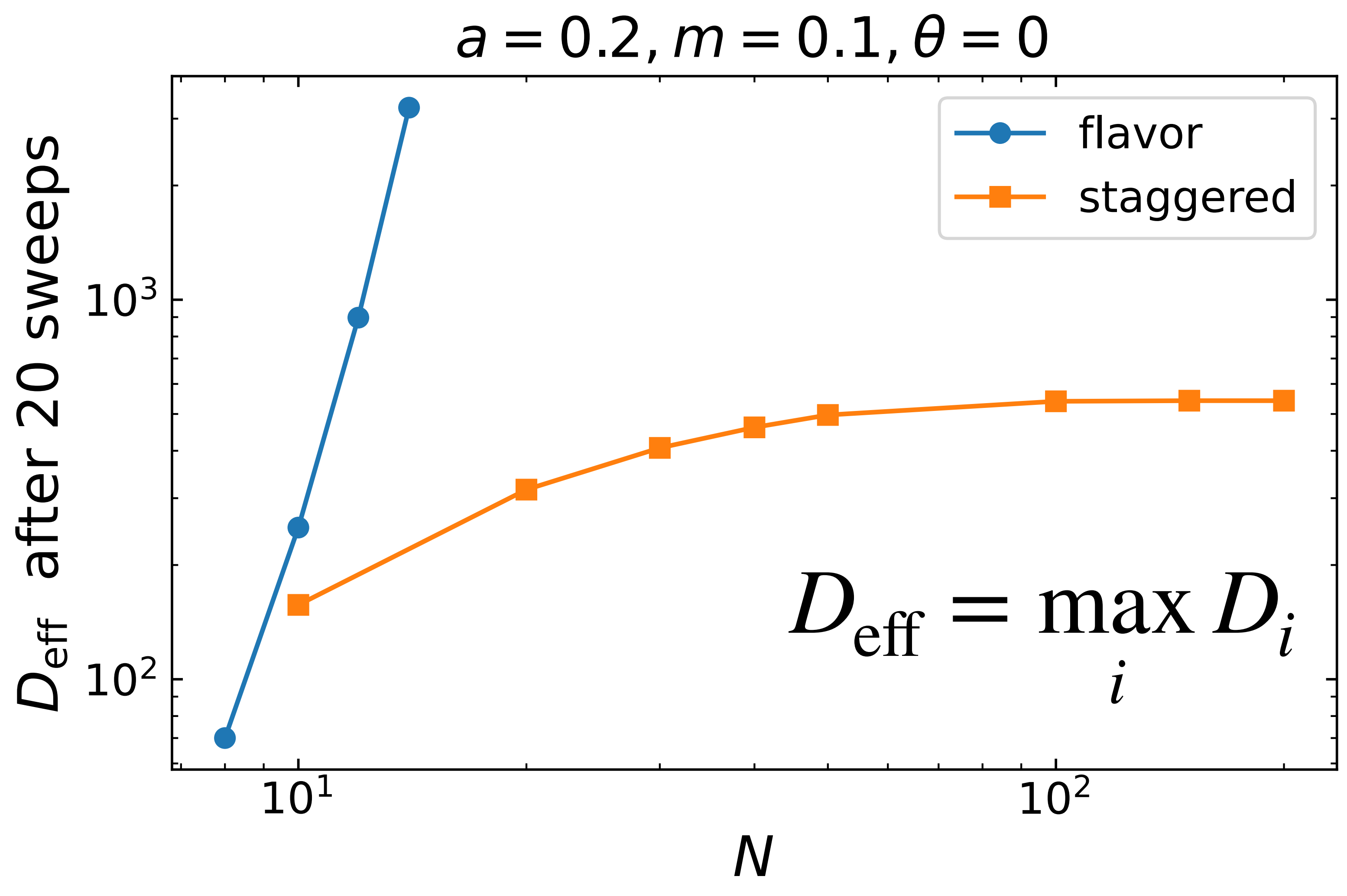


😥 flavor order: $(n, f) \to i = n + N(f - 1)$

👍 staggered order: $(n, f) \to i = nN_f + f - 1$





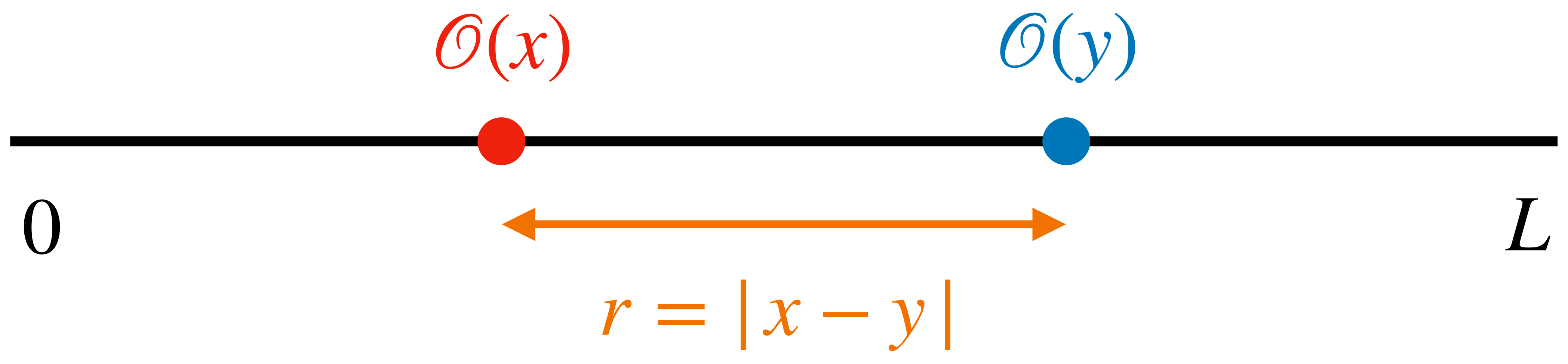


(1) correlation-function scheme

The meson mass should be extracted from the spatial correlation function as we do in Lattice QCD.

- 2-point spatial connected correlation function:

$$C_{\mathcal{O}}(r) = \langle \mathcal{O}(x)\mathcal{O}(y) \rangle - \langle \mathcal{O}(x) \rangle \langle \mathcal{O}(y) \rangle \qquad x = \frac{L-r}{2} \qquad y = \frac{L+r}{2}$$



- generate the ground state using DMRG and measure $C_{\mathcal{O}}(r)$ by changing r

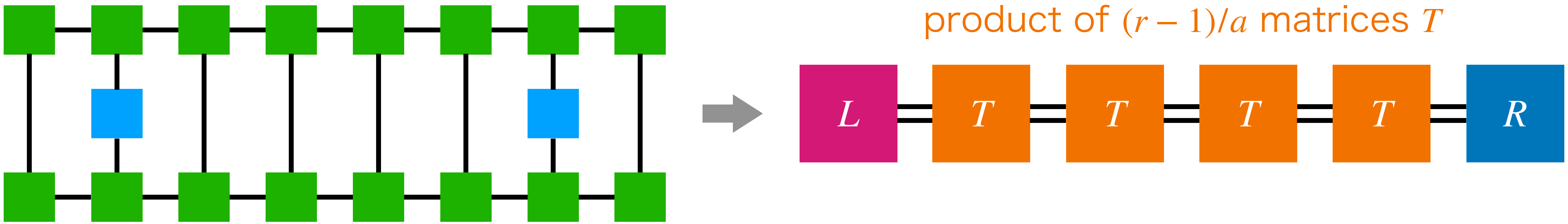
Correlation function from MPS

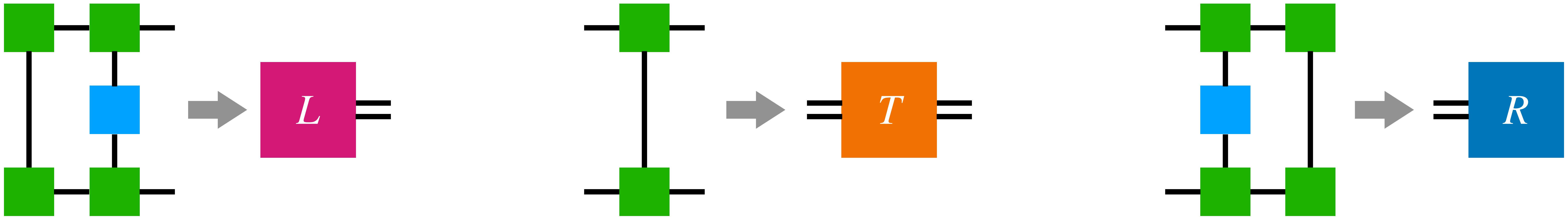
Assuming the translational invariance of MPS;

correlation functions $C(r)$ ∼ linear sum of exponential functions of $r = |x - y|$

$$C(r) = \langle \mathcal{O}(x)\mathcal{O}(y)\rangle \sim L\,T^{(r-1)/a}\,R \sim \sum_{i=1}^{D^2} C_i\,\xi_i^{r/a}$$

ξ_i : eigenvalue of “transfer matrix” T





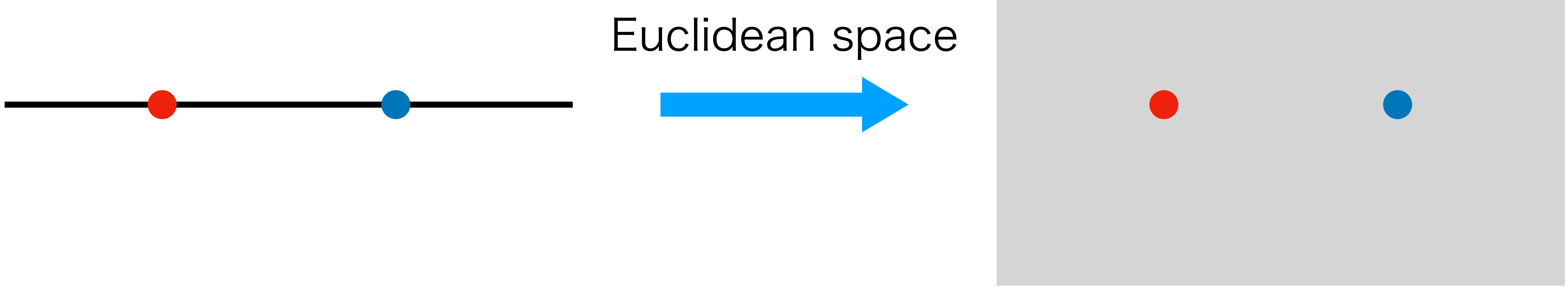
(2) one-point-function scheme

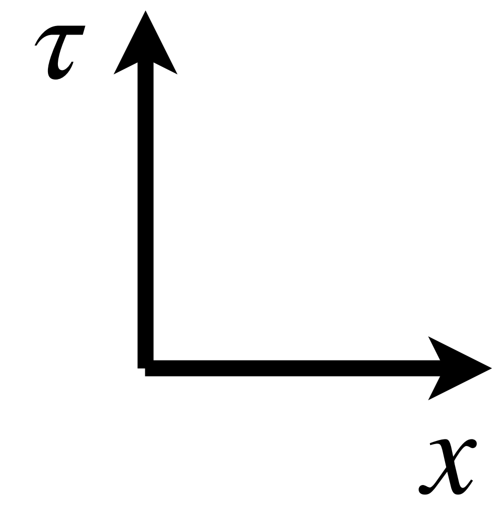
The open boundary (defect) can be a source of mesons,
which makes the one-point function nonzero.

- cf.) zero momentum projection, wall source in lattice QCD
- one-point function near the boundary $\sim$ correlation with the boundary state
- decays exponentially $\langle \mathcal{O}(x) \rangle_{\rm obc} \sim \langle {\rm bdry} | \mathcal{O}(x) | 0 \rangle_{\rm bulk} \sim \exp(-Mx)$

Zero-momentum projection

- correlation function

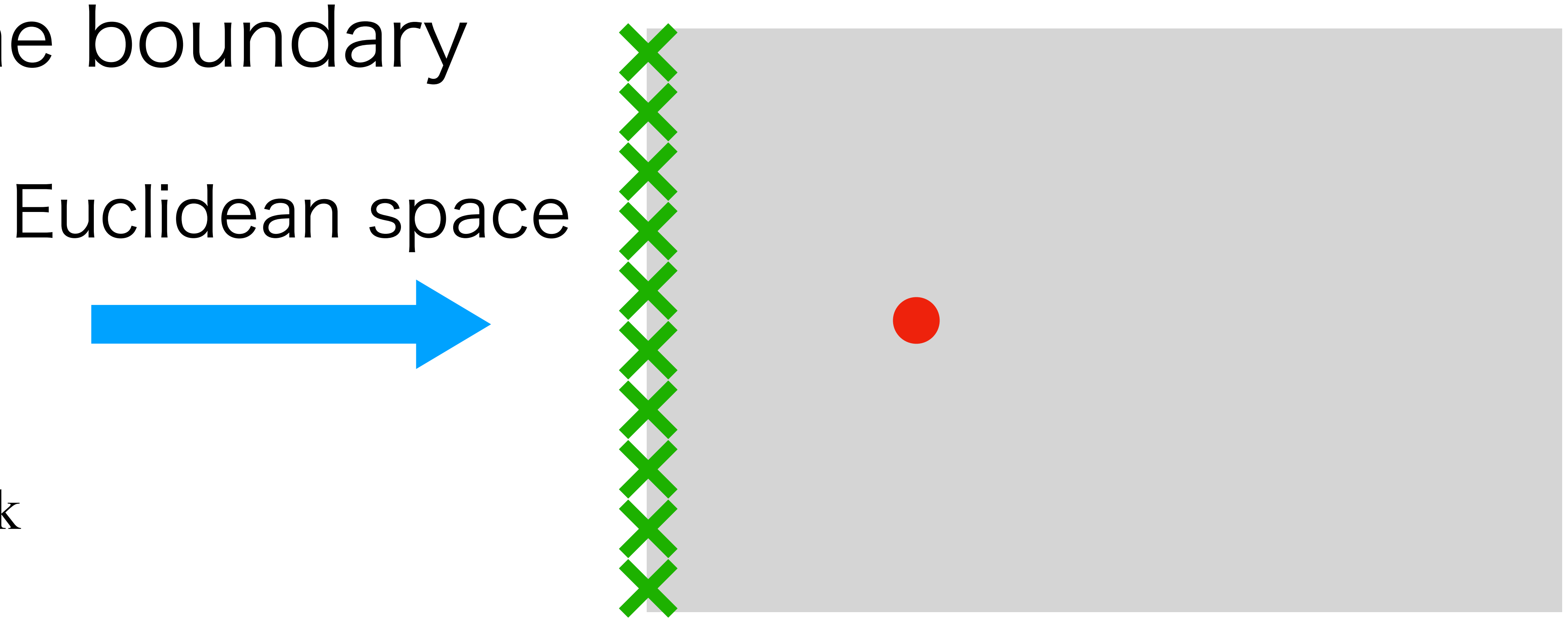




$$\langle 0 | \mathcal{O}(x,0) \mathcal{O}(y,0) | 0 \rangle$$

- 1pt function around the boundary

$$\langle \mathcal{O}(x) \rangle_{\rm obc} \sim \langle {\rm bdry} | \mathcal{O}(x,0) | 0 \rangle_{\rm bulk}$$

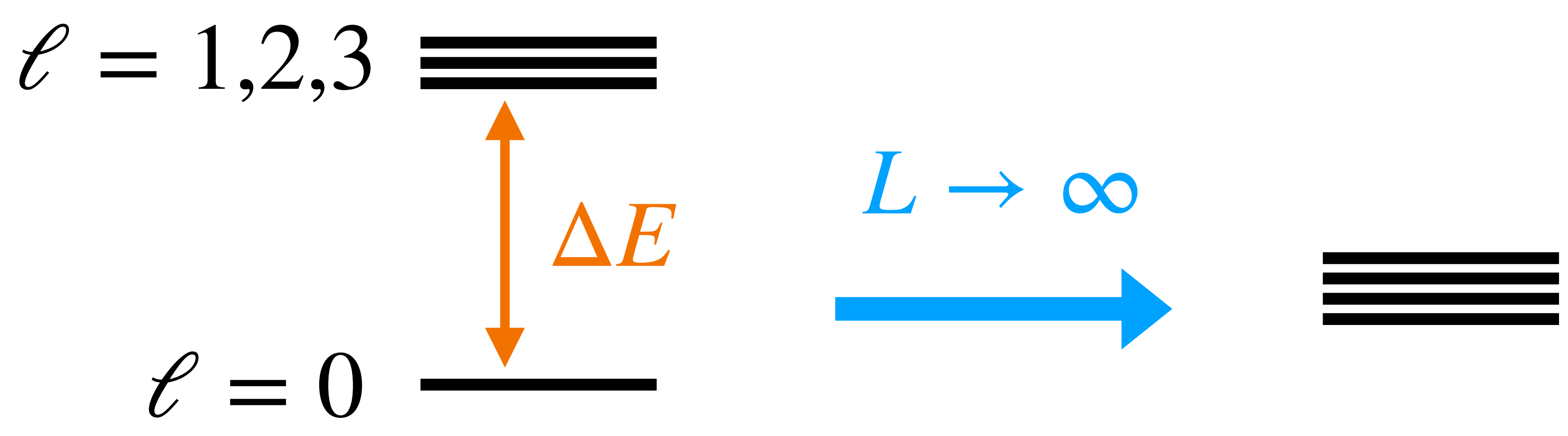


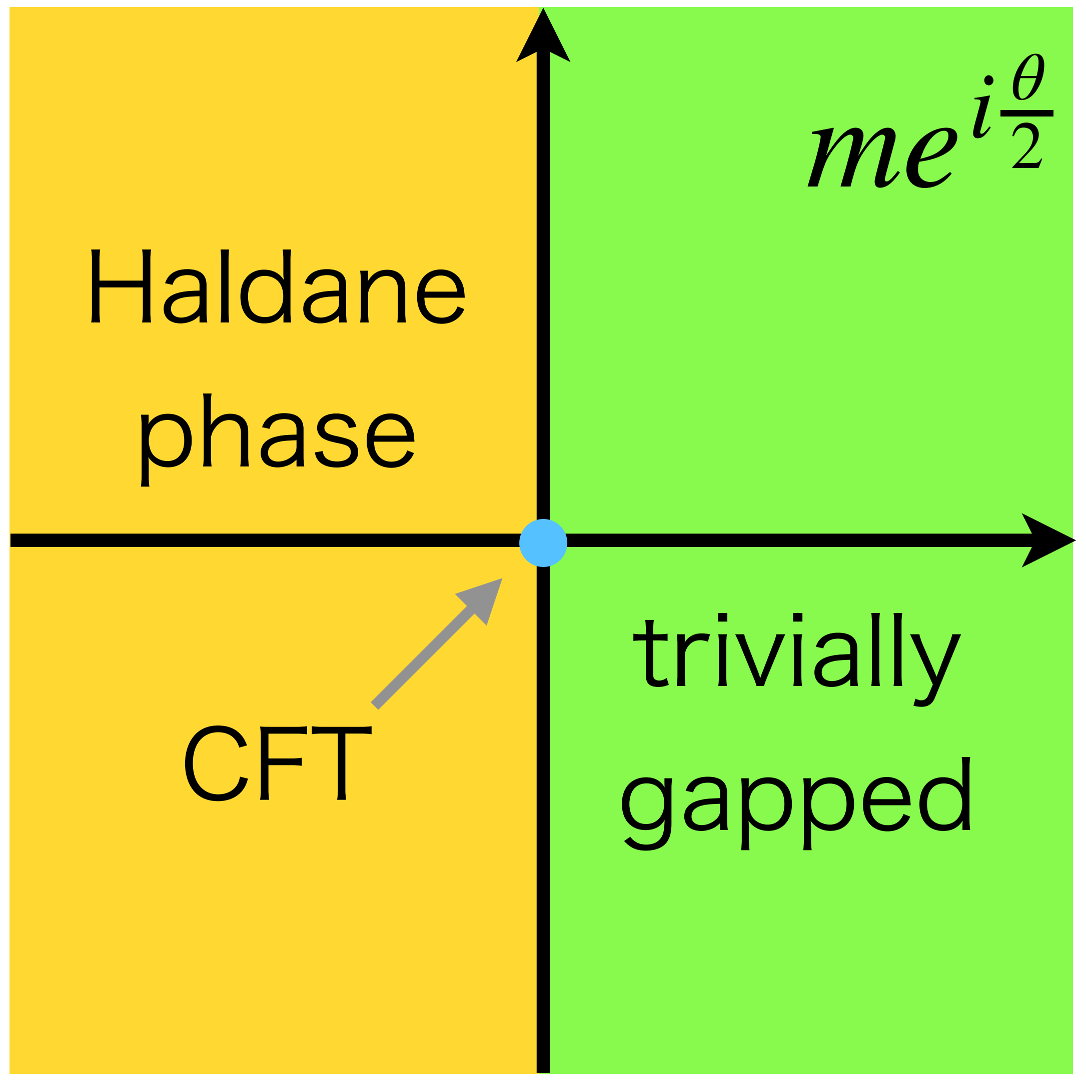
$$\int dr\, e^{ipr} = \delta(p)$$

$$\int d\tau \, \langle 0 | \mathcal{O}(0,\tau) \mathcal{O}(x,0) | 0 \rangle$$

SPT phase of Nf=2 Schwinger model

- The ground state at $\theta = 2\pi$ is a nontrivial SPT state protected by the isospin symmetry.
- The Dirac fermions (J = 1/2) are excited as edge modes at the boundaries.
- **finite L**: interaction between edge modes via pion
  → 1 ground state + 3 degenerated excited states
- **infinite L**: edge modes are decoupled
  → 4 degenerated ground states



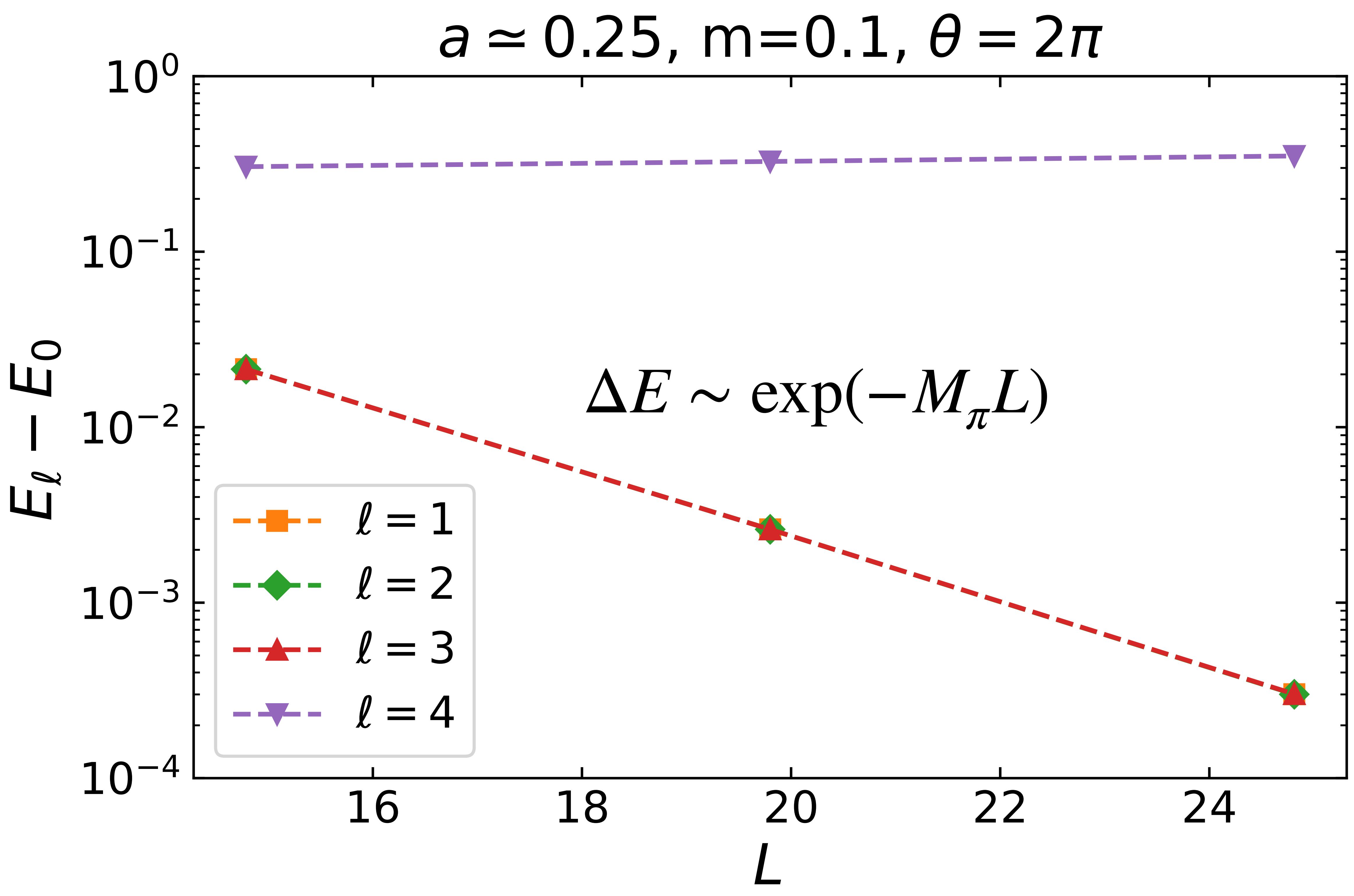


Degeneracy of the ground states

- one ground state + three 1st excited states are observed by DMRG at $\theta = 2\pi$.
- energy gap $\sim \exp(-M_\pi L) \to 0$
- solve $\Delta E_\ell = C_0 + \exp(-ML + C_1)$ for $\ell = 1$;

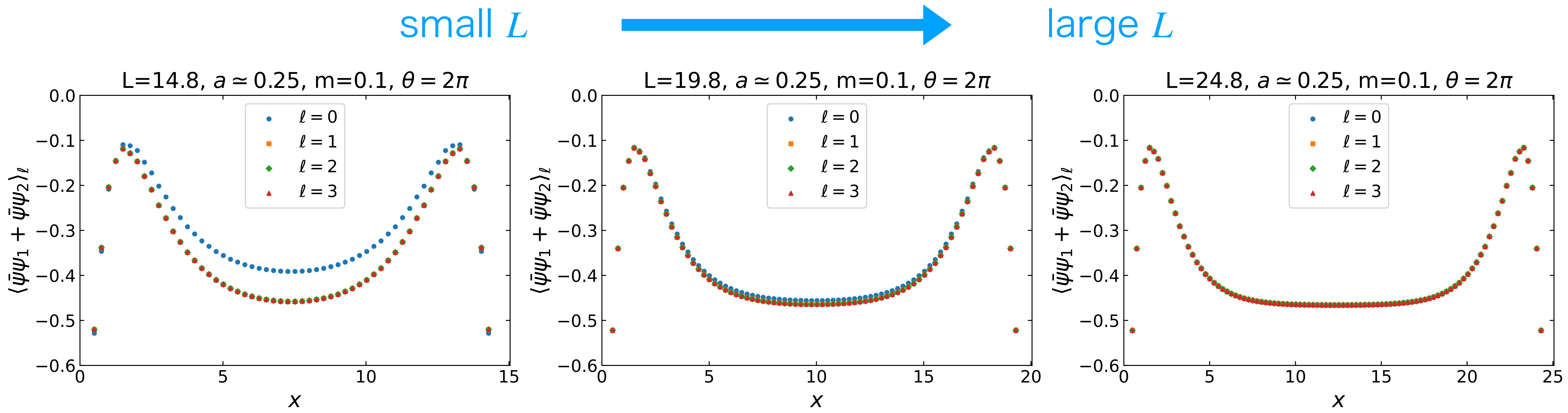
  $M = 0.41767,\ C_0 = -0.00002,\ C_1 = 2.33326$
- cf.) $M_\pi = 0.4175(9)$ by 1pt-fn. scheme
- DMRG is hard when L is small or $\theta \to \pi+$

energy gap of the ℓ-th excited state



Local observables

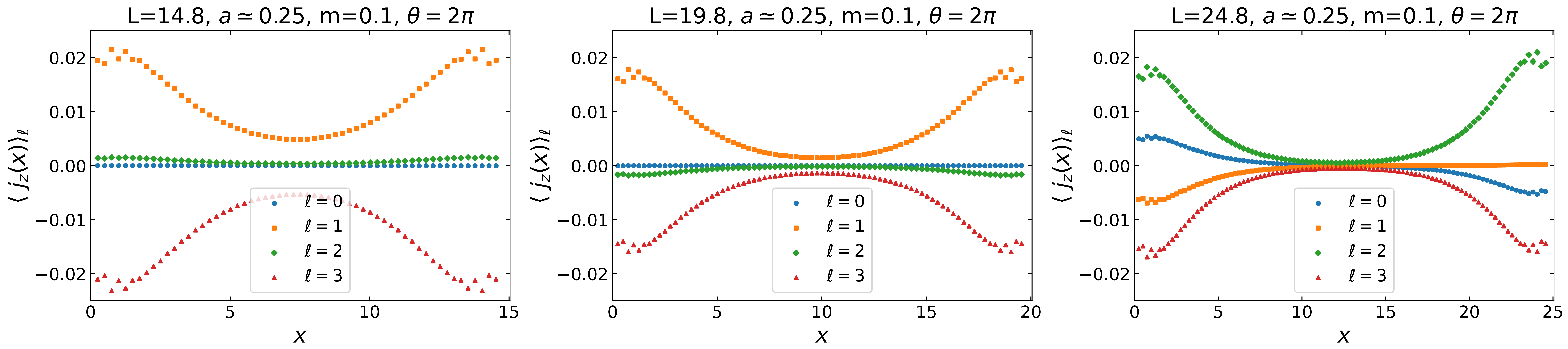
- local scalar condensate $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2$ (isospin singlet) at $\theta = 2\pi$
- degeneracy in $L \to \infty$



Local isospin

- local isospin $j_z(x) = \frac{1}{2}(\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2)$ at $\theta = 2\pi$
- finite L : singlet + triplet —> $L \to \infty$: doublet × doublet
  interaction is suppressed exponentially and the edge modes are decoupled

small L → large L



Electric charge and electric field

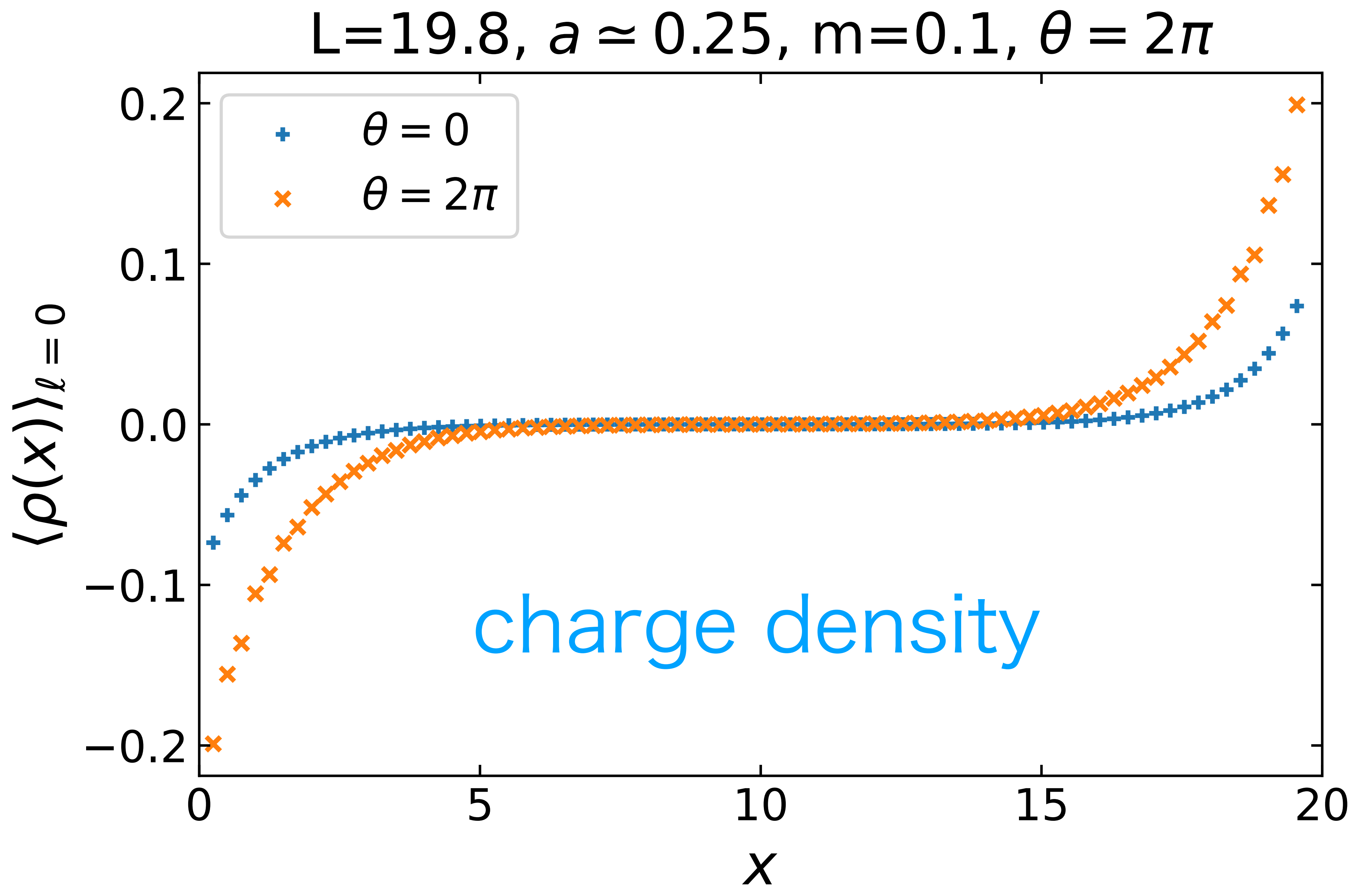
- charge density: $\rho(x) = \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2$

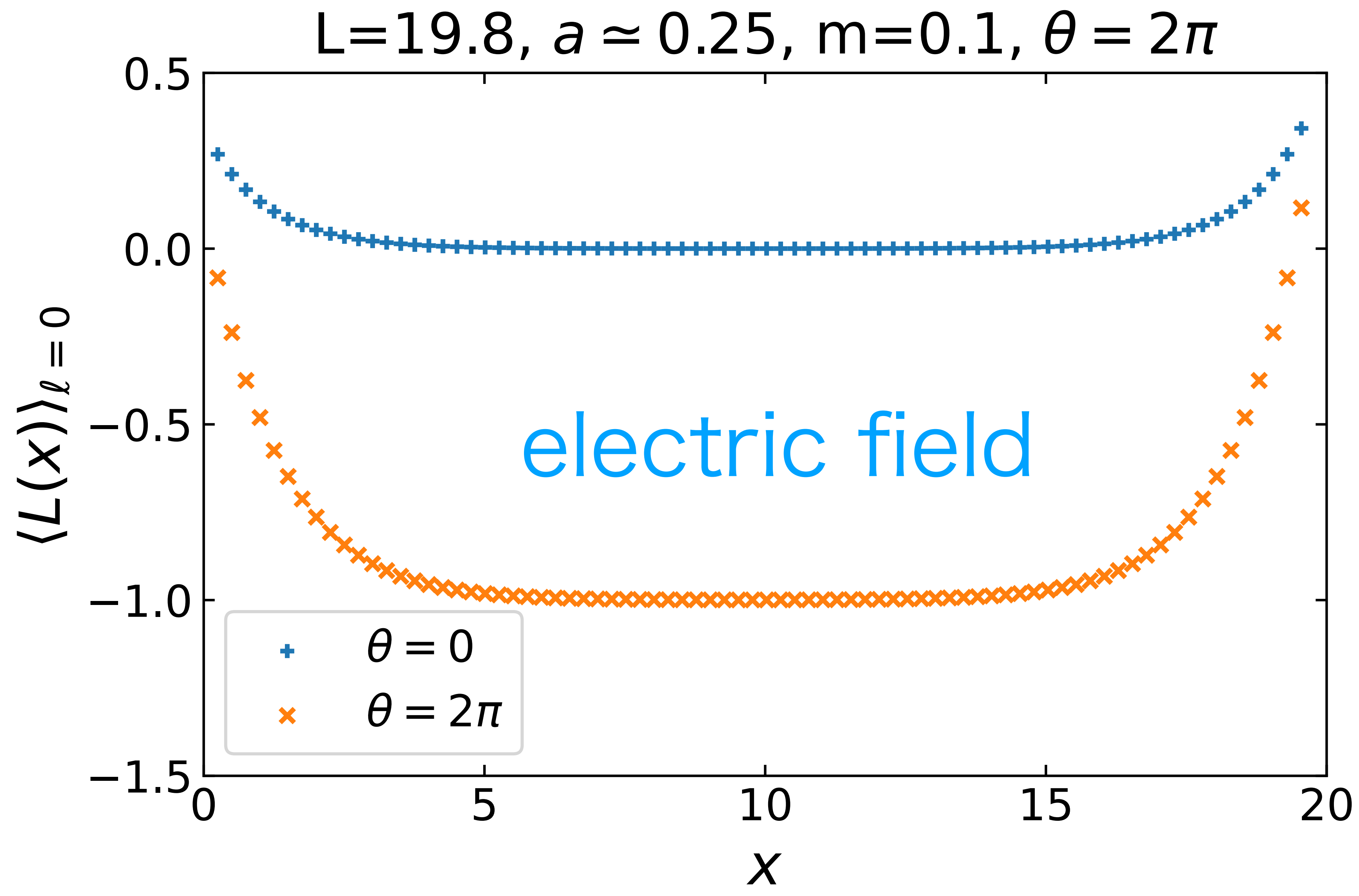
- induced electric field: $L(x) = \int_0^x dy\, \rho(y)$

$L(x) = -1$

cancel the background electric field

$E = \theta/2\pi = +1$ from θ term





(3) Dispersion-relation scheme

The dispersion relation $E = \sqrt{K^2 + M^2}$ can be obtained from the excited states (momentum excitations of mesons).

- generate the excited states using DMRG
- compute the energy E and the total momentum

$$K = \sum_{f=1}^{N_f} \int dx\, \psi_f^\dagger (i\partial_x - A_1)\psi_f \to \frac{i}{4a} \sum_{f=1}^{N_f} \sum_{n=1}^{N-2} (\chi_{f,n-1}^\dagger \chi_{f,n+1} - \chi_{f,n+1}^\dagger \chi_{f,n-1})$$

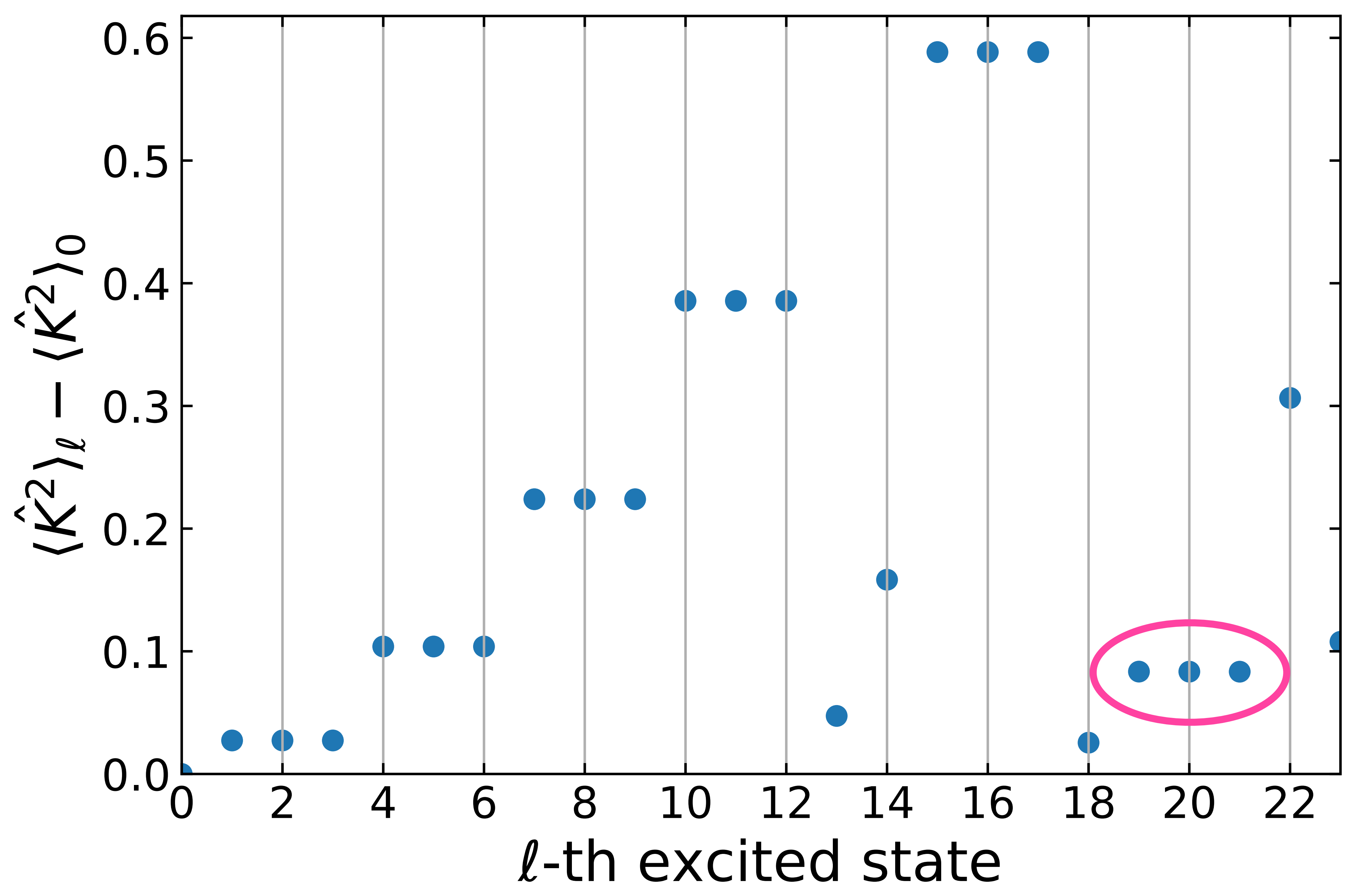
  - $[H, K] \neq 0$ due to the absence of translational invariance but it is still useful as an approximated operator.

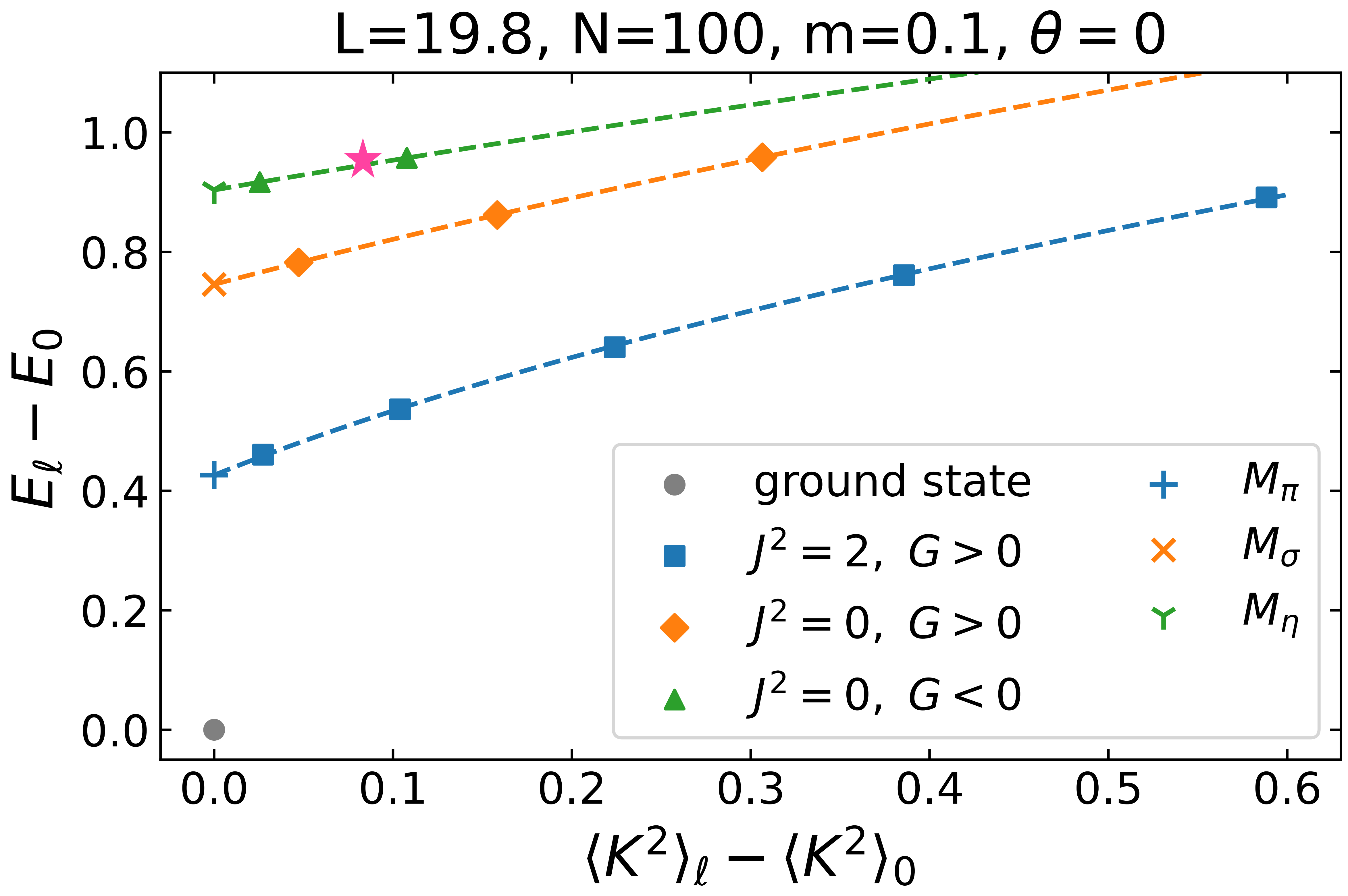
Scattering states

★ triplet of $\ell = 19,20,21$ seems to be a two-pion scattering state $\Delta E > 2M_\pi$

information of scattering (phase shift) may be extracted

[Harada et al. (1994)]





Isospin

- isospin operators: conserved charge of the isospin symmetry

$$J_a = \int dx \sum_{f,f'} \psi_f^\dagger (\tau^a)_{f,f'} \psi_{f'} \qquad \tau^a = \frac{\sigma^a}{2} \text{ : SU(2) generator} \qquad a \in \{x, y, z\}$$

- lattice version

$$J_z = \frac{1}{2} \sum_{n=0}^{N-1} \left(\chi_{1,n}^\dagger \chi_{1,n} - \chi_{2,n}^\dagger \chi_{2,n} \right), \quad J_+ = \sum_{n=0}^{N-1} \chi_{1,n}^\dagger \chi_{2,n} = (J_-)^\dagger, \quad \mathbf{J}^2 = \frac{1}{2}(J_+ J_- + J_+ J_-) + J_z^2$$

- exactly commute with the lattice Hamiltonian

$$[H, J_z] = [H, J_\pm] = [H, \mathbf{J}^2] = 0$$

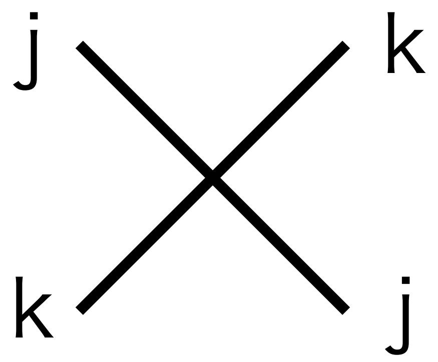
G-parity

- G-parity operator: $G = C\exp(i\pi J_y)$

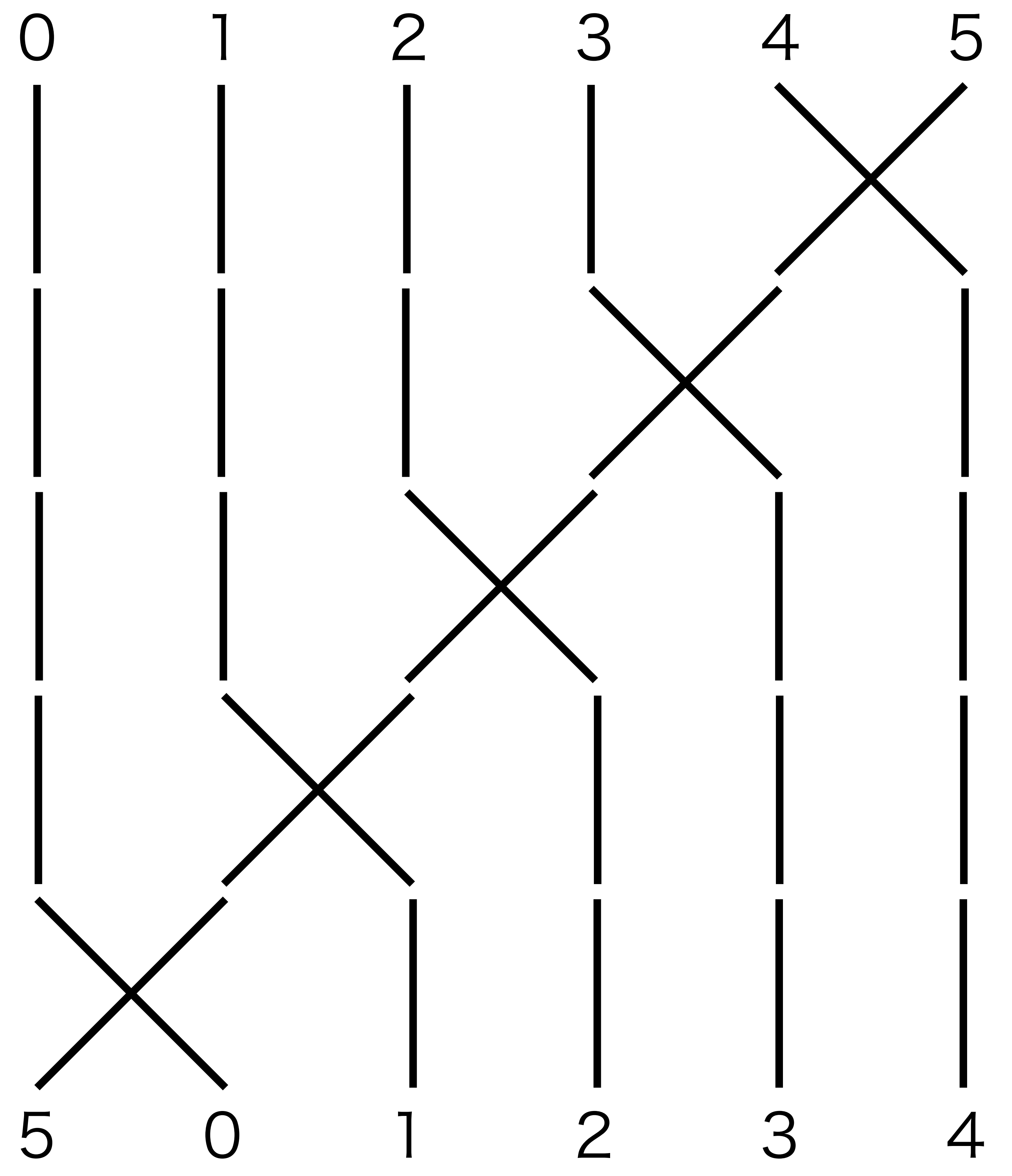
- C = exchange particles/anti-particles
  = exchange even/odd sites and flip each spin
  = 1-site translation and σ^x operators

$$C := \prod_{f=1}^{N_f}\left(\prod_{n=0}^{N-1}\sigma^x_{f,n}\right)\left(\prod_{n=0}^{N-2}(\mathrm{SWAP})_{f;N-2-n,N-1-n}\right)$$

$$(\mathrm{SWAP})_{f;j,k} = \frac{1}{2}\left(\mathbf{1}_{f,j}\mathbf{1}_{f,k} + \sum_a \sigma^a_{f,j}\sigma^a_{f,k}\right) \longrightarrow$$

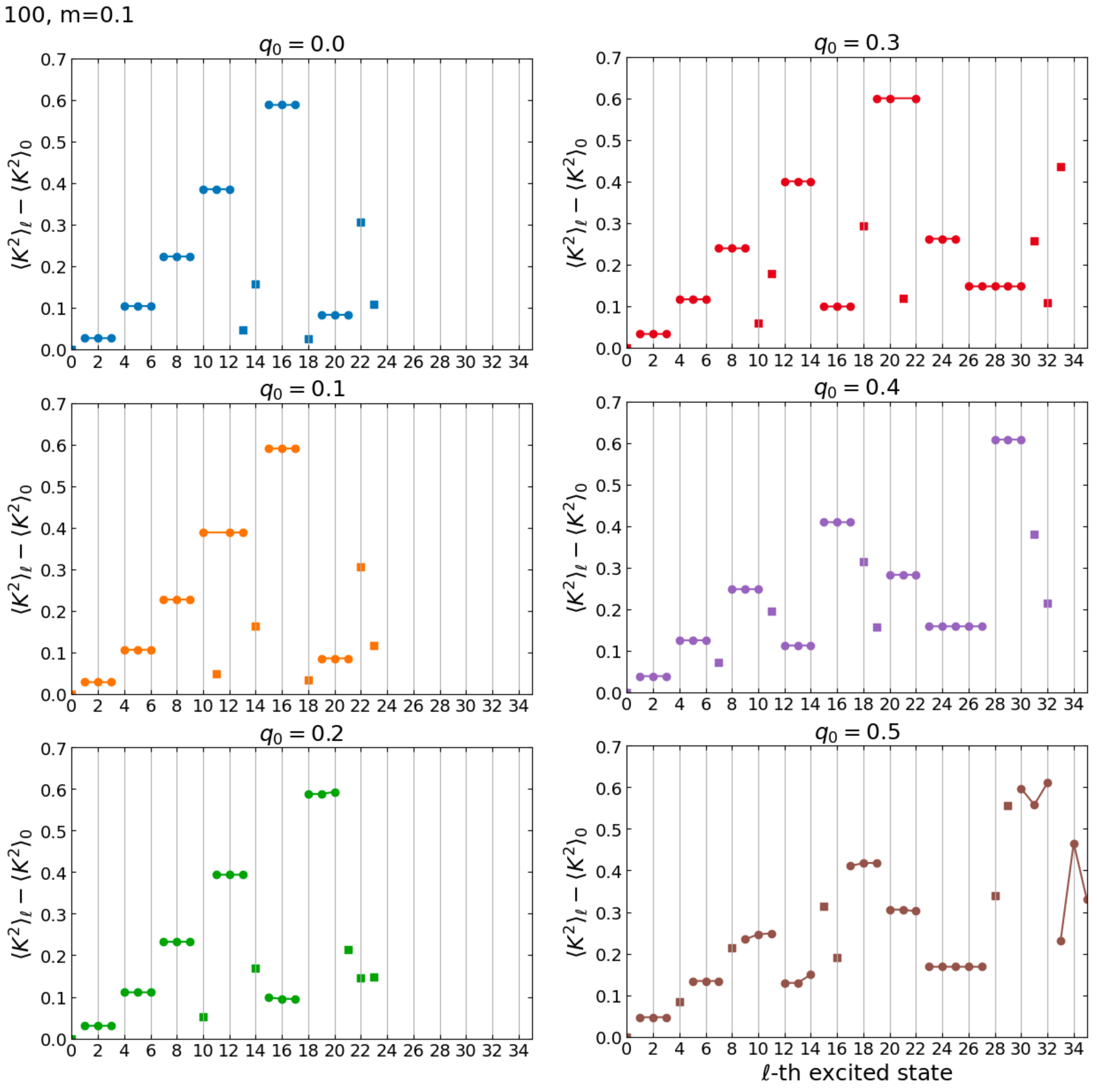


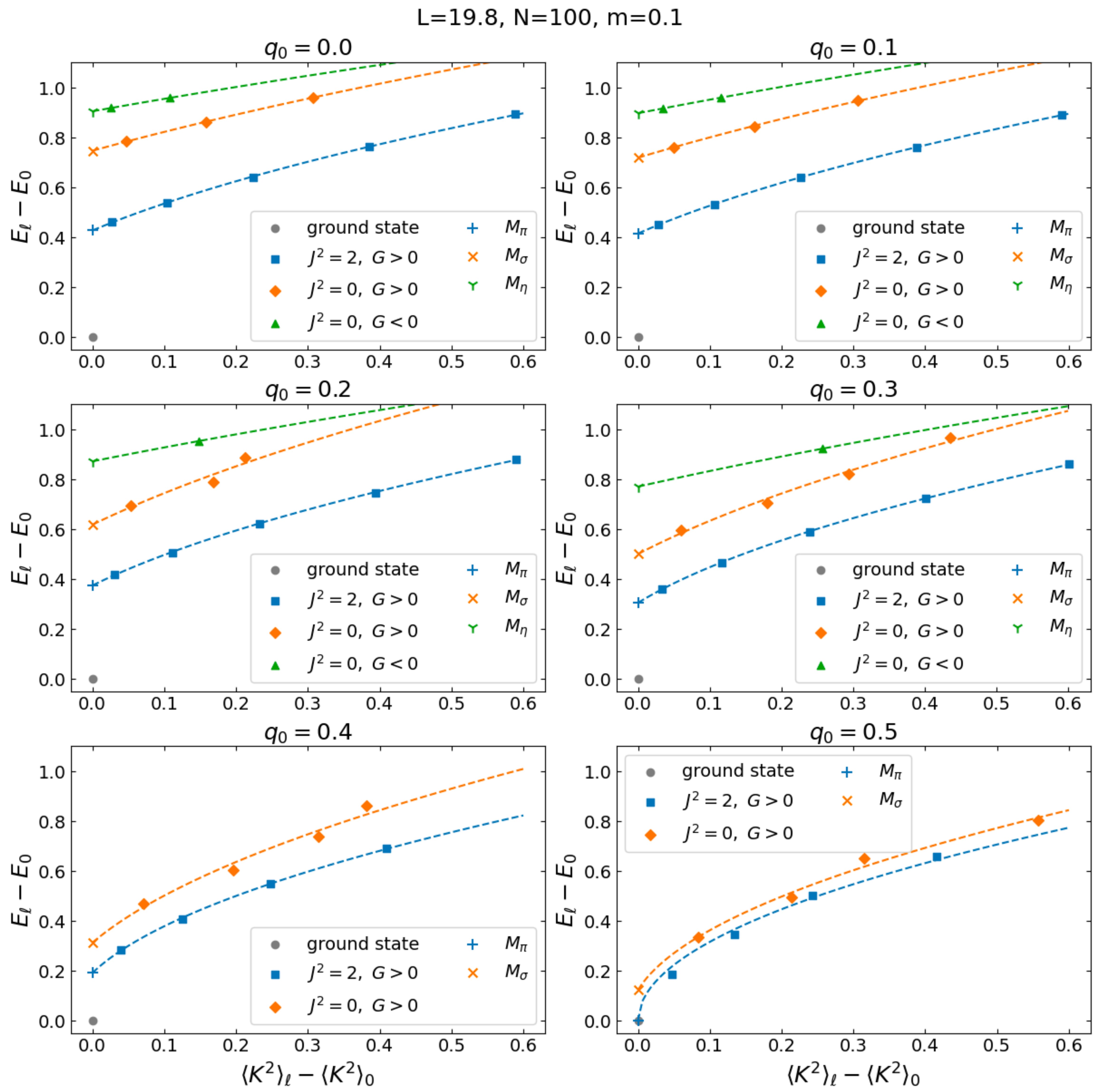
1-site translation



$[H, C] \neq 0$ due to the boundary

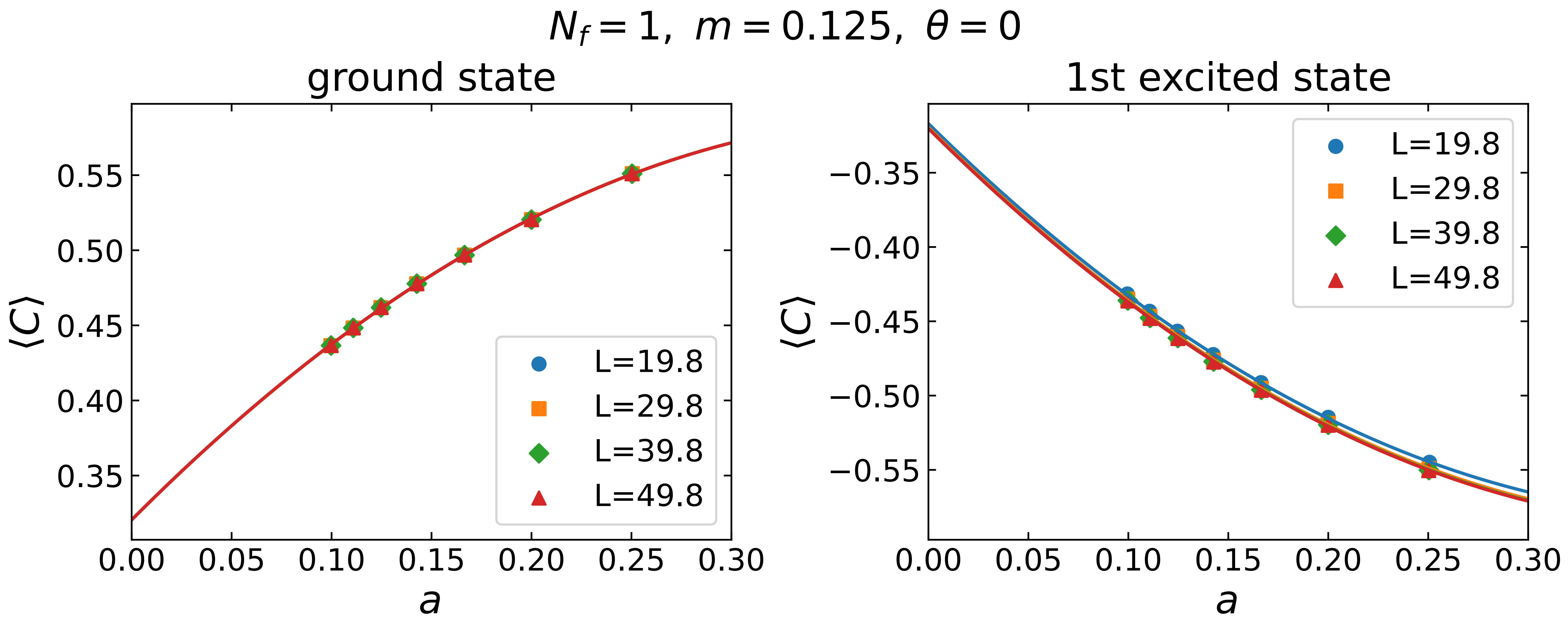
Result at $\theta \neq 0$





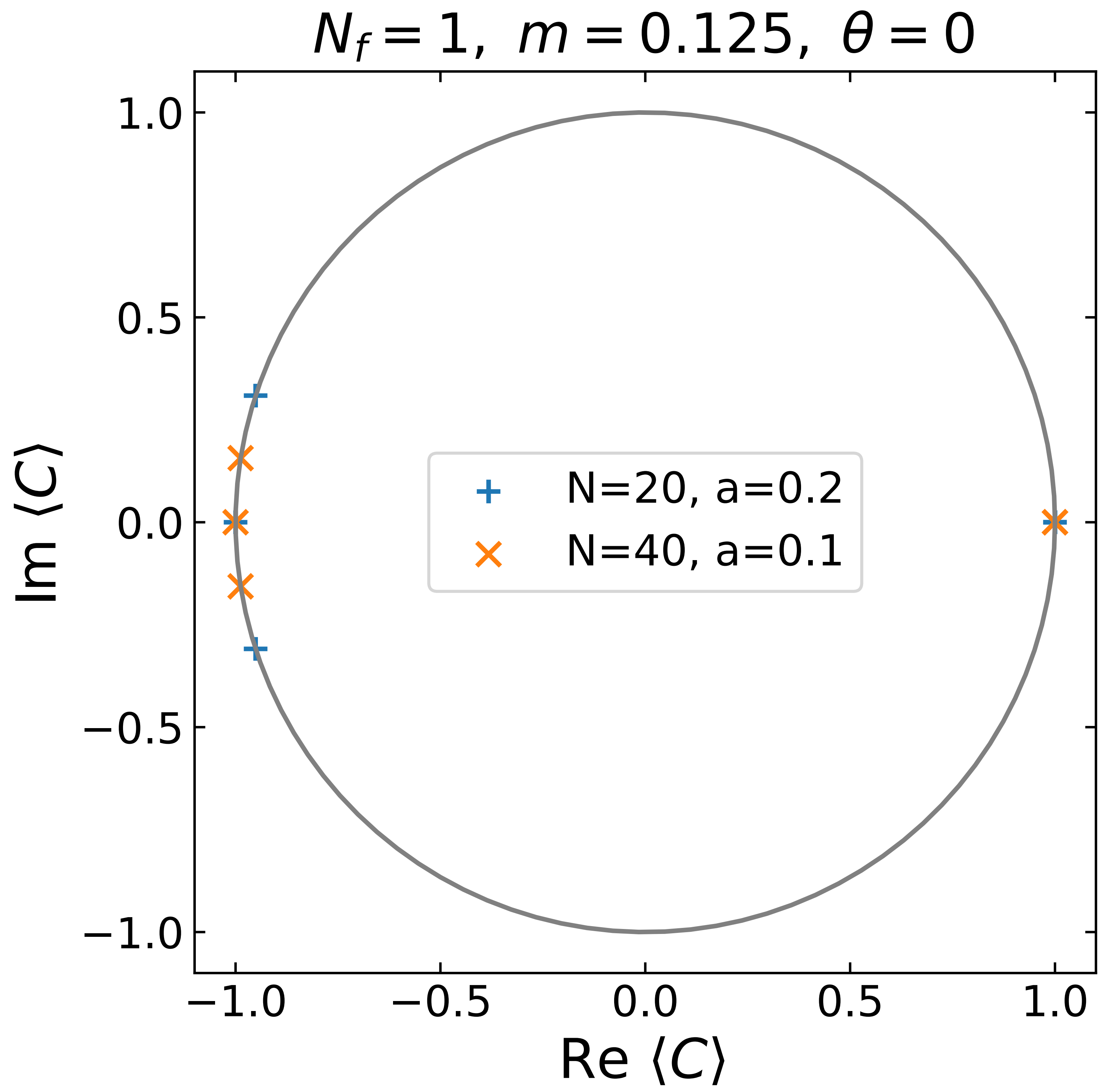
Charge conjugation

continuum limit of $\langle C \rangle$ for various L



1-flavor Schwinger model

boundary effect on $\langle C \rangle$



free fermion with p.b.c

Parity

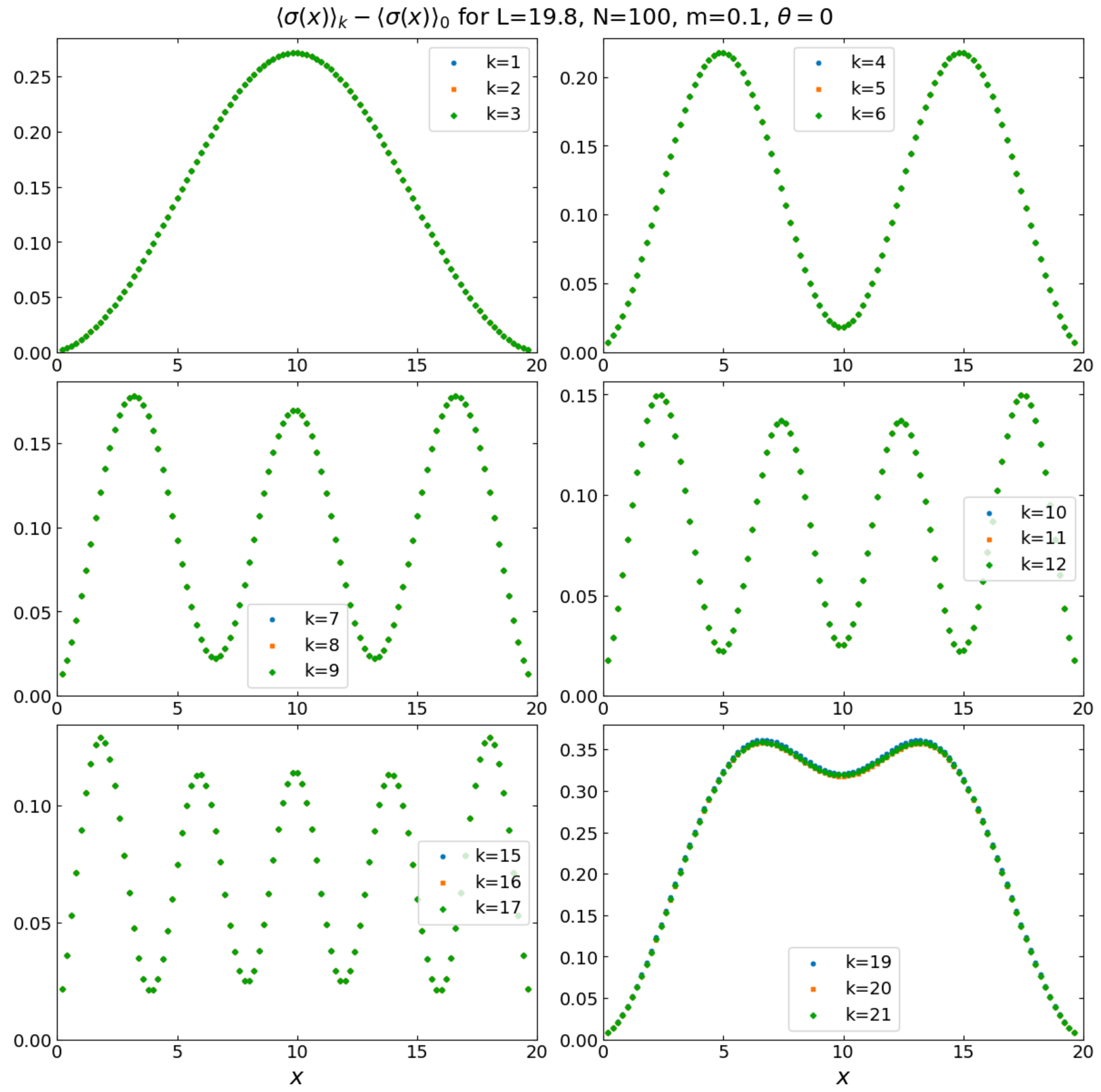
- $$P = \prod_{f=1}^{N_f} \left(\prod_{j=0}^{N/2-1} \sigma^z_{f,2j+1} \right) \left(\prod_{n=0}^{N-2} (\text{SWAP})_{f;N-2-n,N-1-n} \right) \left(\prod_{n=0}^{N/2-1} (\text{SWAP})_{f;n,N-1-n} \right)$$

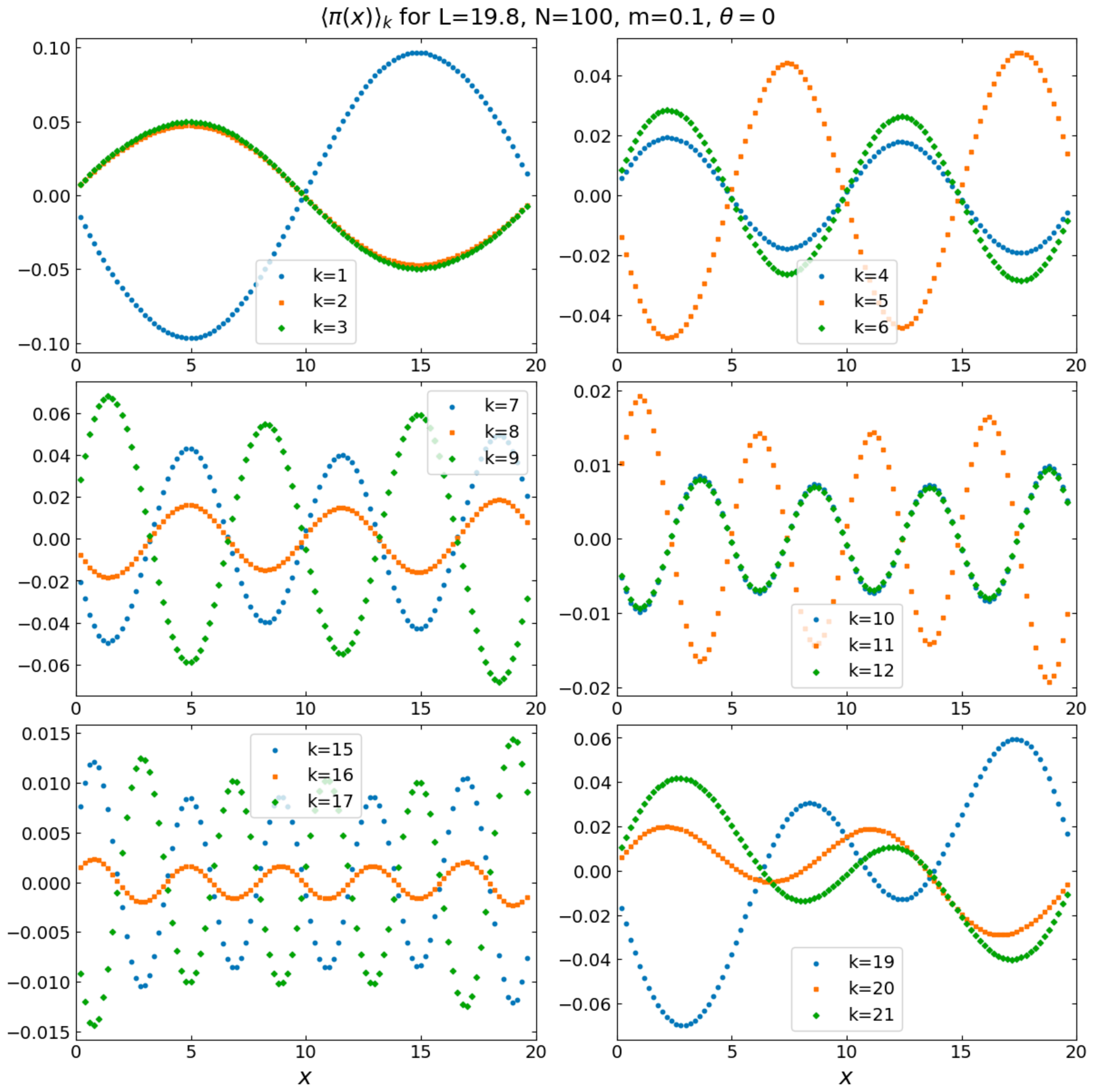
k	$\mathbf{J}^2$	J_z	G	P
0	0.00000003	-0.00000000	0.27984227	3.896×10^{-7}
13	0.00000003	0.00000000	0.27865844	1.273×10^{-7}
14	0.00000003	0.00000000	0.27508176	-2.765×10^{-8}
18	0.00000028	0.00000006	-0.27390909	-6.372×10^{-7}
22	0.00001537	0.00000115	0.26678987	7.990×10^{-8}
23	0.00003607	-0.00000482	-0.27664779	5.715×10^{-7}

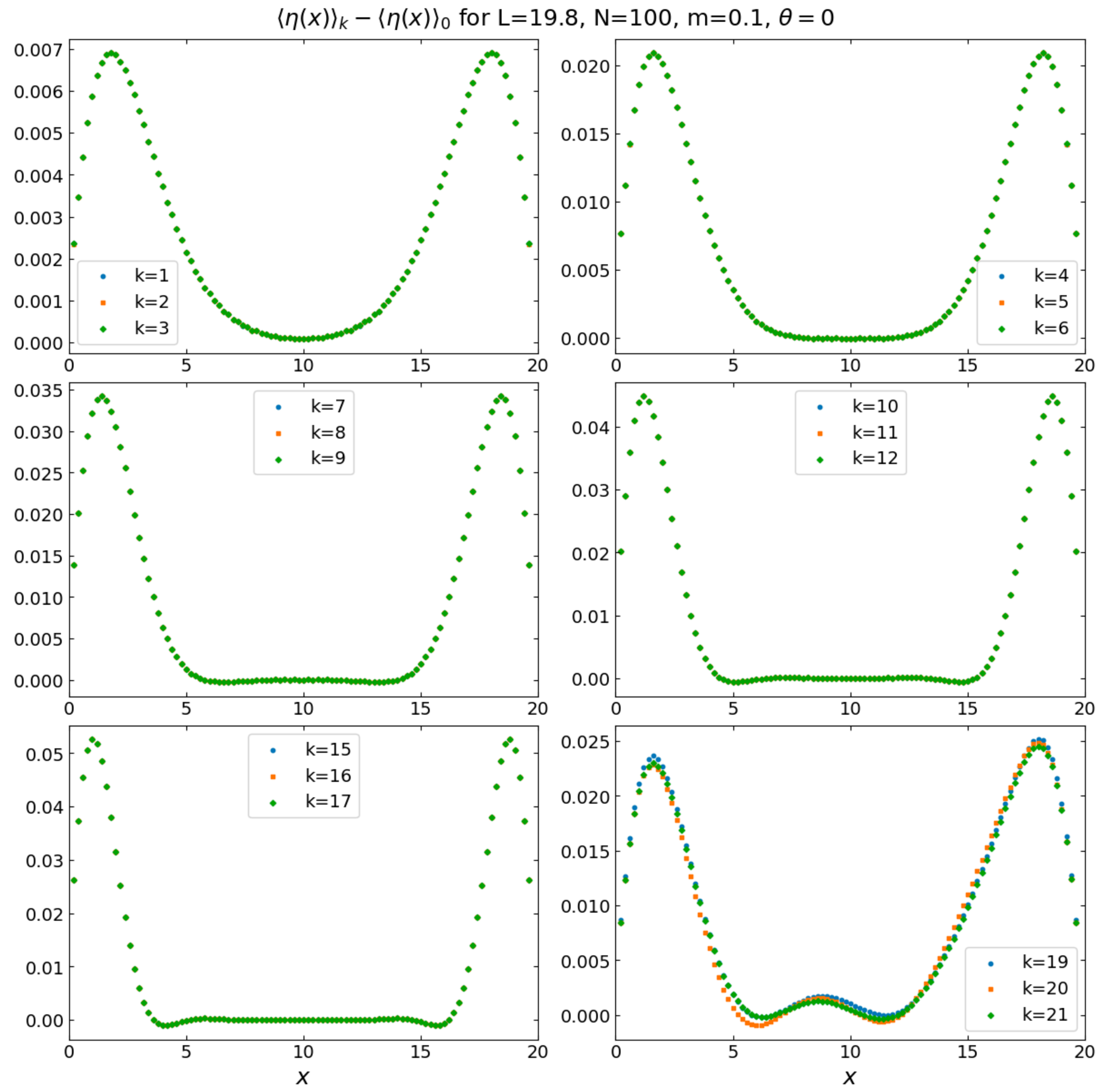
k	$\mathbf{J}^2$	J_z	G	P
1	2.00000004	0.99999997	0.27872443	-6.819×10^{-8}
2	2.00000012	-0.00000000	0.27872416	-6.819×10^{-8}
3	2.00000004	-0.99999996	0.27872443	-6.819×10^{-8}
4	2.00000007	0.99999999	0.27736066	7.850×10^{-8}
5	2.00000006	0.00000000	0.27736104	7.850×10^{-8}
6	2.00000009	-0.99999998	0.27736066	7.850×10^{-8}
7	2.00000010	1.00000000	0.27536687	-8.838×10^{-8}
8	2.00000002	0.00000000	0.27536702	-8.837×10^{-8}
9	2.00000007	-0.99999998	0.27536687	-8.838×10^{-8}
10	2.00000007	0.99999998	0.27356274	9.856×10^{-8}
11	2.00000005	0.00000001	0.27356277	9.856×10^{-8}
12	2.00000007	-0.99999999	0.27356274	9.856×10^{-8}
15	1.99999942	0.99999966	0.27173470	-1.077×10^{-7}
16	2.00000052	0.00000000	0.27173482	-1.077×10^{-7}
17	2.00000015	-1.00000003	0.27173470	-1.077×10^{-7}
19	2.00009067	1.00004377	0.27717104	-3.022×10^{-8}
20	2.00002578	-0.00000004	0.27717020	-3.023×10^{-8}
21	2.00003465	-1.00001622	0.27717104	-3.023×10^{-8}

local observable

- local observables for the triplets
- scalar: $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 = \sigma$ pseudo-scalar: $\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2 = \pi,\quad \bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2 = \eta$







local observable

- local observables for singlets
- scalar: $\bar{\psi}_1\psi_1 + \bar{\psi}_2\psi_2 = \sigma$
- pseudo-scalar:

  $\bar{\psi}_1\gamma^5\psi_1 - \bar{\psi}_2\gamma^5\psi_2 = \pi,$

  $\bar{\psi}_1\gamma^5\psi_1 + \bar{\psi}_2\gamma^5\psi_2 = \eta$