Eclectic Flavor Symmetry in Type IIB String Landscape

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Introduction

Flavor structure

The quark and lepton mixing matrices have been obtained with good accuracy by various experiments.

→ The origin of these structures remains a mystery.

We can explain flavor structure by introducing flavor symmetries, which are $A_4, S_4, \ldots,$ into the Standard Model.

⬇

However we do not know the origin of the flavor symmetries such as non-abelian discrete symmetries.

Introduction

<u>Discrete symmetries and extra dimensional spaces</u>

The Kaluza–Klein reduction of 6D Majorana–Weyl spinor λ (6D Super Yan–Mills theory on T^2) is given by

$$\lambda(x,z)=\sum_n \phi_n(x)\otimes\psi_n(z)\,,$$

The Yukawa couplings of chiral zero–modes are obtained by integral of three wavefunctions:

$$Y^{\tilde{\alpha}\tilde{\beta}\tilde{\gamma}}=\int_{T^2} d^2z\,\psi^{\tilde{\alpha},|M|}(z,\tau)\psi^{\tilde{\beta},|M'|}(z,\tau)\left(\psi^{\tilde{\alpha},|M|+|M'|}(z,\tau)\right)^*.$$

Introduction

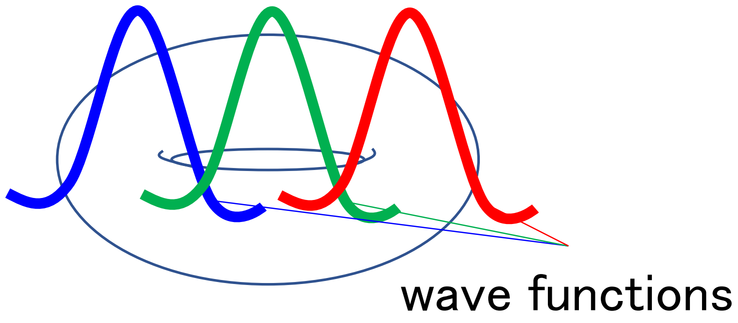
Yukawa couplings are known to be described by wave functions in compact spaces.

→ The origin of the discrete symmetry could be described by modular symmetry arising from compact spaces.

The complex structure moduli τ that determine the geometry of the torus have $SL(2,\mathbb{Z})$ symmetry.

Yukawa coupling

$$Y_{ijk}(\tau) =$$



We discuss the unification of flavor, modular and CP symmetries in Type IIB chiral 4D flux vacua.

Introduction

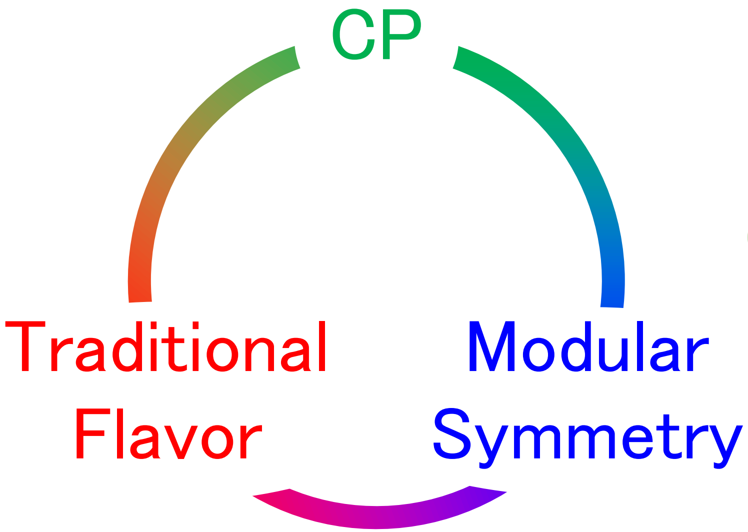
- Eclectic flavor symmetry

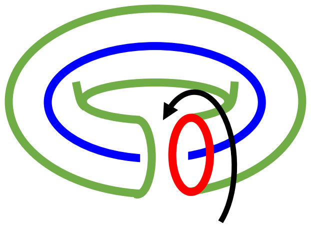
<u>A hybrid picture</u> where the traditional flavor group, the finite modular group and CP are combined to <u>a generalized flavor group</u>

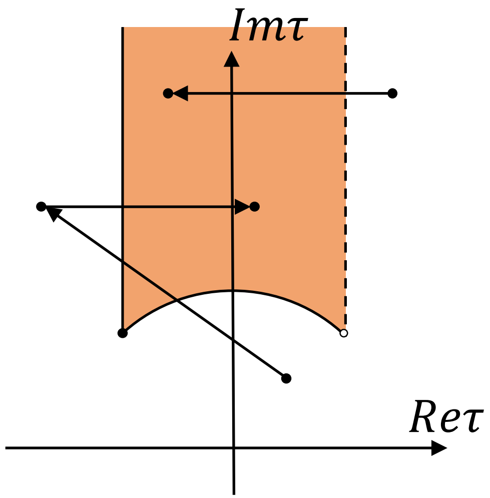
Nilles, Ramos-Sánchez, Vaundrevange 2001.01736

$$K = \alpha_0(-i\tau + i\bar{\tau})^{-1}(\bar{L}L)_{\mathbf{1}}$$

$$+ \sum_i \beta_i(-i\tau + i\bar{\tau})^{-k_L - k_\varphi}(\varphi L \bar{\varphi} \bar{L})_{\mathbf{1},i}$$







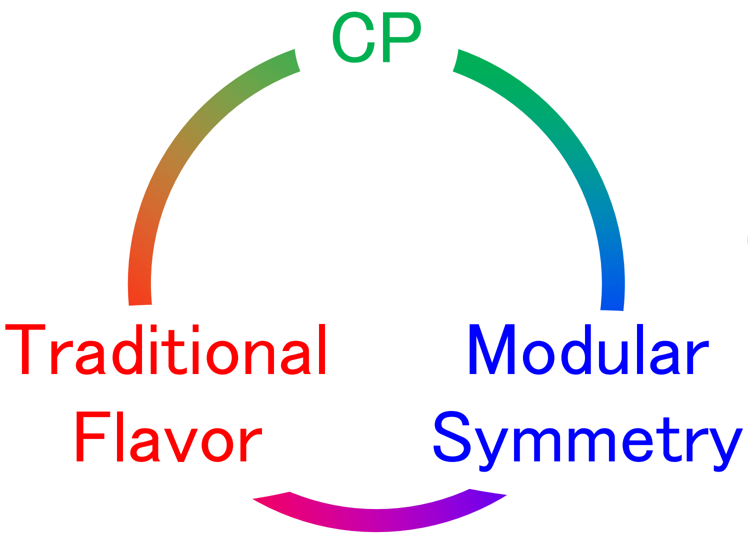
Introduction

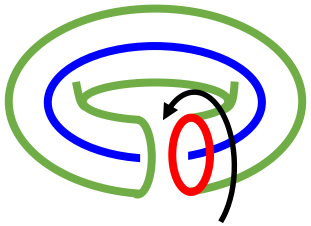
Top-down model building motivated by string theory

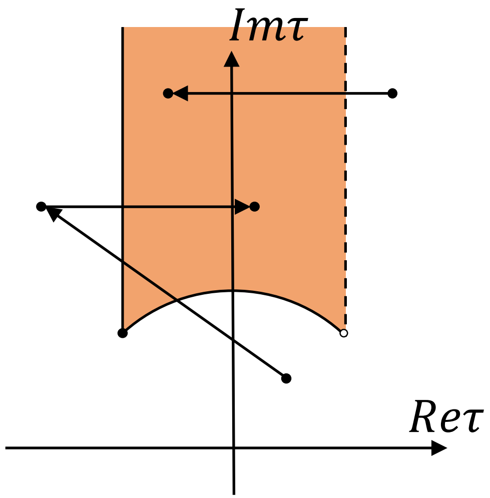
This symmetry potentially incorporates a different flavor structure for quark- and lepton-sector of the Standard Model

$$K = \alpha_0(-i\tau + i\bar{\tau})^{-1}(\bar{L}L)_{\mathbf{1}}$$

$$+\sum_i \beta_i(-i\tau + i\bar{\tau})^{-k_L-k_\varphi}(\varphi L\bar{\varphi}\bar{L})_{\mathbf{1},i}$$





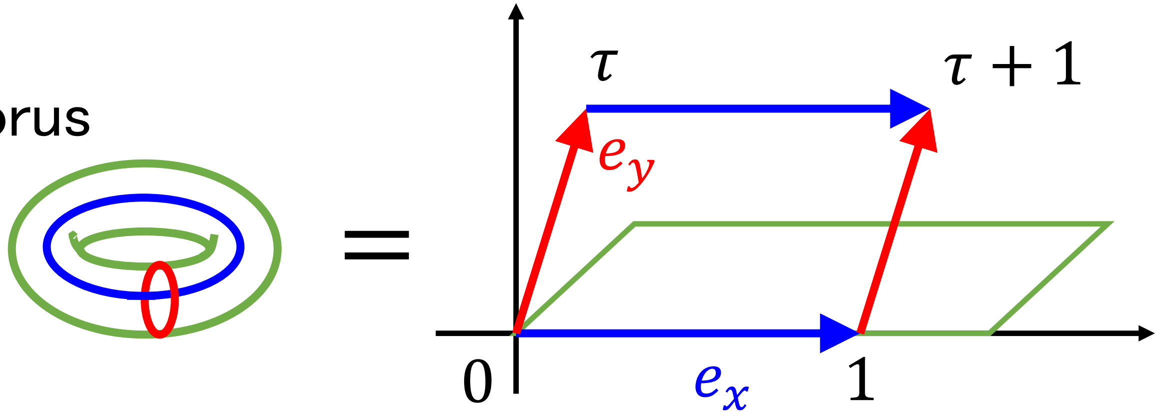


Outline

- Introduction
- Modular symmetry
- Eclectic Flavor Symmetry
- Summary

Modular symmetry

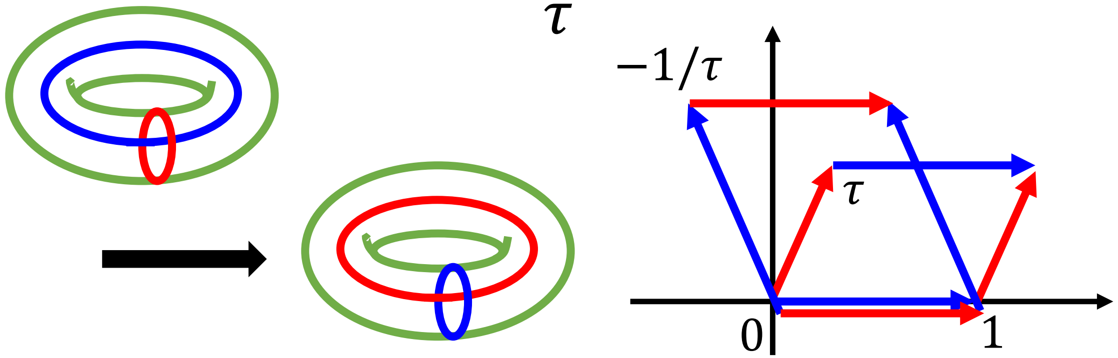
torus



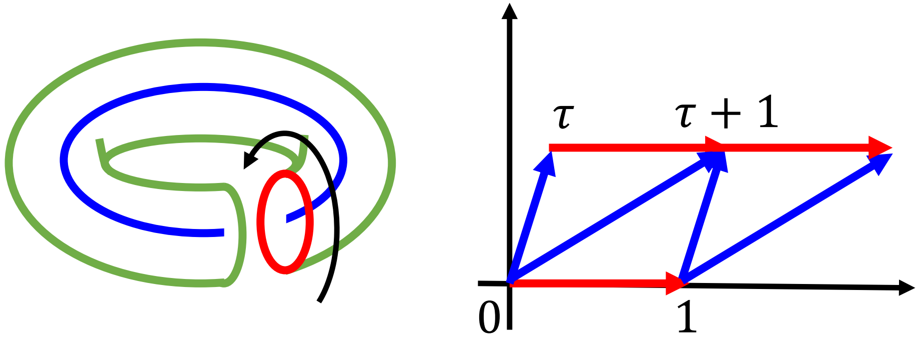
- complex structure moduli : τ

$$\tau \equiv \frac{e_y}{e_x} \to \tau' \equiv \frac{e'_y}{e'_x} = \frac{p\tau + q}{s\tau + t}$$

S: $\tau \to -\frac{1}{\tau}$



T: $\tau \to \tau + 1$



T^2 with magnetic fluxes

Zero-mode wave functions

$$\psi_{\pm}^{\tilde{\alpha},|M|}(z+\zeta,\tau) = \left(\frac{|M|}{A^2}\right)^{1/4} e^{i\pi|M|(z+\zeta)\frac{Im(z+\zeta)}{Im\tau}} \vartheta\begin{bmatrix}\frac{\tilde{\alpha}}{M}\\ 0\end{bmatrix}(|M|(z+\zeta),|M|\tau),$$

(magnetic flux : M, $\tilde{\alpha} = 0,\dots,|M|-1$)

S transformation

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau) \to \psi^{\tilde{\alpha},|M|}\left(-\frac{z+\zeta}{\tau},-\frac{1}{\tau}\right) = (-\tau)^{1/2} e^{i\pi/4} \sum_{\tilde{\beta}=0}^{|M|-1} \frac{1}{\sqrt{|M|}} e^{2\pi i\frac{\tilde{\alpha}\tilde{\beta}}{|M|}} \psi^{\tilde{\beta},|M|}(z+\zeta,\tau),$$

automorphy factor

$= \rho(S)$

T transformation

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau) \to \psi^{\tilde{\alpha},|M|}(z+\zeta,\tau+1) = e^{i\pi|M|\frac{Im\,(z+\zeta)}{2Im\,\tau}} \sum_{\tilde{\beta}=0}^{|M|-1} e^{i\pi\frac{\tilde{\alpha}^2}{|M|}} \delta_{\tilde{\alpha},\tilde{\beta}}\, \psi^{\tilde{\beta},|M|}(z+\zeta,\tau),$$

$= \rho(T)$

Metaplectic modular symmetry

Metaplectic modular group $Mp(2,\mathbb{Z}) : \tilde{\Gamma}$

automorphy factor

$$Mp(2,\mathbb{Z}) = \left\{\tilde{\gamma} = (\gamma, \varphi(\gamma,\tau)) \middle| \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \varphi(\gamma,\tau)^2 = (c\tau + d)\right\}$$

Liu, Yao, Qu, Ding, 2007.13706

$$Mp(2,\mathbb{Z})/Z_2^{\tilde{R}^2} \cong SL(2,\mathbb{Z}), \qquad \tilde{S}^2 = \tilde{R}, \qquad Z_2^{\tilde{R}^2} = \{1, \tilde{R}^2\}$$

The principal congruence subgroup

$$\Gamma(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \middle| a \equiv d \equiv 1, \qquad b \equiv c \equiv 0 (\mathrm{mod}\ N)\right\},$$

$$\tilde{\Gamma}(4N) = \left\{\tilde{\gamma} = (\gamma, v(\gamma) J_{1/2}(\gamma,\tau)) \middle| \gamma \in \Gamma(4N)\right\},$$

$\left(v(\gamma) = \left(\frac{c}{d}\right)\right.$: the Kronecker symbol, $\left.\varphi(\gamma,\tau) = \pm J_{1/2}(\gamma,\tau)\right)$

The finite metaplectic modular groups are given by $\tilde{\Gamma}_{4N} \equiv \tilde{\Gamma}/\tilde{\Gamma}(4N)$.

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Eclectic Flavor Symmetry

The eclectic flavor group is a nontrivial product of a traditional flavor group, a finite modular group and a CP–like transformation.

Define an operation for a cartesian product. $(N \times H)$

- direct product $(G = N \times H)$

$$(n,h)(n',h') = (nn',hh'), \qquad \text{for} \;\; n,n' \in N, \qquad h,h' \in H$$

- semi direct product $(G = N \rtimes H)$

$$(n,h)(n',h') = (nhn'h^{-1},hh'), \qquad \text{for} \;\; n,n' \in N, \qquad h,h' \in H$$

By checking whether each generator is commutative or not, we can determine what kind of group product it is.

Eclectic Flavor Symmetry (models)

- Traditional flavor group $G_{\rm flavor} \equiv \mathbb{Z}_4 \times \mathbb{Z}_2^P \times \mathbb{Z}_2^C \times \mathbb{Z}_2^Z$

  Generators : $\{Z', P, C, Z\}$

Abe, Choi, Kobayashi, Ohki 0904.2631

- Generalized CP group $G_{\rm CP} \equiv \mathbb{Z}_2^{\rm CP}$

  Generator : $\{{\rm CP}\}$

The CP transformation

$$\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{e}_2 \\ \bar{e}_1 \end{pmatrix}.$$

- Finite metaplectic modular group $G_{\rm modular} \equiv \tilde{\Gamma}_8$

In order to discuss the action of the full modular group with the half-integral modular weights (such as Yukawa coupling and wave function on T^2)

Eclectic Flavor Symmetry (results)

Traditional flavor and Modular flavor

$$\tilde{S}_{\rm even} C_{\rm even} \tilde{S}_{\rm even}^{-1} = Z_{\rm even}, \qquad \tilde{S}_{\rm even} Z_{\rm even} \tilde{S}_{\rm even}^{-1} = C_{\rm even},$$

$$\tilde{T}_{\rm even} C_{\rm even} \tilde{T}_{\rm even}^{-1} = C_{\rm even} Z_{\rm even} (Z'_{\rm even})^2, \qquad \tilde{T}_{\rm even} Z_{\rm even} \tilde{T}_{\rm even}^{-1} = Z_{\rm even},$$

The modular transformation is regarded as the outer automorphism of the traditional flavor group.

$\rightarrow G_{\rm flavor} \rtimes G_{\rm modular}$ (Semi–direct product)

Eclectic Flavor Symmetry (results)

Traditional flavor, Modular flavor and CP

$$\widetilde{CP} Z'_{\text{even}} \widetilde{CP}^{-1} = (Z'_{\text{even}})^{-1}, \qquad \widetilde{CP} Z'_{\text{odd}} \widetilde{CP}^{-1} = (Z'_{\text{odd}})^{-1},$$

$$\widetilde{CP} \tilde{S}_{\text{even}} \widetilde{CP}^{-1} = (\tilde{S}_{\text{even}})^{-1}, \qquad \widetilde{CP} \tilde{T}_{\text{even}} \widetilde{CP}^{-1} = \tilde{T}_{\text{even}}{}^{-1},$$

$$\widetilde{CP} \tilde{S}_{\text{odd}} \widetilde{CP}^{-1} = (\tilde{S}_{\text{odd}})^{-1}, \qquad \widetilde{CP} \tilde{T}_{\text{odd}} \widetilde{CP}^{-1} = \tilde{T}_{\text{odd}}{}^{-1},$$

We can construct the outer automorphism

$$u_{\text{CP}} : G_{\text{CP}} \to \text{Aut}(G_{\text{flavor}} \rtimes G_{\text{modular}})$$

$$\to (G_{\text{flavor}} \rtimes G_{\text{modular}}) \rtimes G_{\text{CP}}$$ (Semi−direct product)

Eclectic Flavor Symmetry ($\mathbb{Z}_3$ fixed point)

Type IIB compactifications on $T^6/(\mathbb{Z}_2 \times \mathbb{Z}'_2)$

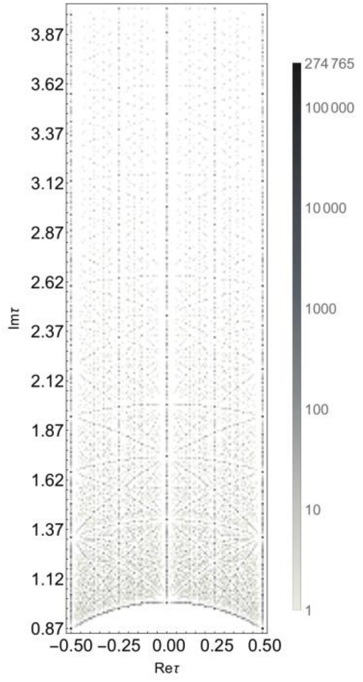
Background three form fluxes $\{a^0, a^i, b_i, b_0\}$

$\Rightarrow$ many VEVs of complex structure (τ)

The distribution of complex structure moduli VEVs clusters at fixed points of $SL(2,\mathbb{Z})$ modular symmetry.

($\tau = i, \omega, i\infty$ with $\omega = \frac{-1+\sqrt{3}i}{2}$, $\mathbb{Z}_3$ fixed point : $\tau = \omega$)

Especially, $\mathbb{Z}_3$ fixed point has the largest part.



$\mathbb{Z}_3$: 40.3%

The numbers of stable flux vacua on the fundamental domain of τ

Ishiguro, Kobayashi, Otsuka 2011.09154

Eclectic Flavor Symmetry ($\mathbb{Z}_3$ fixed point)

$\mathbb{Z}_3$ modular symmetry generated by $\{1, ST, (ST)^2\}$

$$(\tilde{S}\tilde{T})C_{\text{even}}(\tilde{S}\tilde{T})^{-1} = C_{\text{even}}Z_{\text{even}}(Z'_{\text{even}})^2, \qquad (\tilde{S}\tilde{T})Z_{\text{even}}(\tilde{S}\tilde{T})^{-1} = Z_{\text{even}},$$

The discrete non-abelian symmetry $(G_{\text{flavor}} \rtimes \mathbb{Z}_3) \rtimes G_{\text{CP}}$ remains in the low-energy 4D effective action.

Eclectic Flavor Symmetry ($\mathbb{Z}_3$ fixed point)

$\mathbb{Z}_3$ modular symmetry generated by $\{1, ST, (ST)^2\}$

$$(\tilde{S}\tilde{T})C_{\rm even}(\tilde{S}\tilde{T})^{-1} = C_{\rm even}Z_{\rm even}(Z'_{\rm even})^2, \qquad (\tilde{S}\tilde{T})Z_{\rm even}(\tilde{S}\tilde{T})^{-1} = Z_{\rm even},$$

The discrete non-abelian symmetry $(G_{\rm flavor} \rtimes \mathbb{Z}_3) \rtimes G_{\rm CP}$ remains in the low-energy 4D effective action.

The coefficient of 4D higher-dimensional operators will be described by the product of modular forms with half-integral modular weights.

→ The eclectic flavor symmetry would control the flavor structure of higher-dimensional operators.

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Summary

By introducing flavor symmetries, which are $A_4, S_4, \ldots$ and modular flavor symmetry, into the Standard Model, we can get clues to explain the flavor structure.

The traditional flavor, modular flavor and CP symmetries are uniformly described in the context of eclectic flavor symmetry

$$(G_{\text{flavor}} \rtimes G_{\text{modular}}) \rtimes G_{\text{CP}}$$

A part of eclectic flavor symmetry would control the flavor structure of 4D higher-dimensional operators. (at $\mathbb{Z}_3$ fixed point)

Back up

Data

$$|V_{\rm CKM}| = \begin{pmatrix} 0.97435 \pm 0.00016 & 0.22500 \pm 0.00067 & 0.00369 \pm 0.00011 \\ 0.22486 \pm 0.00067 & 0.97349 \pm 0.00016 & 0.04182^{+0.00085}_{-0.00074} \\ 0.00857^{+0.00020}_{-0.00018} & 0.04110^{+0.00083}_{-0.00072} & 0.999118^{+0.000031}_{-0.000036} \end{pmatrix},$$

particle data group (2022)

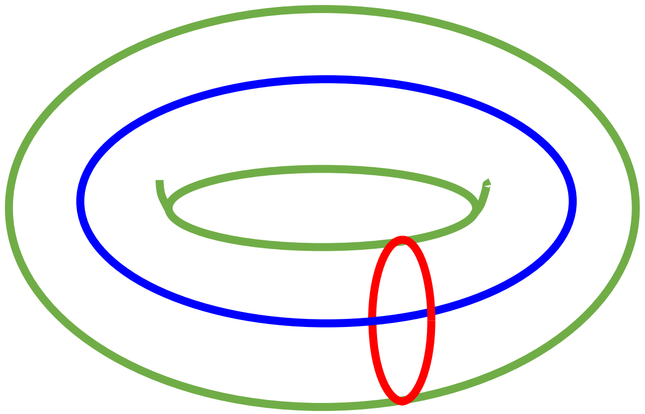
$$|U|_{3\sigma}^{\text{with SK-atm}} = \begin{pmatrix} 0.803 \to 0.845 & 0.514 \to 0.578 & 0.143 \to 0.155 \\ 0.244 \to 0.498 & 0.502 \to 0.693 & 0.632 \to 0.768 \\ 0.272 \to 0.517 & 0.473 \to 0.672 & 0.623 \to 0.761 \end{pmatrix}$$

NuFIT 5.2 (2022)

Tri-Bi-Maximal matrix: $U \sim \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}$

Modular symmetry

torus



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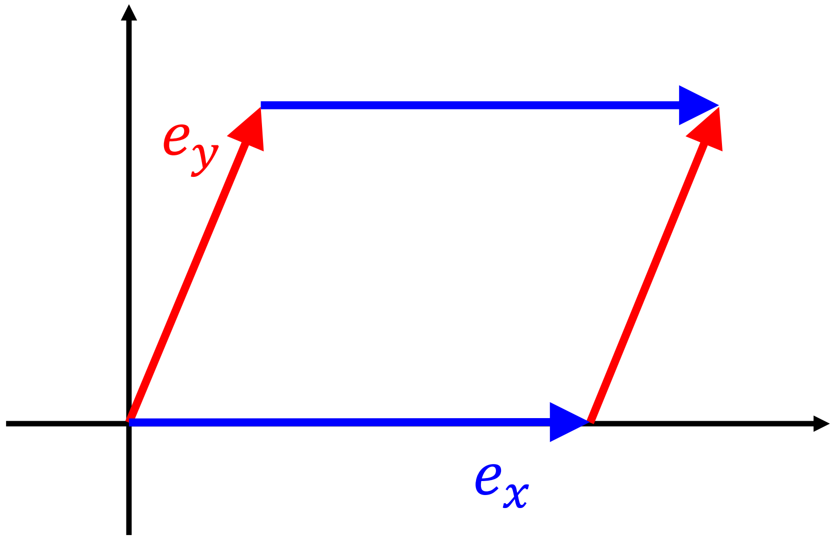


$SL(2,\mathbb{Z})$ transformation

$$\begin{pmatrix} e'_y \\ e'_x \end{pmatrix} = \begin{pmatrix} p & q \\ s & t \end{pmatrix} \begin{pmatrix} e_y \\ e_x \end{pmatrix}$$

$$p, q, s, t, \in \mathbb{Z}, \qquad pt - qs = 1$$

two generators

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$S^4 = (ST)^3 = I$$

T^2 with magnetic fluxes

The background $U(1)_a$ gauge field strength F_a on $(T^2)_i$,

$$\frac{m_a^i}{2\pi}\int_{T_i^2} F_a^i = n_a^i,$$

m_a^i : wrapping number on $(T^2)_i$
n_a^i : units of magnetic flux on $(T^2)_i$

The number of chiral zero-modes on T^6 $(T^2 \times T^2 \times T^2)$

$$I_{ab} = \prod_{i=1}^{3} \left(n_a^i m_b^i - n_b^i m_a^i \right), \qquad (\equiv M \in \mathbb{Z})$$

MSSM

D-brane configurations leading to left-right symmetric Minimal Supersymmetric Standard Model (MSSM).

N_α	Gauge group	(n_α^1, m_α^1)	(n_α^2, m_α^2)	(n_α^3, m_α^3)
$N_a = 6$	$SU(3)_C$	(1,0)	$(g, 1)$	$(g, -1)$
$N_b = 2$	$USp(2)_L$	(0,1)	$(1, 0)$	$(0, -1)$
$N_c = 2$	$USp(2)_R$	(0,1)	$(0, -1)$	$(1, 0)$
$N_d = 2$	$U(1)_d$	(1,0)	$(g, 1)$	$(g, -1)$

m_a^i: wrapping number on $(T^2)_i$

n_a^i: units of magnetic flux on $(T^2)_i$

Marchesano, Shiu hep-th/0409132

The magnetic flux g determines the generations of quark and lepton chiral multiplets in the visible sector

MSSM

- Tadpole cancellation condition (D3−brane charge)

$$D3: \sum_a N_a n_a^1 n_a^2 n_a^3 + \frac{1}{2} N_{\text{flux}} = 16,$$

- The existence of magnetized D9−branes in the hidden sector

$$8g^2 = -Q_{D3}^{\text{hid}} + 16 - \frac{N_{\text{flux}}}{2},$$

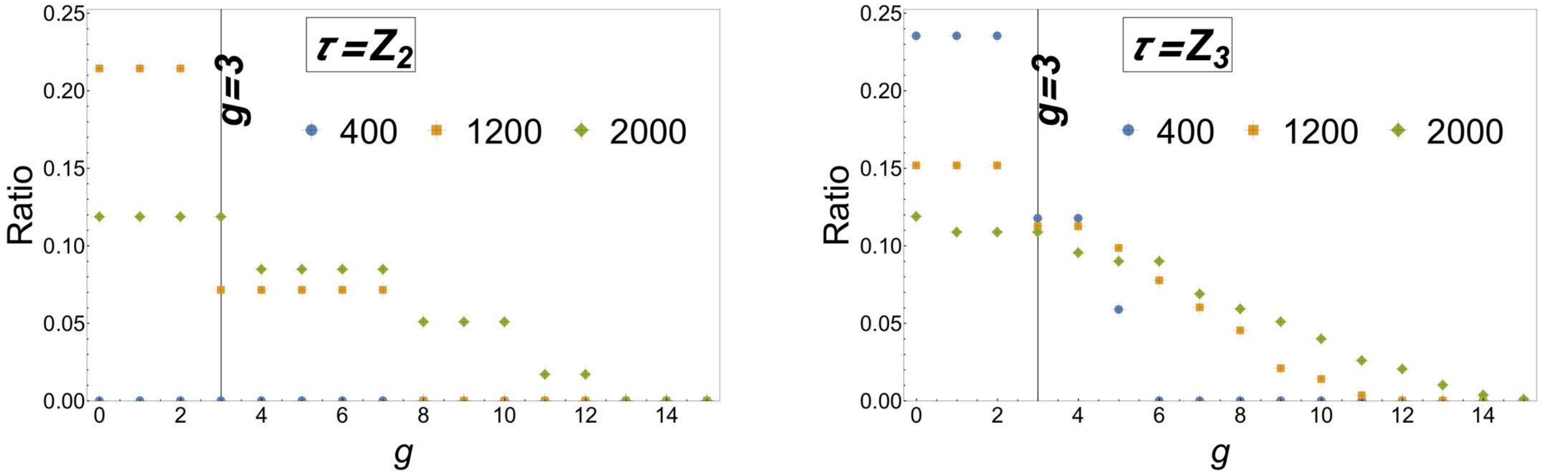
(Q_{D3}^{hid} : D3−brane charge induced by the magnetic flux on D9−branes)

We freely change the value of Q_{D3}^{hid} to reveal the mutual relation between the generation number g and the flux quanta N_{flux}.

MSSM (result)

The numbers of flux vacua as a function of the generation number g at $\tau = i$ and $\tau = \omega$ respectively.

We change the maximum value of Q_{D3}^{hid} as $\left|Q_{D3}^{\text{hid}}\right| = 400, 1200, 2000$



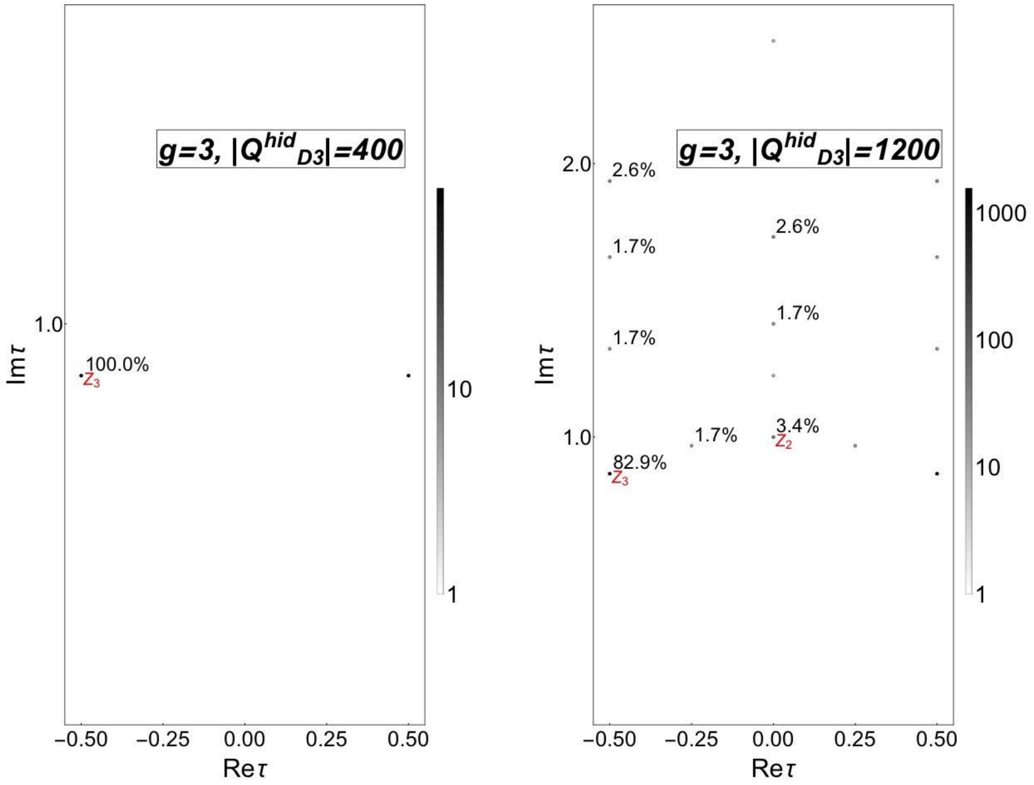
The small generation number is favored in the string landscape

MSSM (result)

The numbers of stable flux vacua with $g = 3$ generation of quarks/leptons on the fundamental domain of τ.

MSSM-like models are still peaked at the $\mathbb{Z}_3$ fixed point

Similar results are obtained for Pati-Salam-like model



Pati–Salam–like model on $T^6/Z_2 \times Z_2'$

D–brane configuration leading to Pati–Salam–like model.

N_α	Gauge group	(n_α^1, m_α^1)	(n_α^2, m_α^2)	$(\tilde{n}_\alpha^3, m_\alpha^3)$
$N_a = 8$	$U(4)_C$	$(0, -1)$	$(1, 1)$	$(1/2, 1)$
$N_b = 4$	$U(2)_L$	$(g, 1)$	$(1, 0)$	$(1/2, -1)$
$N_c = 4$	$U(2)_R$	$(g, -1)$	$(0, 1)$	$(1/2, -1)$

m_a^i: wrapping number on $(T^2)_i$
n_a^i: units of magnetic flux on $(T^2)_i$

$$\tilde{n}_\alpha^3 = n_\alpha^3 + \frac{1}{2} m_\alpha^3$$

The magnetic flux g on the first torus determines the generations of quark and lepton chiral multiplets in the visible sector

Pati–Salam–like model

- Tadpole cancellation conditions (D3–brane charge)

$$D3: \sum_a N_a n_a^1 n_a^2 \tilde{n}_a^3 + \frac{1}{2} N_{\text{flux}} = 8\,,$$

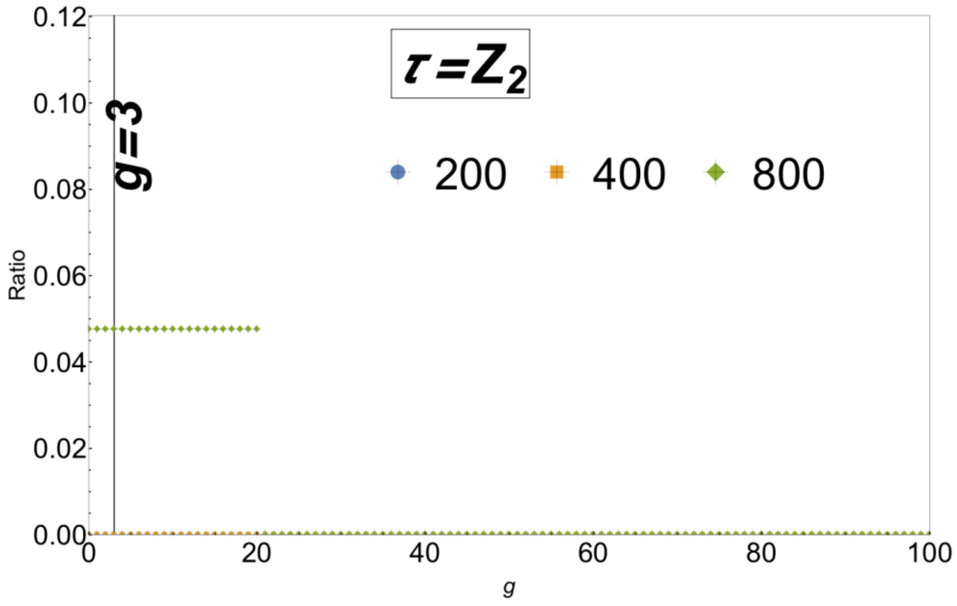
- The existence of magnetized D9–branes in the hidden sector

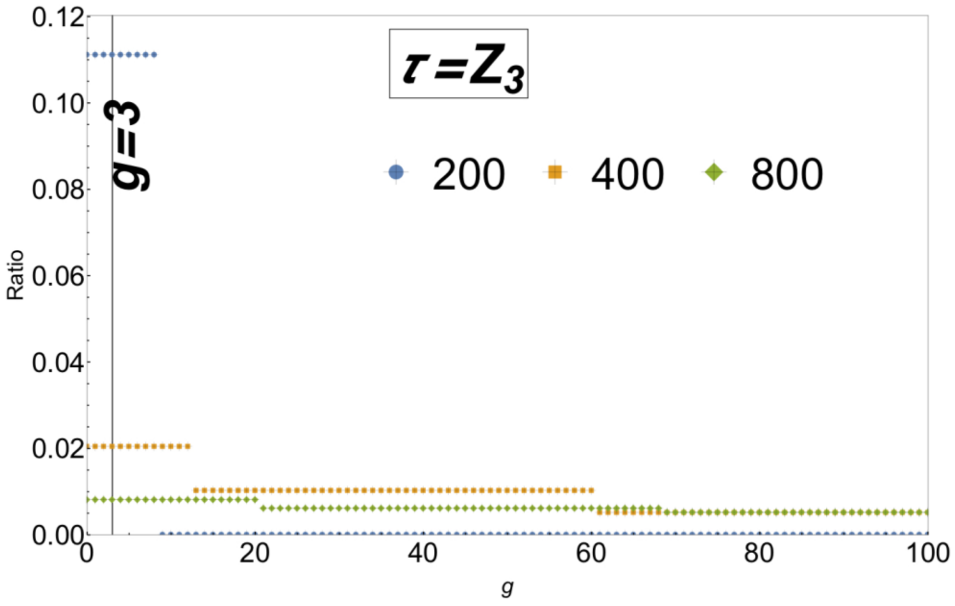
$$2g = -Q_{D3}^{\text{hid}} + 8 - \frac{N_{\text{flux}}}{2}\,,$$

From this equation, one can count the number of g generation models.

Pati–Salam–like model

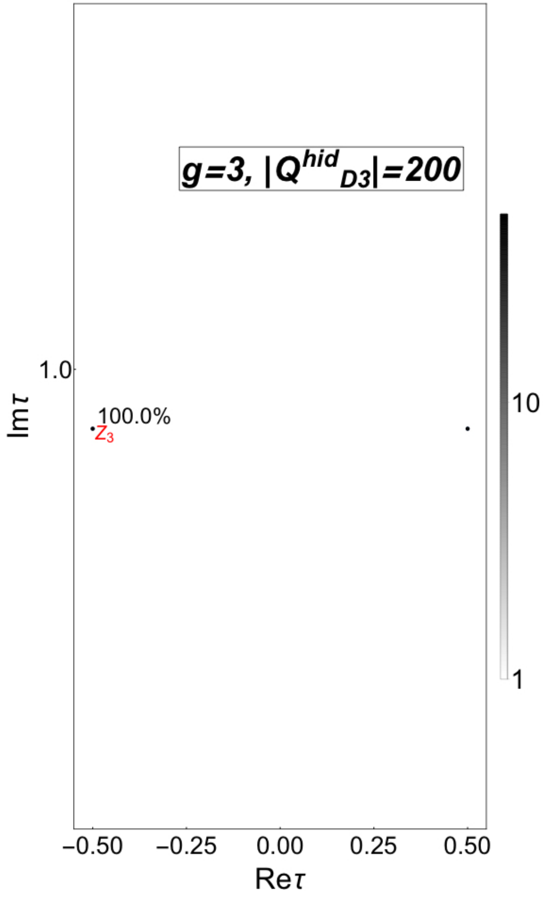
The numbers of models as a function of the generation number g at $\tau = i$ and $\tau = \omega$ respectively.

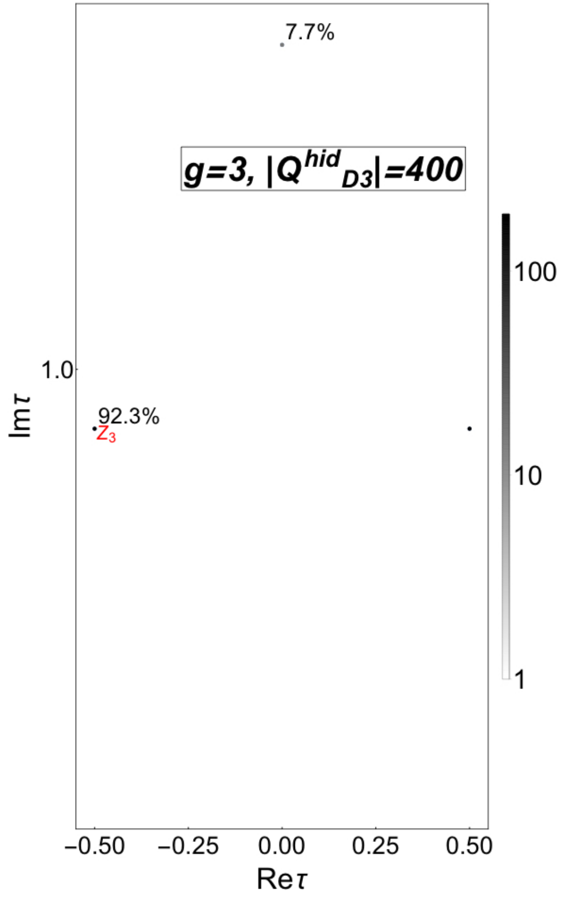




Pati–Salam–like model

The numbers of stable flux vacua with $g = 3$ generation of quarks/leptons on the fundamental domain of τ.





Metaplectic modular symmetry

Metaplectic group $Mp(2,\mathbb{Z})$: $\tilde{\Gamma}$

$$Mp(2,\mathbb{Z}) = \left\{\tilde{\gamma} = (\gamma, \varphi(\gamma,\tau)) \middle| \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \quad \varphi(\gamma,\tau)^2 = (c\tau + d)\right\}$$

Liu, Yao, Qu, Ding, 2007.13706

The generators of $Mp(2,\mathbb{Z})$

$$\tilde{S} = (S, -\sqrt{-\tau}) = \left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, -\sqrt{-\tau}\right), \qquad \tilde{T} = (T, -1) = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, 1\right),$$

$$\tilde{R} = \tilde{S}^2 = (S^2, -i) = \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, -i\right),$$

T^2 with magnetic fluxes

S transformation

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau) \to \psi^{\tilde{\alpha},|M|}\left(-\frac{z+\zeta}{\tau},-\frac{1}{\tau}\right) = (-\tau)^{1/2}\sum_{\tilde{\beta}=0}^{|M|-1}\frac{1}{\sqrt{|M|}}e^{i\pi/4}e^{2\pi i\frac{\tilde{\alpha}\tilde{\beta}}{|M|}}\psi^{\tilde{\beta},|M|}(z+\zeta,\tau),$$

Representation matrix

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}}\mathrm{e}^{i\pi/4}\mathrm{e}^{2\pi i\frac{\tilde{\alpha}\tilde{\beta}}{|M|}}, \qquad (\,\varphi(\gamma,\tau) = -\sqrt{-\tau}\,)$$

T^2 with magnetic fluxes

T transformation

$$\psi^{\tilde{\alpha},|M|}(z+\zeta,\tau)$$
$$\to \psi^{\tilde{\alpha},|M|}\left(z+\zeta+\frac{1}{2},\tau+1\right) = e^{i\pi|M|\frac{Im\,(z+\zeta)}{2Im\,\tau}} \sum_{\tilde{\beta}=0}^{|M|-1} e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)} \delta_{\tilde{\alpha},\tilde{\beta}}\, \psi^{\tilde{\beta},|M|}(z+\zeta,\tau),$$

Representation matrix

$$\rho(\tilde{T})_{\tilde{\alpha}\tilde{\beta}} = e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)} \delta_{\tilde{\alpha},\tilde{\beta}}, \qquad (\,\varphi(\gamma,\tau)=1\,)$$

T^2 with magnetic fluxes

Redefinition of the representation matrix (S transformation)

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}} e^{i\pi\frac{3|M|+1}{4}} e^{2\pi i\frac{\tilde{\alpha}\tilde{\beta}}{|M|}},$$

(Additional factor $e^{3i\pi|M|/4}$ $(I_{ab} + I_{bc} + I_{ca} = 0)$)

The representation matrix $\rho(\tilde{\gamma})$ satisfies following conditions.

$$\rho(\tilde{S})^2 = \rho(\tilde{R}), \qquad \left(\rho(\tilde{S})\rho(\tilde{T})\right)^3 = \rho(\tilde{R})^4 = \mathbb{I},$$

$$\rho(\tilde{T})\rho(\tilde{R}) = \rho(\tilde{R})\rho(\tilde{T}), \qquad \rho(\tilde{T})^{2|M|} = \mathbb{I}$$

Metaplectic modular symmetry

The generators of $\tilde{\Gamma}_{4N}$ satisfy the following relations.

$$\tilde{S}^2 = \tilde{R}, \qquad (\tilde{S}\tilde{T})^3 = \tilde{R}^4 = \mathbb{I}, \qquad \tilde{T}\tilde{R} = \tilde{R}\tilde{T}, \qquad \tilde{T}^{4N} = \mathbb{I},$$

For $N > 1$, additional relations are required to ensure the finiteness.

$$\tilde{S}^5\tilde{T}^6\tilde{S}\tilde{T}^4\tilde{S}\tilde{T}^2\tilde{S}\tilde{T}^4 = \mathbb{I}, \qquad (\text{for } N = 2),$$

($\tilde{\Gamma}_8$ is the discrete group of order 768)

<u>Representation matrix</u>

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}} e^{i\pi\frac{3|M|+1}{4}} e^{2\pi i\frac{\tilde{\alpha}\tilde{\beta}}{|M|}}, \qquad \rho(\tilde{T})_{\tilde{\alpha}\tilde{\beta}} = e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)}\delta_{\tilde{\alpha},\tilde{\beta}}$$

(Additional factor $e^{3i\pi|M|/4}$ $(I_{ab} + I_{bc} + I_{ca} = 0)$)

Almumin, Chen, Knapp-Pérez, Ramos-Sánchez, Ratz, Shukla 2102.11286

Pati–Salam–like model on $T^6/Z_2 \times Z_2'$

$\mathbb{Z}_2 \times \mathbb{Z}'_2$ projection

$$\theta : \psi(z_1, z_2, z_3) \rightarrow s_1 s_2 \psi(-z_1, -z_2, z_3),$$
$$\theta : \psi(z_1, z_2, z_3) \rightarrow s_2 s_3 \psi(z_1, -z_2, -z_3),$$

$(s_i = \mathrm{sign}(I^i_{ab}))$

The number of $\mathbb{Z}_2$–even and –odd zero modes

$$I^i_{even} = \frac{1}{2}(I^i_{ab} + s_i f_i), \qquad I^i_{odd} = \frac{1}{2}(I^i_{ab} - s_i f_i),$$

$(f_i = 1 \text{ for odd}, \ \ f_i = 2 \text{ for even})$

$$I_{ab} = \prod_{i=1}^{3}(I^i_{\mathrm{even}} + I^i_{\mathrm{odd}}),$$

Three generations are realized in $\mathbb{Z}_2$–even zero mode. $(|I_{ab}| = g = 4)$

$T^2/\mathbb{Z}_2$ with magnetic fluxes

The modular transformations on $T^2/\mathbb{Z}_2$ orbifold

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}} \mathrm{e}^{i\pi\frac{3|M|+1}{4}} \cos\left(\frac{2\pi\tilde{\alpha}\tilde{\beta}}{|M|}\right), \qquad \rho(\tilde{T})_{\tilde{\alpha}\tilde{\beta}} = e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)} \delta_{\tilde{\alpha},\tilde{\beta}}$$

($\mathbb{Z}_2$−even mode with $\tilde{\alpha}, \tilde{\beta} = 0,1, \ldots I_{\mathrm{even}}$)

$$\rho(\tilde{S})_{\tilde{\alpha}\tilde{\beta}} = -\frac{1}{\sqrt{|M|}} \mathrm{e}^{i\pi\frac{3|M|+1}{4}} \sin\left(\frac{2\pi\tilde{\alpha}\tilde{\beta}}{|M|}\right), \qquad \rho(\tilde{T})_{\tilde{\alpha}\tilde{\beta}} = e^{i\pi\,\tilde{\alpha}\left(\frac{\tilde{\alpha}}{|M|}+1\right)} \delta_{\tilde{\alpha},\tilde{\beta}}$$

($\mathbb{Z}_2$−odd mode with $\tilde{\alpha}, \tilde{\beta} = 0,1, \ldots I_{\mathrm{odd}}$)

Generalized CP

The CP transformation

$$\begin{pmatrix} e_2 \\ e_1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \bar{e}_2 \\ \bar{e}_1 \end{pmatrix}.$$

The CP transformation enlarges $SL(2,\mathbb{Z})$ modular group to $GL(2,\mathbb{Z}) \simeq SL(2,\mathbb{Z}) \rtimes \mathbb{Z}_2^{CP}$.

Conditions to be satisfied

$$CP^2 = 1, \qquad (CP)S(CP)^{-1} = S^{-1}, \qquad (CP)T(CP)^{-1} = T^{-1}$$

Flavor symmetry and Modular symmetry

Traditional flavor group $(A_4, S_4 \ldots)$: F

Finite modular group $(\Gamma_N, \tilde{\Gamma}_{4N}, \ldots)$: M

When both representations act on same space ⋯

$$\rho(F)\,\rho(M)\,\psi_{\pm}^{\tilde{\alpha},|M|}(z+\zeta,\tau) \overset{?}{=} \rho(M)\,\rho(F)\,\psi'^{\tilde{\alpha}',|M|}_{\pm}(z+\zeta,\tau)$$

The nontrivial relation between flavor symmetry and modular symmetry

Eclectic Flavor Symmetry (models)

Traditional flavor symmetry ($G_{\text{flavor}} \equiv \mathbb{Z}_4 \times \mathbb{Z}_2^P \times \mathbb{Z}_2^C \times \mathbb{Z}_2^Z$)

$\mathbb{Z}_2$−even mode

$$\rho(Z'_{\text{even}}) = i\mathbb{I}_3, \qquad \rho(P_{\text{even}}) = \mathbb{I}_3, \qquad \rho(C_{\text{even}}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \rho(Z_{\text{even}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\mathbb{Z}_2$−odd mode

$$\rho(Z'_{\text{odd}}) = i, \qquad \rho(P_{\text{odd}}) = -1, \qquad \rho(C_{\text{odd}}) = -1, \qquad \rho(Z_{\text{odd}}) = -1,$$

Eclectic Flavor Symmetry (models)

Generalized CP group ($G_{\text{CP}} \equiv \mathbb{Z}_2^{\text{CP}}$)

The CP transformation of matter wavefunction on T^2

$$\psi^{\tilde{\alpha},|M|} \to \overline{\psi^{\tilde{\alpha},|M|}(z,\tau)},$$

$$\psi^{\tilde{\alpha},|M|} \to \varphi(\widetilde{CP},\tau)\rho(\widetilde{CP})_{\tilde{\alpha}\tilde{\beta}}\overline{\psi^{\tilde{\beta},|M|}(z,\tau)},$$

with

$$\varphi(\widetilde{CP},\tau) = i, \qquad \rho(\widetilde{CP})_{\tilde{\alpha}\tilde{\beta}} = -i\delta_{\tilde{\alpha},\tilde{\beta}}.$$

Eclectic Flavor Symmetry (models)

Finite metaplectic modular group ($G_{\text{modular}} \equiv \tilde{\Gamma}_8$)

The explicit representations of the modular group on $T^2/\mathbb{Z}_2$ ($M = 4$)

$$\rho(\tilde{S}_{\text{even}}) = -\frac{1}{2}\begin{pmatrix} (-1)^{1/4} & 1+i & (-1)^{1/4} \\ 1+i & 0 & -1-i \\ (-1)^{1/4} & -1-i & (-1)^{1/4} \end{pmatrix}, \qquad \rho(\tilde{T}_{\text{even}}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -(-1)^{1/4} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\rho(\tilde{S}_{\text{odd}}) = (-1)^{3/4}, \qquad \rho(\tilde{T}_{\text{odd}}) = -(-1)^{1/4},$$