Lefschetz-thimble analysis of the Lorentzian IKKT matrix model around its classical solutions

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Introduction

- Ultimate goal: numerically(nonperturbative) investigating the emergence of (3+1)D expanding spacetime. Lorentzian IKKT matrix model is a promising candidate.

$$Z = \int dA d\psi d\bar{\psi} e^{i(S_b+S_f)}$$

(not absolutely convergent)

$$S_b = -\frac{N}{4}\mathrm{tr}\{[A_\mu, A_\nu][A^\mu, A^\nu]\}$$

A_μ(0,1,...9) and ψ_α(α=1,2....16) are N×N Hermitian matrices

$$S_f = -\frac{N}{2}\mathrm{tr}\{\bar{\psi}_\alpha(\Gamma^\mu)_{\alpha\beta}[A_\mu, \psi_\beta]\}$$

- Integrand involves a pure phase factor e^{iS_b}, usual Monte Carlo methods is not applicable. This is the sign problem!

- Numerical simulation is difficult!

- In this talk: We study N=2 bosonic case of the model using generalized Lefschetz thimble method.

- N=2 model → a prototype of emerging space-time + some nice analytical predictions from 1/D expansion (N.Yamamori's talk)

Plan of the talk

1. Introduction

2. How to deal with the Lorentzian IKKT matrix model (numerically)

3. Results for the N=2 bosonic model

4. Summary and future prospects

Regularization and sign problem

partition function

Kim-JN-Tsuchiya Phys.Rev.Lett. 108 (2012) 011601,1108.1540 [hep-th]

$$Z_L = \int dA\, d\psi\, e^{i(S_b+S_f)} = \int dA\, e^{iS_b} \mathrm{Pf} M(A)$$

polynomial in A
real valued unlike Euclidean

pure phase factor
(oscillating weight)

(model not well defined as it is) | (sign problem)

As a regularization, we introduce a Lorentz invariant mass term

$$Z = \int dA e^{i(\mathrm{S_b}+\mathrm{S_m})} \mathrm{Pf} M(A)$$

$$S_m = \frac{1}{2} N\gamma \{ e^{i\epsilon} \mathrm{tr}(A_0)^2 - e^{-i\epsilon} \mathrm{tr}(A_i)^2 \}$$

convergence factor

Anagnostopoulos-Azuma-Hatakeyama-Hirasawa-J.N.-Papadoudis-Tsuchiya, work in progress

we use generalized Lefschetz thimble method

deform the integration contour

$$\frac{\partial}{\partial \sigma} z_k(x,\sigma) = \overline{\frac{\partial S(z(x,\sigma))}{\partial z_k}}$$

fluctuation of action supresses along the flow

large flow time: milder sign problem

Sign of the mass term

$$Z = \int dA e^{-S_{eff}},$$

partition function we study using GTM

no fermions!

$$S_{eff} = -iN\left(\frac{1}{2}\mathrm{tr}[\mathrm{A}_0, \mathrm{A}_\mathrm{I}]^2 - \frac{1}{4}\mathrm{tr}[\mathrm{A}_\mathrm{I}, \mathrm{A}_\mathrm{J}]^2\right) - \frac{i}{2}\underline{N\gamma\{e^{i\epsilon}\mathrm{tr}(A_0)^2 - e^{-i\epsilon}\mathrm{tr}(A_I)^2\}}$$

(mass term)

(sign of γ becomes crucial)

$\gamma < 0, \epsilon \to -0$	Euclidean model(SO(10) symmetry)	$Z < \infty$ (finite)
$\gamma > 0, \epsilon \to +0$	$\neq$ Euclidean (leads to unbounded action)	$Z = \infty$ (divergence) (in $\varepsilon \to 0$ limit)

Classical solutions for N=2 model

(W.Piensuk's talk)

$\gamma < 0$	$A_\mu = 0$		
$\gamma > 0$	$A_\mu = 0$	$A_\mu = \begin{cases} \sqrt{\frac{\gamma}{2}}\sigma_\mu & \mu = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$	$A_\mu = \begin{cases} \sqrt{\gamma}\sigma_\mu & \mu = 1,2 \\ 0 & \text{otherwise} \end{cases}$
	(trivial solution)	(Pauli solution)	(squashed Pauli solution)

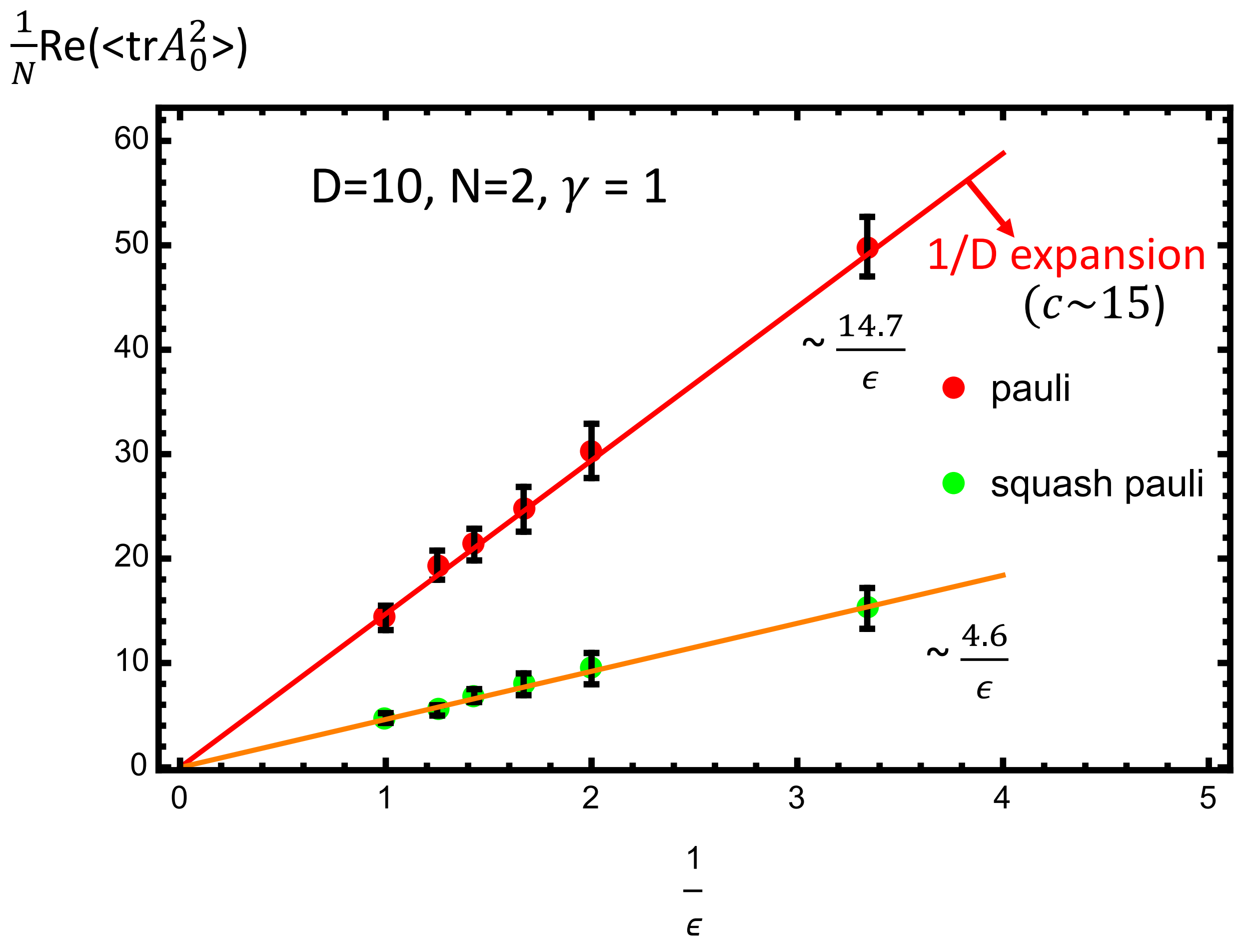
Divergence of the partition function

- **Initial configuration** :

  A_μ = Pauli
  = Squashed pauli

- **Results obtained by the GTM**:

  (sample configurations only on the thimble associated with the Pauli or squashed Pauli solutions)



$$\langle \frac{1}{N}\mathrm{tr}(A_0)^2\rangle \sim \frac{c}{\epsilon} \qquad \text{(in } \epsilon \to 0)$$

$$\langle \frac{1}{N}\mathrm{tr}(A_0)^2\rangle \sim -\frac{\partial}{\partial\epsilon}\log Z \Rightarrow Z \sim \epsilon^{-c}$$

(W.Piensuk's talk)

Pauli	c ~14.7
Squashed pauli	c ~ 4.6

$$\langle -\frac{1}{N}\mathrm{tr}(A_0)^2 + \frac{1}{N}\mathrm{tr}(A_i)^2\rangle = (\text{finite}) \sim \frac{3}{4}\gamma \quad \text{(large } \gamma)$$

(in $\epsilon \to 0$)

Partition function diverges faster for Pauli thimble

Pauli thimble dominates in N=2 bosonic model at $\gamma > 0$.

Divergence due to the non-compactness of Lorentz symmetry group is confirmed.

Transition at finite ϵ

Results for the pauli thimble:

(N.Yamamori's talk)

v^+ and new → Pauli solution

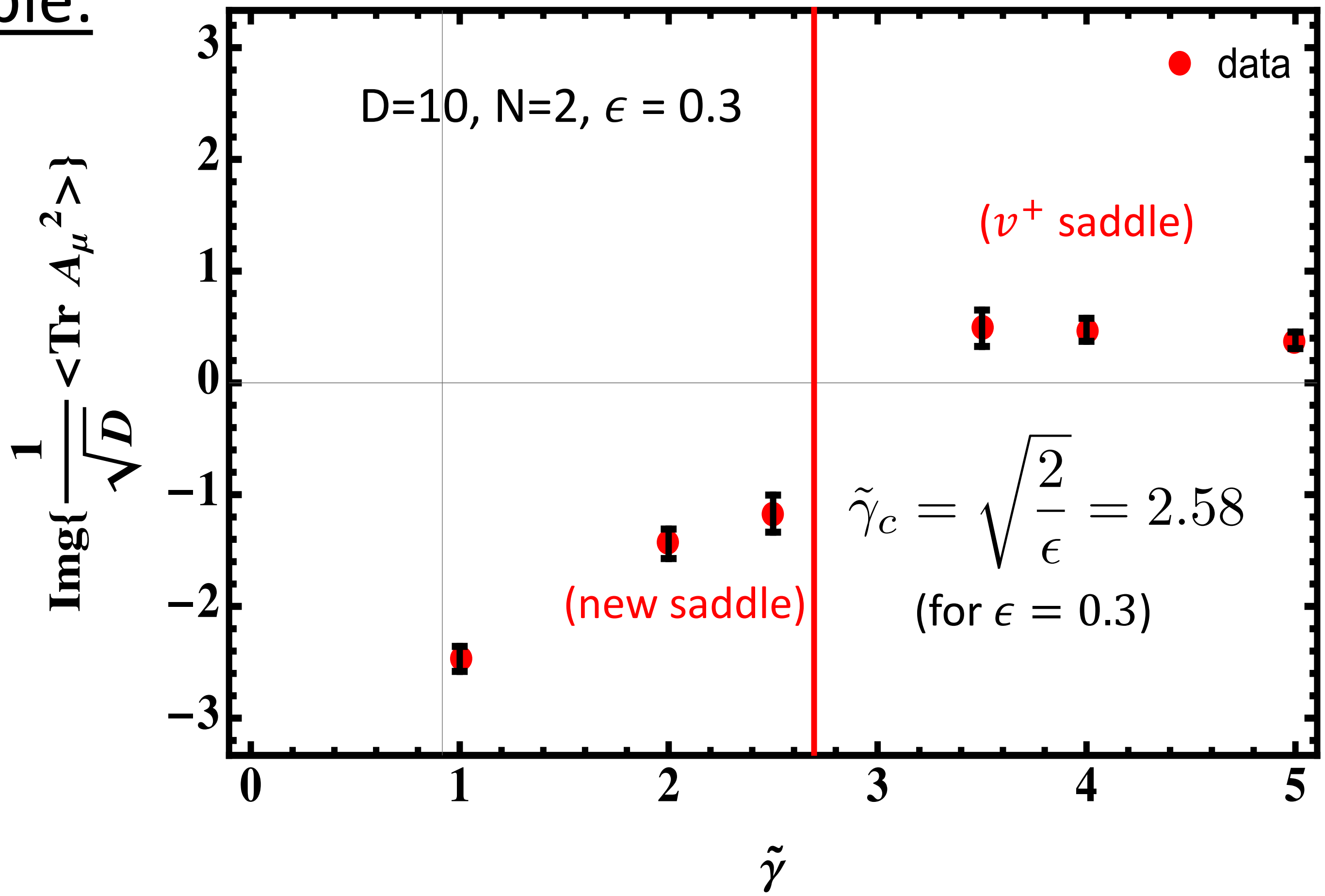
h-theory saddles (1/D analysis)

$$Z(v^{new}) \sim (\tilde{\gamma}\epsilon)^{-\frac{3}{2}D}$$

$$Z(v^+) \sim \left(\frac{\tilde{\gamma}}{2}\right)^{\frac{3}{2}D}$$

$$|Z_{new}| > |Z_{v^+}|$$

$$\text{for} \quad \tilde{\gamma} < \tilde{\gamma}_c \equiv \sqrt{\frac{2}{\epsilon}} \quad \text{(large D)}$$



$$\tilde{\gamma} < \tilde{\gamma}_c \rightarrow \text{new}$$

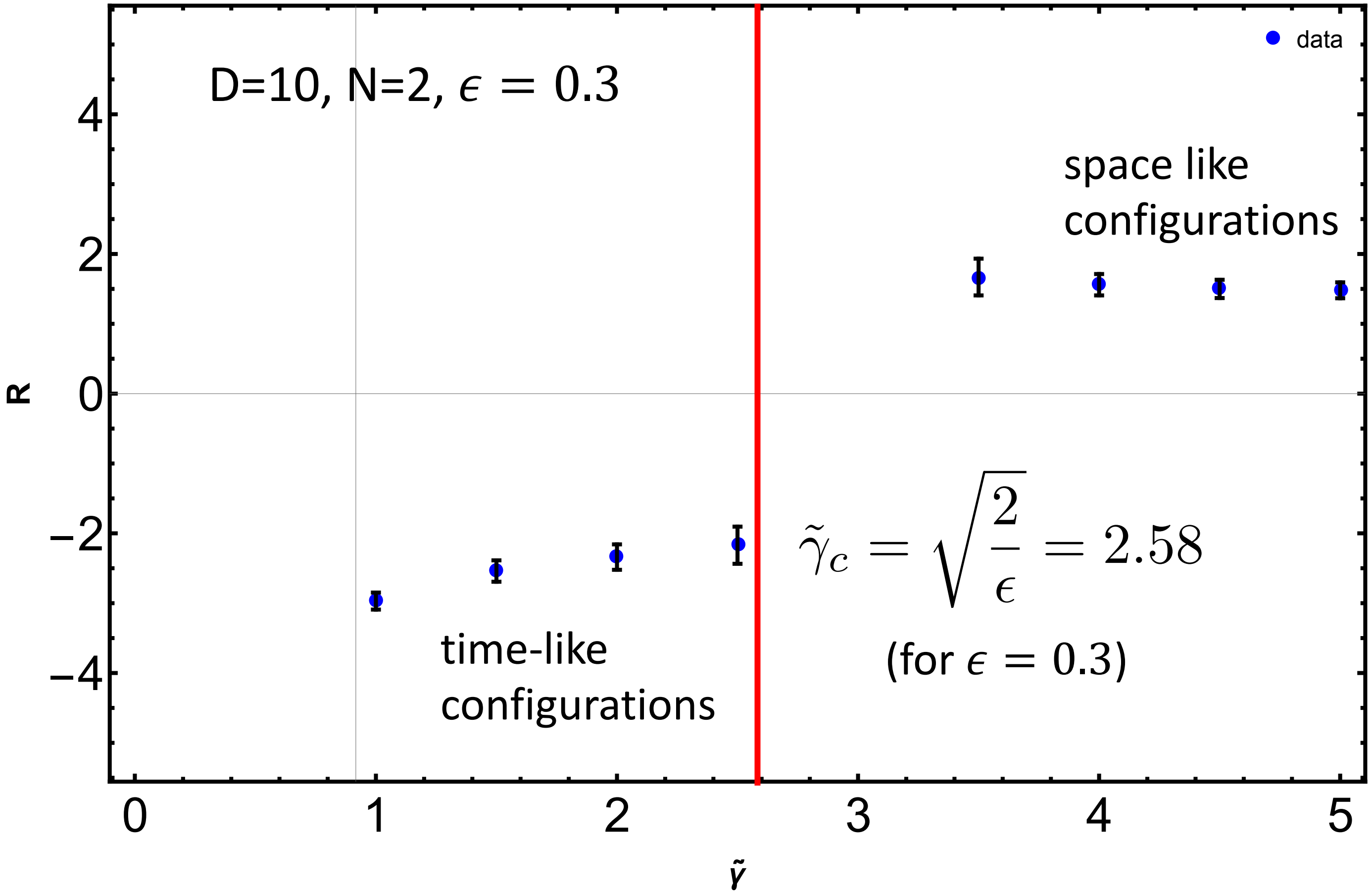
$$\tilde{\gamma} > \tilde{\gamma}_c \rightarrow v^+$$

Thimble calculations confirmed the transition predicted from 1/D expansion analysis for Pauli thimble.

What causes the divergence?

$$R = -\mathrm{tr}(A_0^\dagger A_0) + \mathrm{tr}(A_i^\dagger A_i)$$

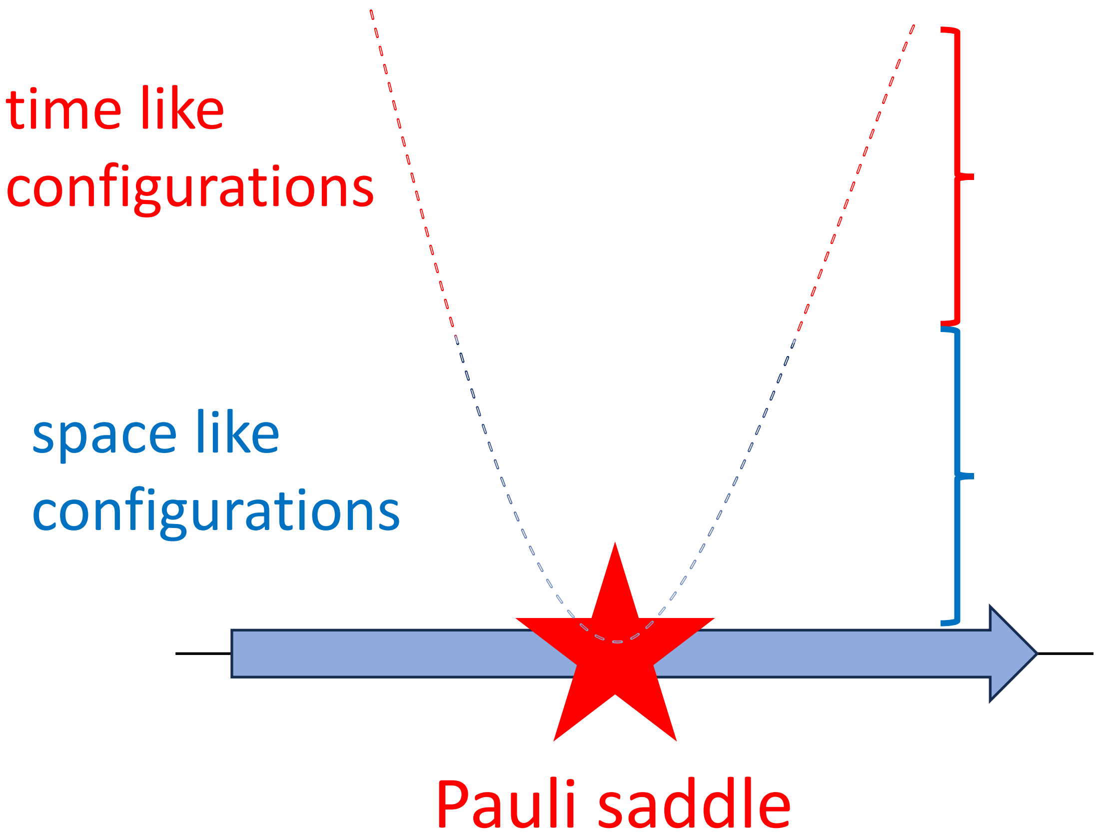
{ R<0 time like configurations
R>0 space like configurations }



Divergence is caused by time-like configurations

dominant saddle		dominant config.
new saddle	$\tilde{\gamma} < \tilde{\gamma}_c$	time-like (R<0)
$v^{(+)}$saddle	$\tilde{\gamma} > \tilde{\gamma}_c$	space-like(R>0)

The interpretation of the transition



Non-compactness of Lorentz symmetry- causes divergence

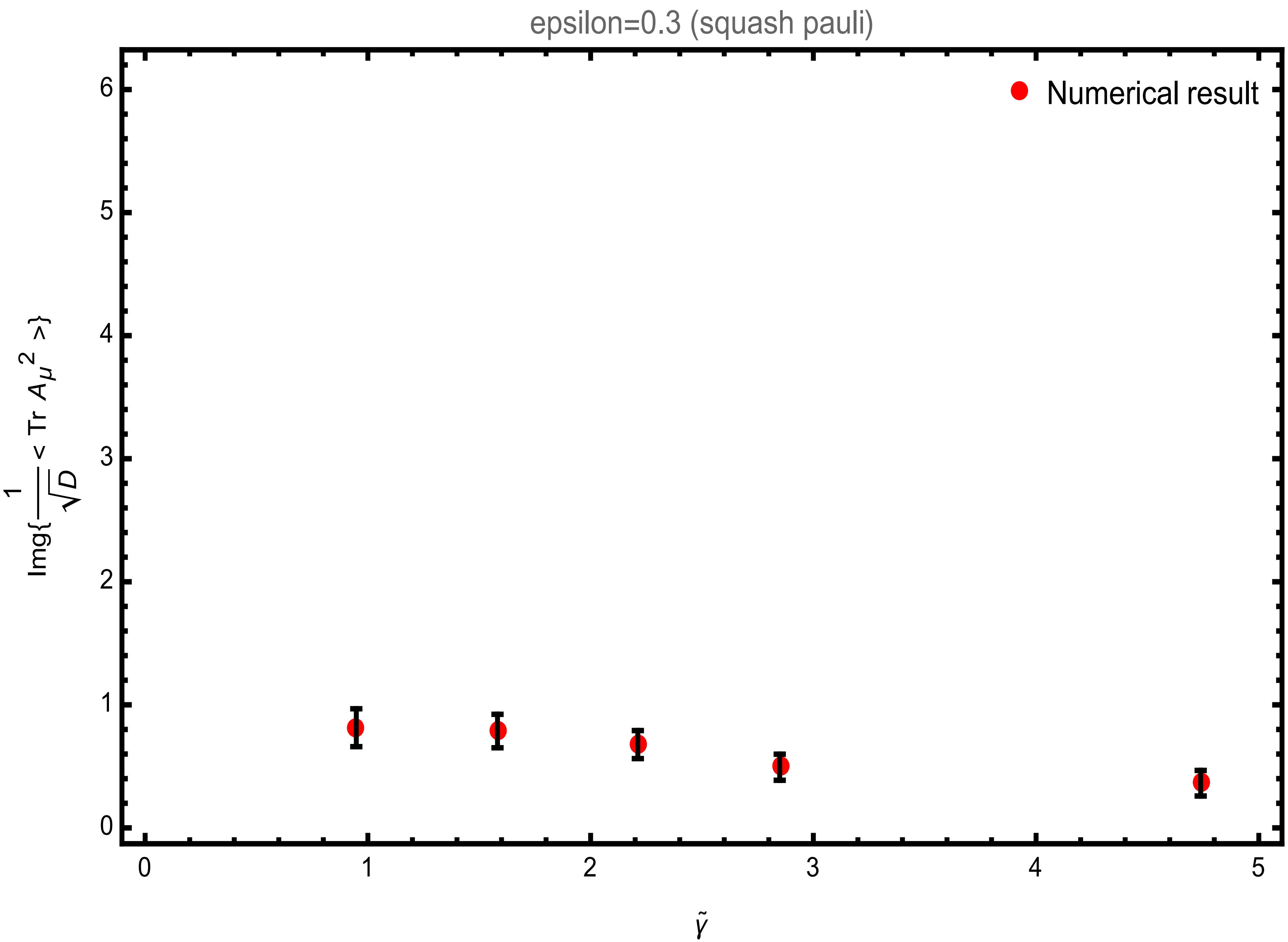
Non-compactness of Lorentz symmetry- does not cause divergence

$\epsilon \to 0$ makes time like configurations dominate.
$\tilde{\gamma} \to \infty$ makes space-like configurations dominate.

Transition point
$\tilde{\gamma}^2 \epsilon = 2$
(large D)

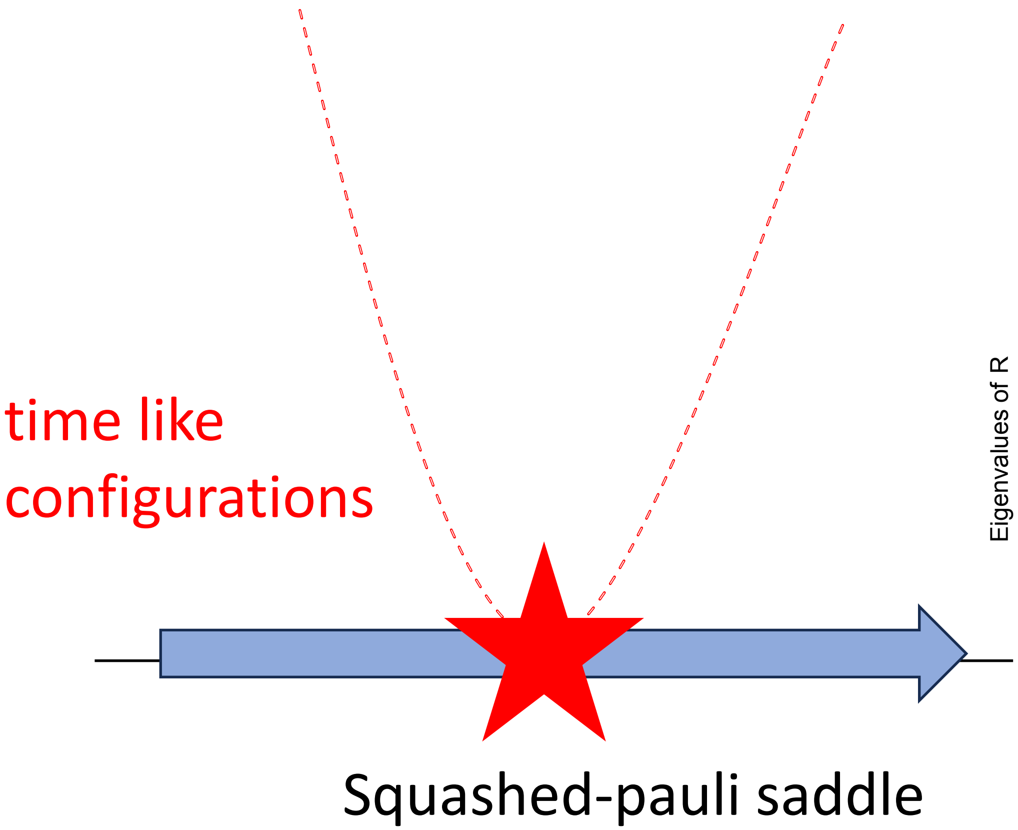
SO(D) symmetric model doesn't show these properties (truly of Lorentzian nature)

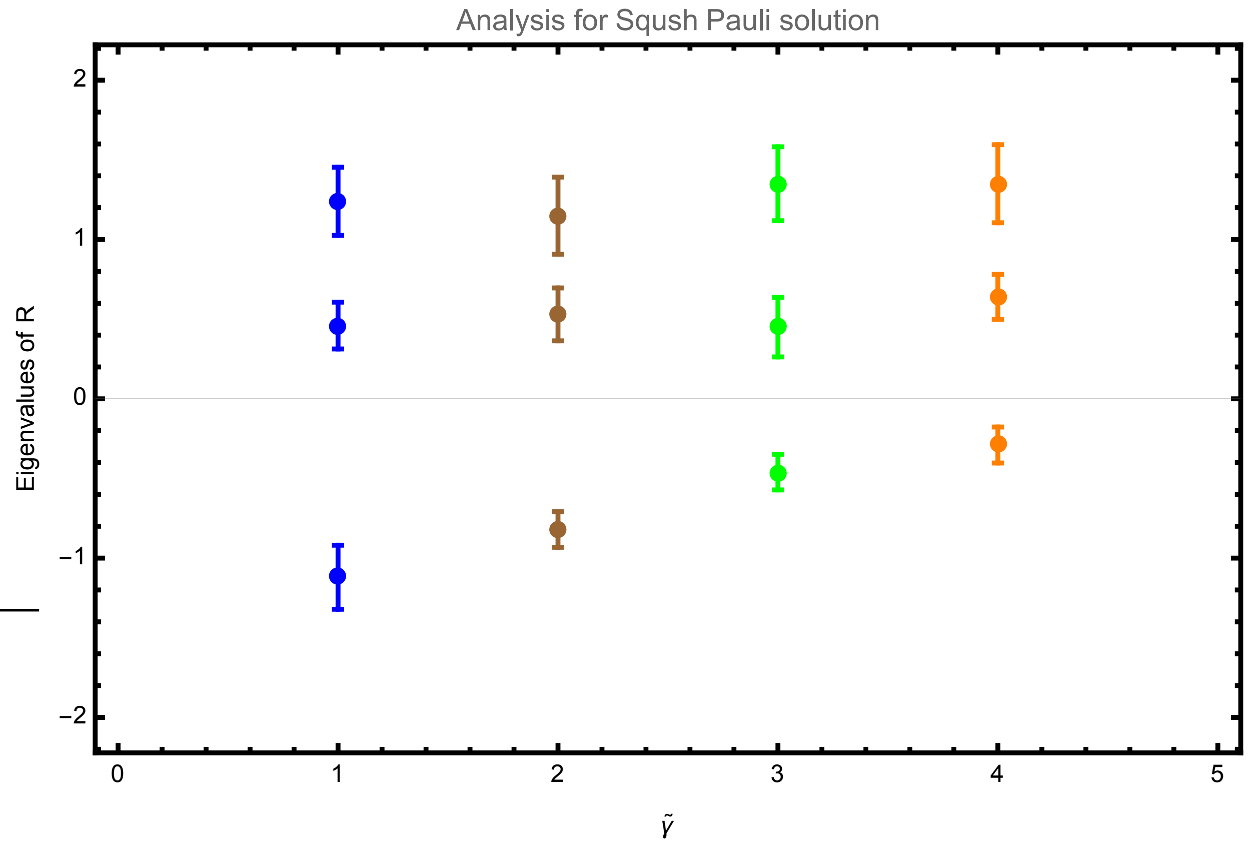
No transition for the squashed pauli solution!



Results for the squashed Pauli thimble

$$R_{ab} = -(A_0^a)^*(A_0^b) + \sum_{i=1}^{D-1}(A_i^a)^*(A_i^b)$$





configurations near squashed Pauli always have one time-like component, since it has one shrunken direction unlike Pauli.

Summary

- Lorentzian IKKT matrix model is not well defined as it is and suffers from severe sign problem.
- We regularized the model by adding a Lorentz invariant mass term. The generalized thimble method enabled us to solve the sign problem.
- The mass term allows interesting classical solutions to appear for $\gamma>0$.
- In the N=2 bosonic case, we show that partition function associated with non-trivial saddles diverges due to the non-compactness of the Lorentz symmetry.
- The divergence occurs due to the "time-like configurations" and it turns out to be stronger for the Pauli solution.

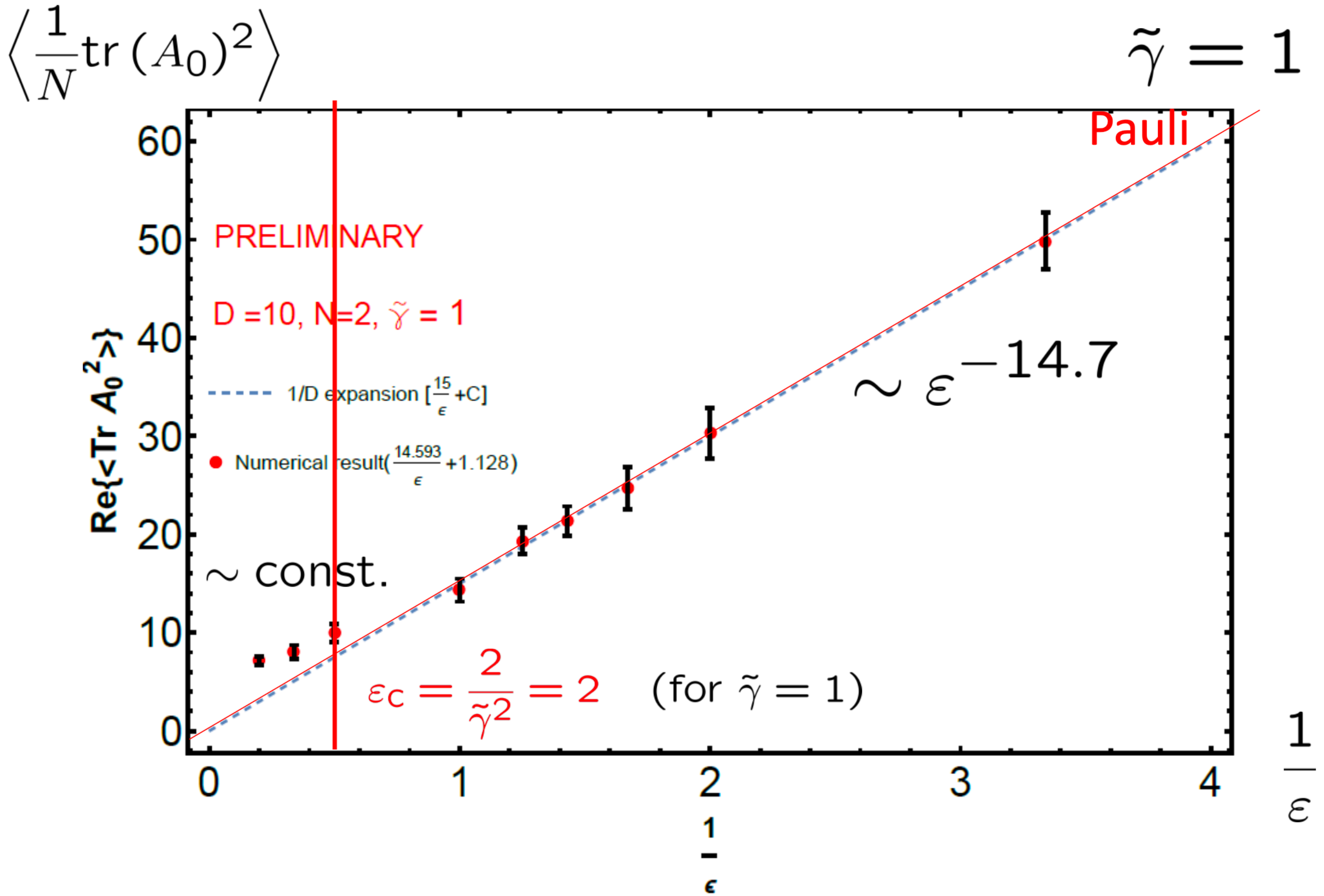
Future work

- SUSY - impact of SUSY, simulations are doable (N=2 case is on-going)
- larger N - Computational cost in generalized Lefschetz thimble method grows with N as O(N^6). But we may still do N=4,8,16.....

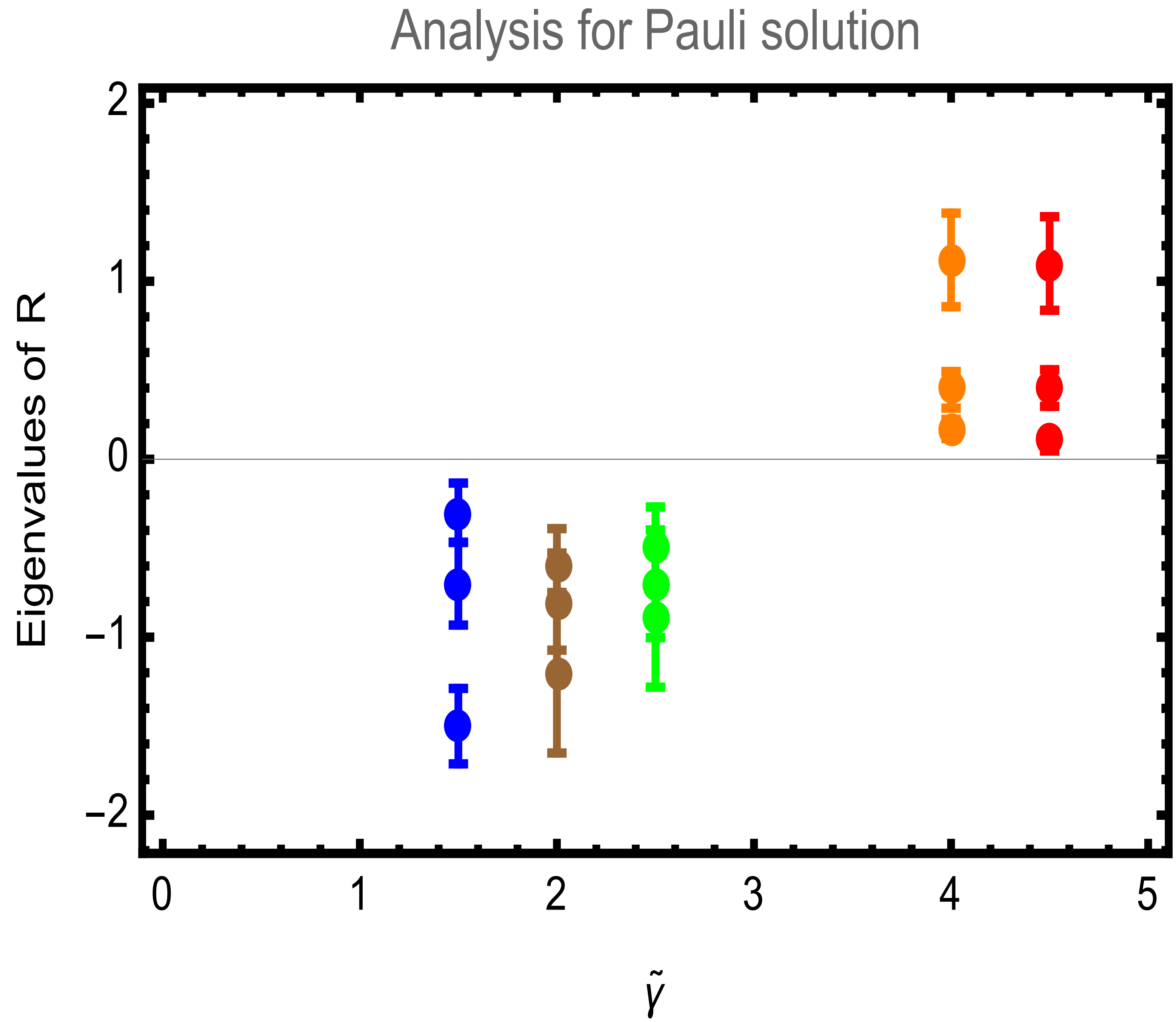
Thank you so much for your attention

Backup slide

No divergence for space-like configs.



For $\tilde{\gamma} = \infty$, we expect to see $\left\langle \frac{1}{N} \mathrm{tr}\,(A_0)^2 \right\rangle \to$ const. as $\varepsilon \to 0$.


Analysis for Pauli solution
Eigenvalues of R
2
1
0
−1
−2
0
1
2
3
4
5
$\tilde{\gamma}$