Ground state degeneracy and module category

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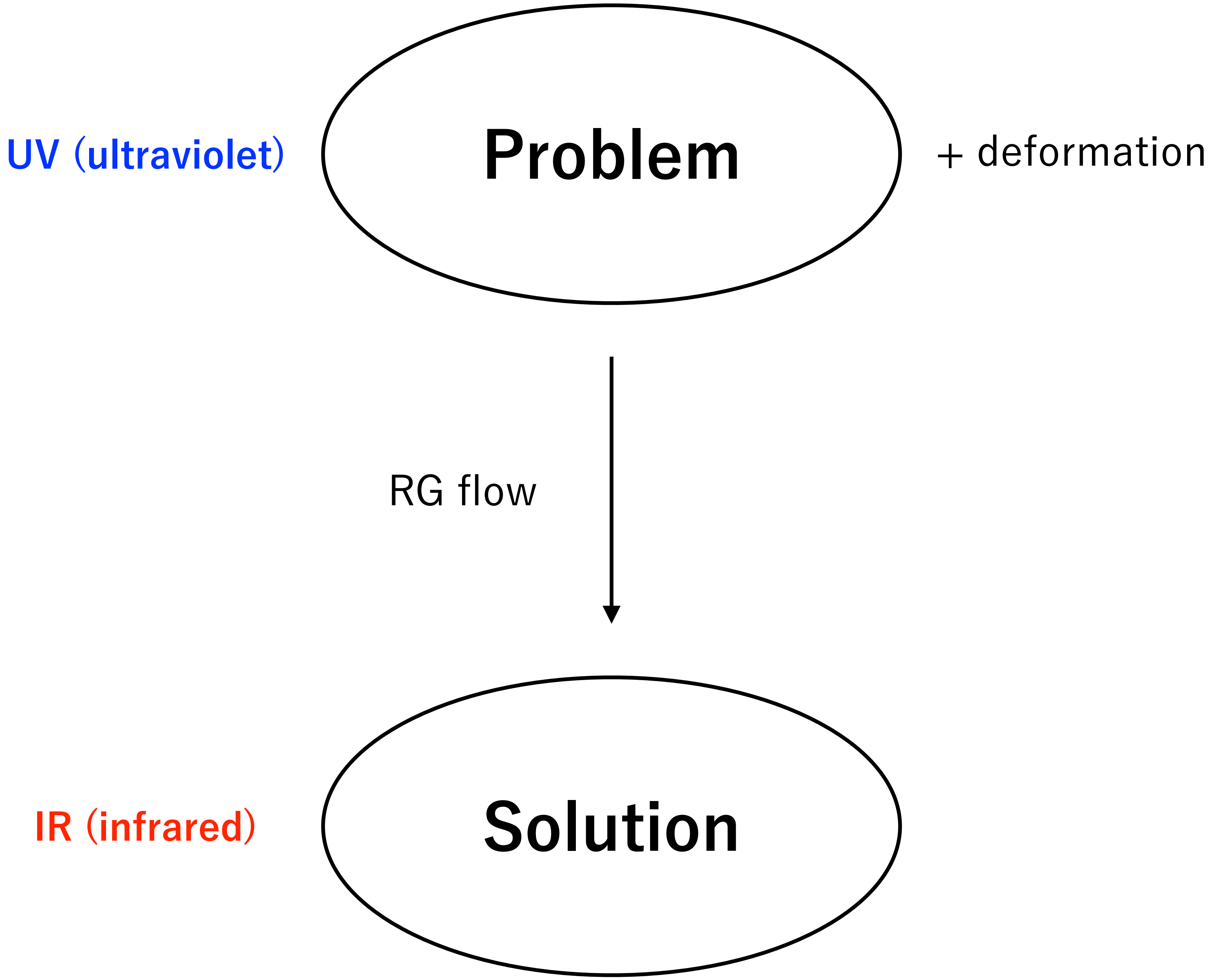
Based on

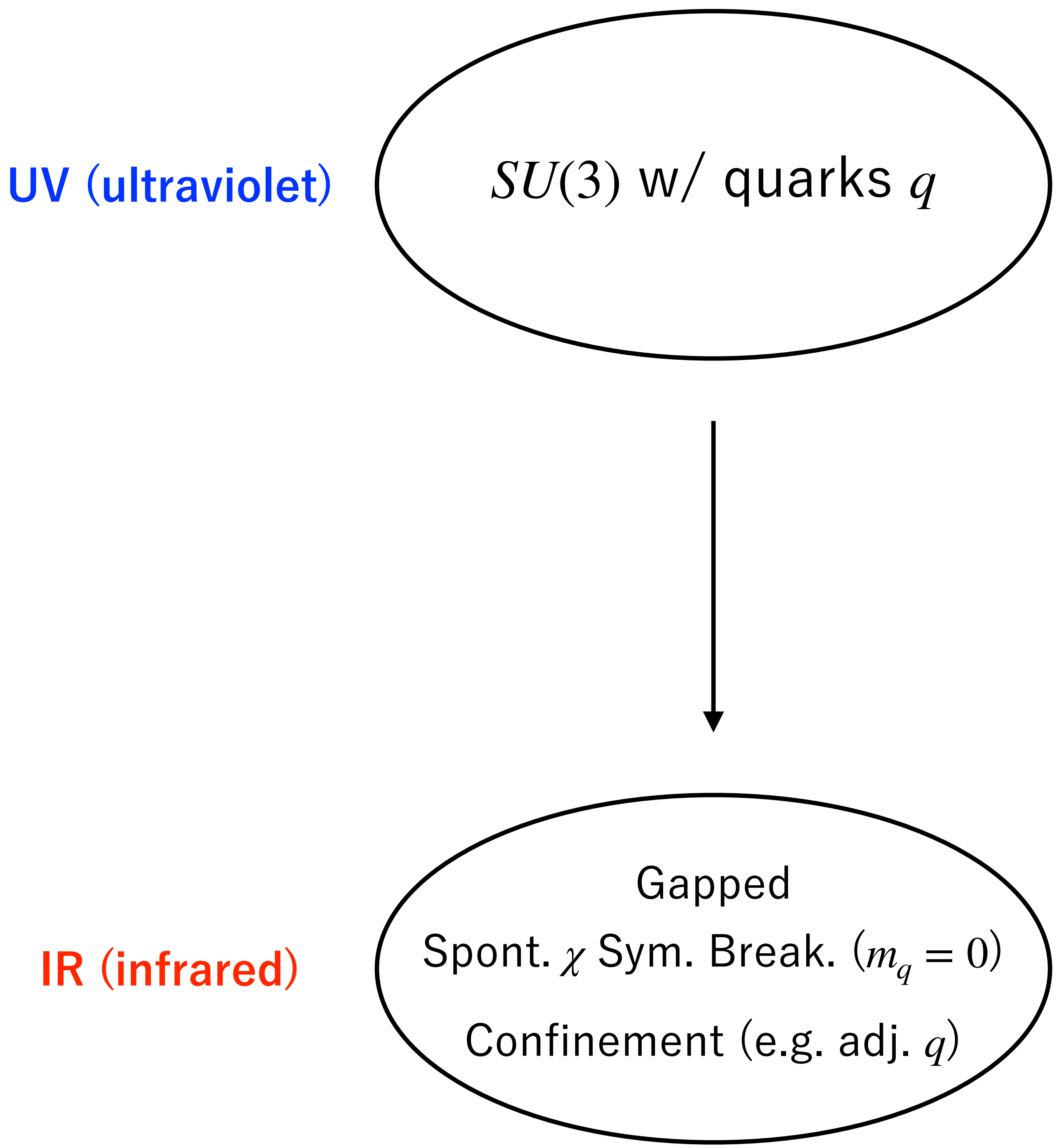
2311.00746 (KK)

2311.15631 (KK)

Renormalization Group (RG) flow



Example: Quantum ChromoDynamics



Possible answers

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved		
Spont. broken		

Example: QCD

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved	Confinement	
Spont. broken	SχSB	

This talk

Symmetry\Gap	Gapped (or TQFT)	Gapless (~CFT)
Preserved		
Spont. broken		

This talk

Goal: Prove SSB

Content

1. Module category

2. Results

Content

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2. Results

Module category

Generalized symmetry

$=$fusion category C

Module category

Generalized symmetry

$=$fusion category C

It is equipped with

$$\otimes\,,\,\oplus\,.$$

(w/ consistency conditions)

Module category

In 2d space(time), we have 1-to-1 correspondence

$$\{\mathcal{C}\text{-symmetric gapped phases}\}$$

$$\cong \{\mathcal{C}\text{-module categories } M\}.$$

[Thorngren-Wang '19]
[Huang-Lin-Seifnashri '21]

Module category

In 2d space(time), we have 1-to-1 correspondence

$$\{C\text{-symmetric gapped phases}\}$$

$$\cong \{C\text{-module categories } M\}.$$

[Thorngren-Wang '19]
[Huang-Lin-Seifnashri '21]

This especially implies

$$\text{GSD} = \text{rank}(M).$$

$\text{rank}(M)$:= # of (simple) objects in M

Module category

Physical implications:

- Classification of C-module categories gives possible GSDs.

- In particular, if $\nexists$ C-module category w/ rank 1,

C is spontaneously broken.

Module category: definition

$\mathcal{C}$-module category

:= `representation of $\mathcal{C}$'

Module category: definition

Representation:

Let G be a group. A rep. of G is a homomorphism

$$\rho : G \to \text{actions on vector sp. } V.$$

Concretely, $\rho(g) \in GL(V)$ sends $v \in V$ to $\rho(g)v \in V$.

Module category: definition

Representation:

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$$\rho : G \to \text{actions on vector sp. } V.$$

Concretely, $\rho(g) \in GL(V)$ sends $v \in V$ to $\rho(g)v \in V$.

Module category:

Let C be a fusion category. A (left) module category M has

$$\triangleright : C \times M \to M.$$

Concretely, $c \in C$ sends $m \in M$ to $c \triangleright m \in M$.

Module category: definition

An important module category:

Any $\mathcal{C}$ has $\mathcal{C}$ itself as a $\mathcal{C}$-module category defined by

$$\triangleright := \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

It is called the regular module category $\mathcal{M} \simeq \mathcal{C}$.

Content

1. Module category

2. Results

How to classify module categories

Method: [2311.00746 (KK)]

1. find **upper bound** on rank(M),
2. list up candidates,
3. check which satisfy axioms.

Upper bound from FPdim

Example: Ising fusion category (FC)

Ising FC$=\{1,\eta,N\}$

i.e., rank=3

It has (fusion) product

$$\eta\otimes\eta=1,\quad \eta\otimes N=N=N\otimes\eta,\quad N\otimes N=1\oplus\eta\,.$$

Upper bound from FPdim

Example: Ising FC

The product $\eta \otimes \eta = 1, \quad \eta \otimes N = N = N \otimes \eta, \quad N \otimes N = 1 \oplus \eta$

is described by (fusion) matrices (in the basis $\{1, \eta, N\}$)

Definition. $(N_j)_{kl} = 1$ if $j \otimes k = l \oplus \cdots$

$$N_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad N_\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Upper bound from FPdim

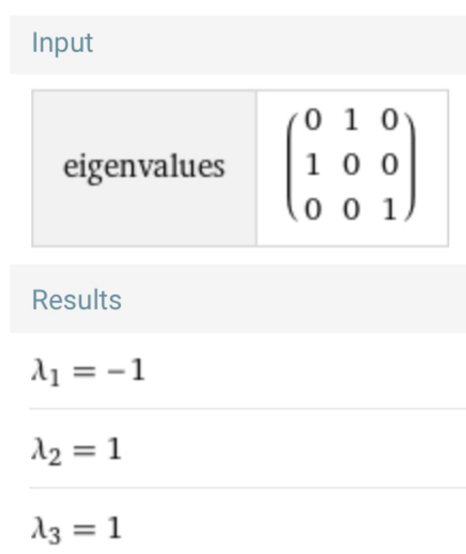
Frobenius-Perron dimension $\mathrm{FPdim}_C(j)$ of N_j (or j)

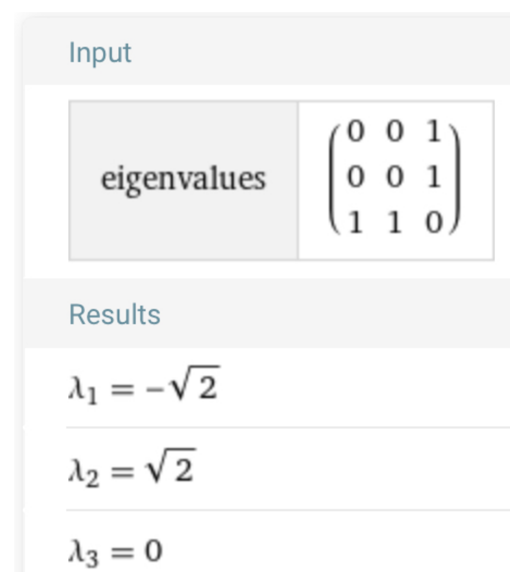
:=largest eigenvalue of N_j

Upper bound from FPdim

<u>Example</u>: Ising FC

$$N_1=\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix},\quad N_\eta=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix},\quad N_N=\begin{pmatrix}0&0&1\\0&0&1\\1&1&0\end{pmatrix}$$ have





$\text{FPdim}_C(1)=1=\text{FPdim}_C(\eta)$, $\text{FPdim}_C(N)=\sqrt{2}$.

Upper bound from FPdim

Frobenius-Perron dimension of an FC C

$$\mathrm{FPdim}(C) := \sum_{j \in C} \mathrm{FPdim}_C(j)^2.$$

Upper bound from FPdim

Frobenius-Perron dimension of an FC $\mathcal{C}$

$$\mathrm{FPdim}(\mathcal{C}) := \sum_j \mathrm{FPdim}_{\mathcal{C}}(j)^2.$$

<u>Example</u>: Ising FC

$$\mathrm{FPdim}(\text{Ising FC}) = 1^2 + 1^2 + \sqrt{2}^2 = 4.$$

Upper bound from FPdim

Some facts:

- For an FC $\mathcal{C}$, $\forall j \in \mathcal{C}$, $\mathrm{FPdim}_{\mathcal{C}}(j) \geq 1$. [Etingof-Nikshych-Ostrik '02]
- For a finite $\mathcal{C}$-module category $\mathcal{M}$, $\exists A \in \mathcal{C}$ such that

$$\mathrm{FPdim}_{\mathcal{C}}(A) = \mathrm{FPdim}(\mathcal{C})/\mathrm{FPdim}(\mathcal{M}).$$

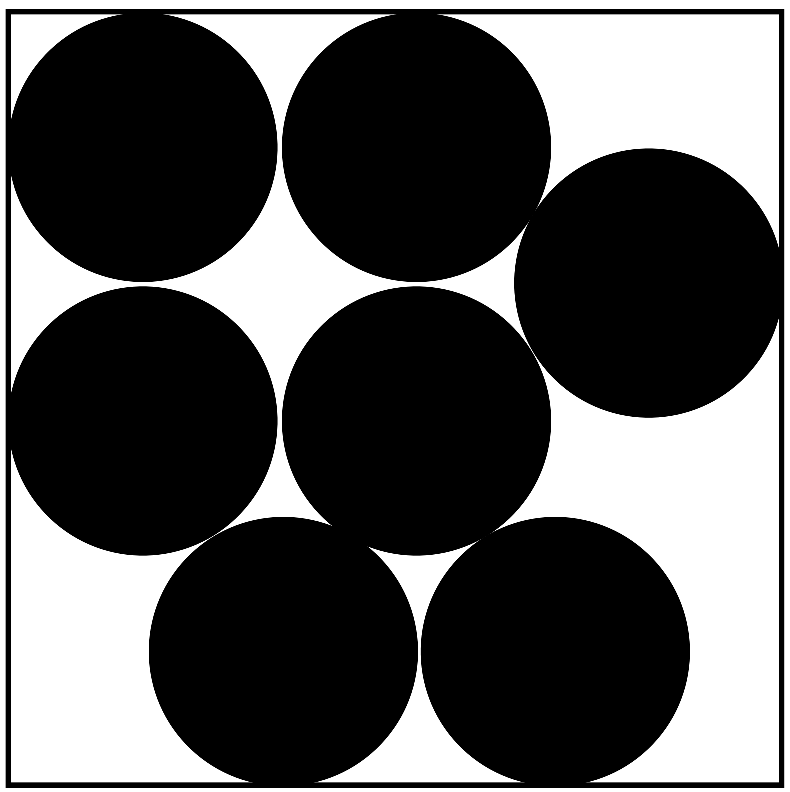
[Davydov-Müger-Nikshych-Ostrik '10]

$$\Rightarrow 1 \leq \mathrm{FPdim}(\mathcal{M}) \leq \mathrm{FPdim}(\mathcal{C}).$$

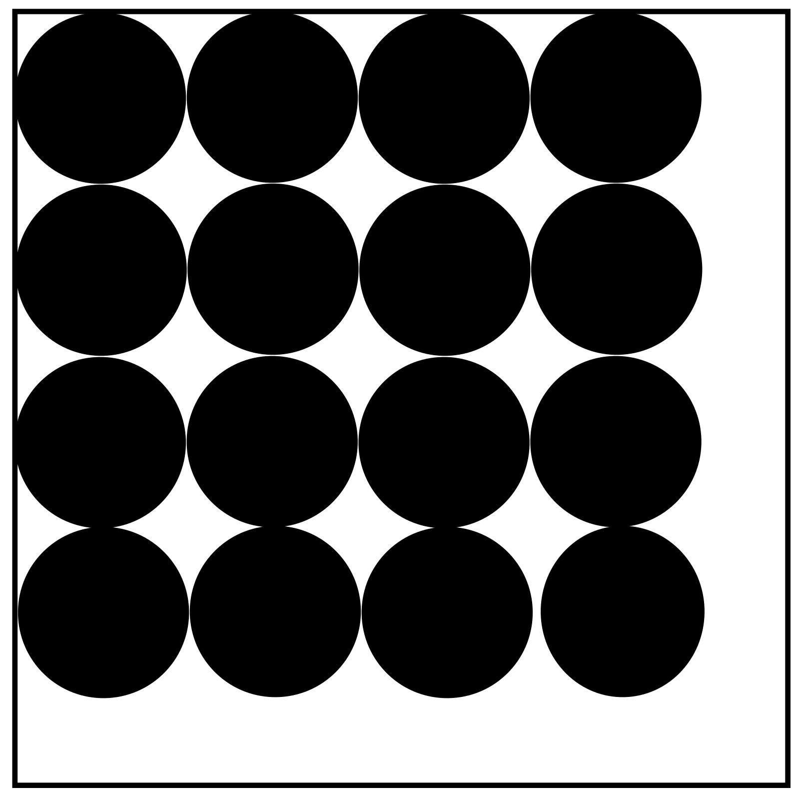
Upper bound from FPdim

Theorem. $\mathrm{rank}(\mathcal{M}) \leq \lfloor \mathrm{FPdim}(\mathcal{C}) \rfloor$. [2311.00746 (KK)]

Proof.



v.s.



Size of box $= \mathrm{FPdim}(\mathcal{M}) \leq \mathrm{FPdim}(\mathcal{C})$, # of balls $= \mathrm{rank}(\mathcal{M})$.

How to classify module categories

Method: [2311.00746 (KK)]

1. find upper bound on rank(M),
2. list up **candidates**,
3. check which satisfy axioms.

Candidates

The upper bound leaves only finitely many candidate C-module categories $\{D\}$;

they have

(i) $\mathrm{rank}(D) \leq \lfloor \mathrm{FPdim}(C) \rfloor$,

(ii) $(1 \leq)\mathrm{FPdim}(D) \leq \mathrm{FPdim}(C)$, and

(iii) solves $\mathrm{FPdim}_C(A) = \sum_{j=1}^{rank(C)} n_j \mathrm{FPdim}_C(j)$ $\quad (n_j \in \mathbb{N})$

$= \mathrm{FPdim}(C)/\mathrm{FPdim}(D)$.

How to classify module categories

Method: [2311.00746 (KK)]

1. find upper bound on rank(M),
2. list up candidates,
3. check which satisfy **axioms**.

Classification: examples

Example 1. $M(5,4) + \phi_{1,3}$ [2311.00746 (KK)]

The deformation preserves Ising FC

$$C = \{1, \eta, N\}.$$

We saw

$\mathrm{FPdim}_C(1) = 1 = \mathrm{FPdim}_C(\eta)$, $\mathrm{FPdim}_C(N) = \sqrt{2}$, $\mathrm{FPdim}(C) = 4$.

Classification: examples

Example 1. $M(5,4) + \phi_{1,3}$ [2311.00746 (KK)]

$\mathrm{FPdim}_C(1) = 1 = \mathrm{FPdim}_C(\eta)$, $\mathrm{FPdim}_C(N) = \sqrt{2}$, $\mathrm{FPdim}(C) = 4$

5 candidates of C-module categories

List of small multiplicity-free fusion rings

The list below contains all multiplicity free fusion rings up to 6 particles. Clicking on a name will red
more information on the formal names $\mathrm{FR}^{r,n}_{m}$ see the page on formal fusion ring names. Here the
For the last 5 columns a value of True indicates that there exists at least 1 way to categorify the fus
There can be multiple categories stemming from the same fusion ring with multiple (possibly disjur

Names	Rank	Number Selfdual Particles	Number Non-zero Structure Constants	$\mathcal{D}^2_{PF}$
$\mathrm{FR}^{1,0}_1$: Trivial	1	1	1	1.
$\mathrm{FR}^{2,0}_1$: $\mathbb{Z}_2 \cong \mathrm{SU}(2)_1$	2	2	4	2.
$\mathrm{FR}^{3,2}_1$: $\mathbb{Z}_3 \cong \mathrm{SU}(3)_1$	3	1	9	3.
$\mathrm{FR}^{2,0}_2$: $\mathrm{Fib} \cong \mathrm{PSU}(2)_3$	2	2	5	3.618
$\mathrm{FR}^{3,0}_1$: $\mathrm{Ising} \cong \mathrm{SU}(2)_2$	3	3	10	4.
$\mathrm{FR}^{4,0}_1$: $\mathbb{Z}_2 \times \mathbb{Z}_2$	4	4	16	4.
$\mathrm{FR}^{4,2}_1$: $\mathbb{Z}_4 \cong \mathrm{SU}(4)_1$	4	2	16	4.
$\mathrm{FR}^{5,4}_1$: $\mathbb{Z}_5 \cong \mathrm{SU}(5)_1$	5	1	25	5.

$$FR^{1,0}_1, FR^{2,0}_1, FR^{3,0}_1, FR^{4,0}_1, FR^{4,2}_1.$$

Classification: examples

Example 1. $M(5,4) + \phi_{1,3}$ [2311.00746 (KK)]

Checking axioms, we find the only C-module category is

$$M \simeq C = \{1, \eta, N\}.$$

Physically, the classification result implies

(i) GSD$=$3,

(ii) C is spontaneously broken in gapped phase.

Classification: examples

Example 2. $M(6,5)+\phi_{2,1}$ [2311.00746 (KK)]

The deformation preserves rank 3 FC

$$C=\{1,\eta,M\}$$

with

$\otimes$	1	η	M
1	1	η	M
η		1	M
M			$1\oplus\eta\oplus M$

,

$\mathrm{FPdim}_C(1)=1=\mathrm{FPdim}_C(\eta)$, $\mathrm{FPdim}_C(M)=2$, $\mathrm{FPdim}(C)=6$.

Classification: examples

Example 2. $M(6,5)+\phi_{2,1}$ [2311.00746 (KK)]

$\text{FPdim}_{\mathcal{C}}(1)=1=\text{FPdim}_{\mathcal{C}}(\eta)$, $\text{FPdim}_{\mathcal{C}}(M)=2$, $\text{FPdim}(\mathcal{C})=6$

7 candidates of $\mathcal{C}$-module categories

Names	Rank	Number Selfdual Particles	Number Non-zero Structure Constants	$\mathcal{D}^2_{PF}$
$\mathrm{FR}_1^{1,0}$: Trivial	1	1	1	1.
$\mathrm{FR}_1^{2,0}$: $\mathbb{Z}_2 \cong \mathrm{SU}(2)_1$	2	2	4	2.
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$\mathrm{FR}_1^{5,4}$: $\mathbb{Z}_5 \cong \mathrm{SU}(5)_1$	5	1	25	5.
$\mathrm{FR}_2^{3,0}$: $\mathrm{Rep}(\mathrm{D}_3) \cong \mathrm{PSU}(2)_4$	3	3	11	6.
$\mathrm{FR}_2^{4,2}$: $\mathrm{Potts} \cong \mathrm{TY}(\mathbb{Z}_3)$	4	2	18	6.
$\mathrm{FR}_1^{6,2}$: D_3	6	4	36	6.
$\mathrm{FR}_1^{6,4}$: $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$	6	2	36	6.
$\mathrm{FR}_2^{4,0}$: $\mathrm{SU}(2)_3 \cong \mathrm{Fib} \times \mathbb{Z}_2$	4	4	20	7.236

$\mathrm{FR}_1^{1,0}, \mathrm{FR}_1^{2,0}, \mathrm{FR}_1^{3,2}, \mathrm{FR}_2^{3,0}, \mathrm{FR}_2^{4,2}, \mathrm{FR}_1^{6,2}, \mathrm{FR}_1^{6,4}.$

Classification: examples

Example 2. $M(6,5) + \phi_{2,1}$ [2311.00746 (KK)]

Checking axioms, we find two C-module categories

Fusion ring	Module category $\mathcal{M}$	rank($\mathcal{M}$)
$\mathrm{FR}_2^{3,0}$	$\mathcal{C}$	3
$\mathrm{FR}_1^{3,2}$	$\mathrm{Vec}_{\mathbb{Z}_3}^1$	3

.

Physically, the classification result implies

(i) GSD$=$3,

(ii) C is spontaneously broken in gapped phase.

Classification: examples

Example 3. $M(3,5)+\phi_{1,2}$ (non-unitary) [2311.00746 (KK)]

The deformation preserves Fibonacci category

$$C=\{1,W\}$$

w/ (fusion) products

$$1\otimes j=j=j\otimes 1,\quad W\otimes W=1\oplus W.$$

They have

$\mathrm{FPdim}_C(1)=1$, $\mathrm{FPdim}_C(W)=\frac{1+\sqrt{5}}{2}$, $\mathrm{FPdim}(C)=\frac{5+\sqrt{5}}{2}\approx 3.6$.

Classification: examples

Example 3. $M(3,5)+\phi_{1,2}$ (non-unitary) [2311.00746 (KK)]

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Candidates of $\mathcal{C}$-module categories

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$$FR_1^{1,0}, FR_2^{2,0}.$$

Classification: examples

Example 3. $M(3,5) + \phi_{1,2}$ (non-unitary) [2311.00746 (KK)]

Checking axioms, we find the only C-module category is

$$M \simeq C = \{1, W\}.$$

Physically, the classification result implies

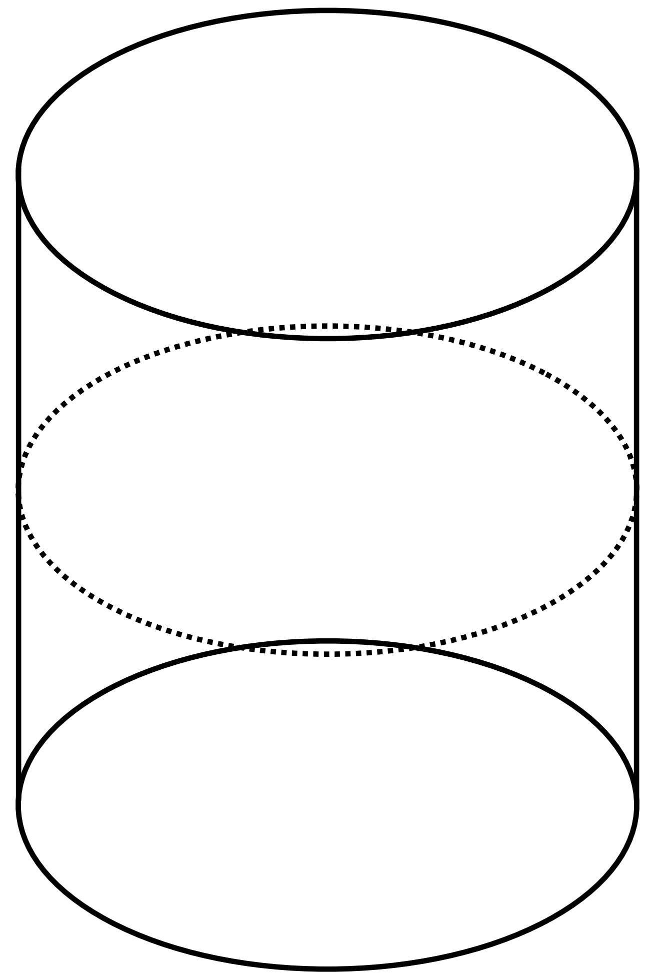
(i) GSD=2,

(ii) C is spontaneously broken in gapped phase.

Numerical check

Truncated Conformal Space Approach

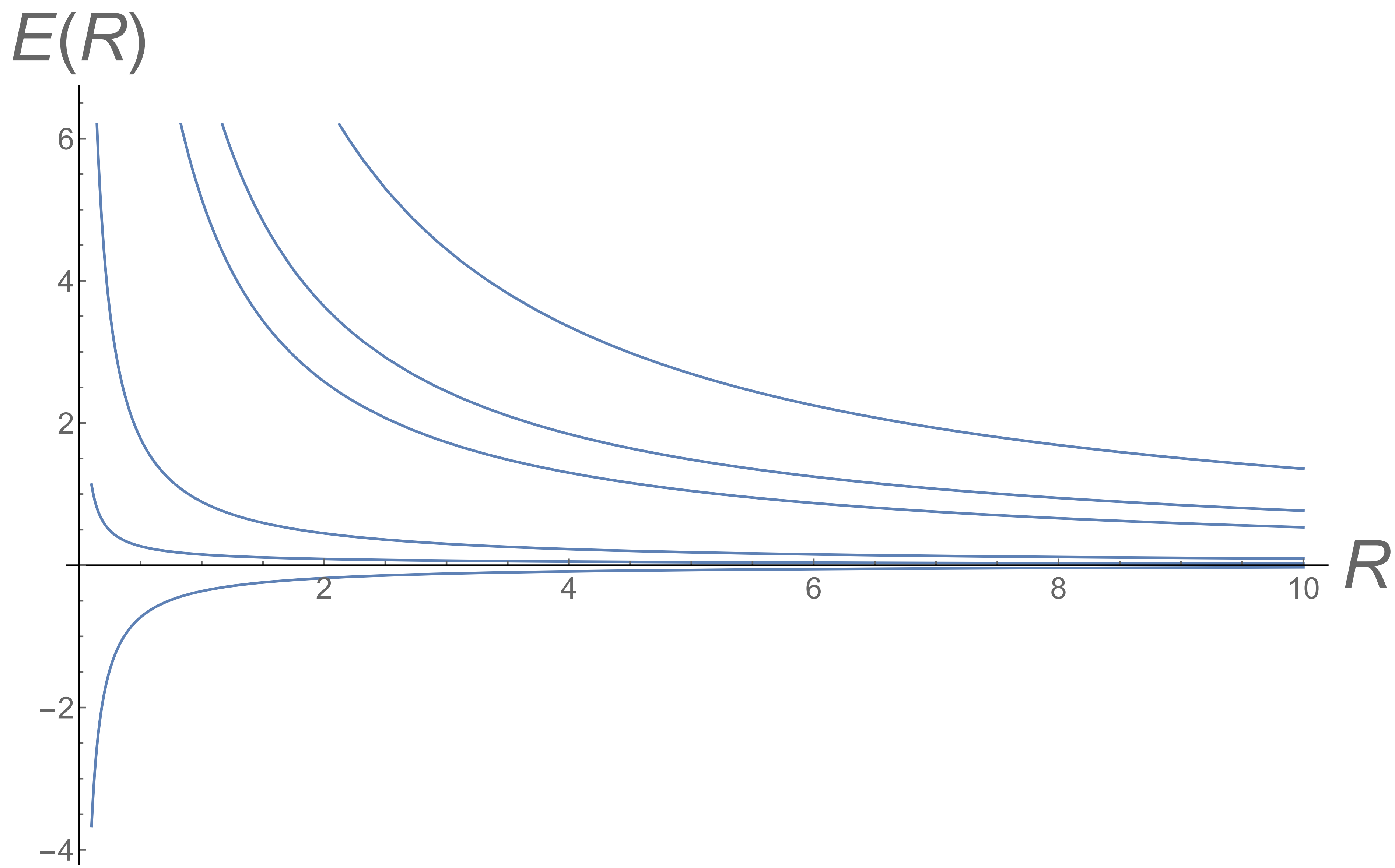
[Yurov-Zamolodchikov '89]



$$H_{deform} = H_{CFT_{UV}} - \lambda \int_{circle} O$$

Classification: examples

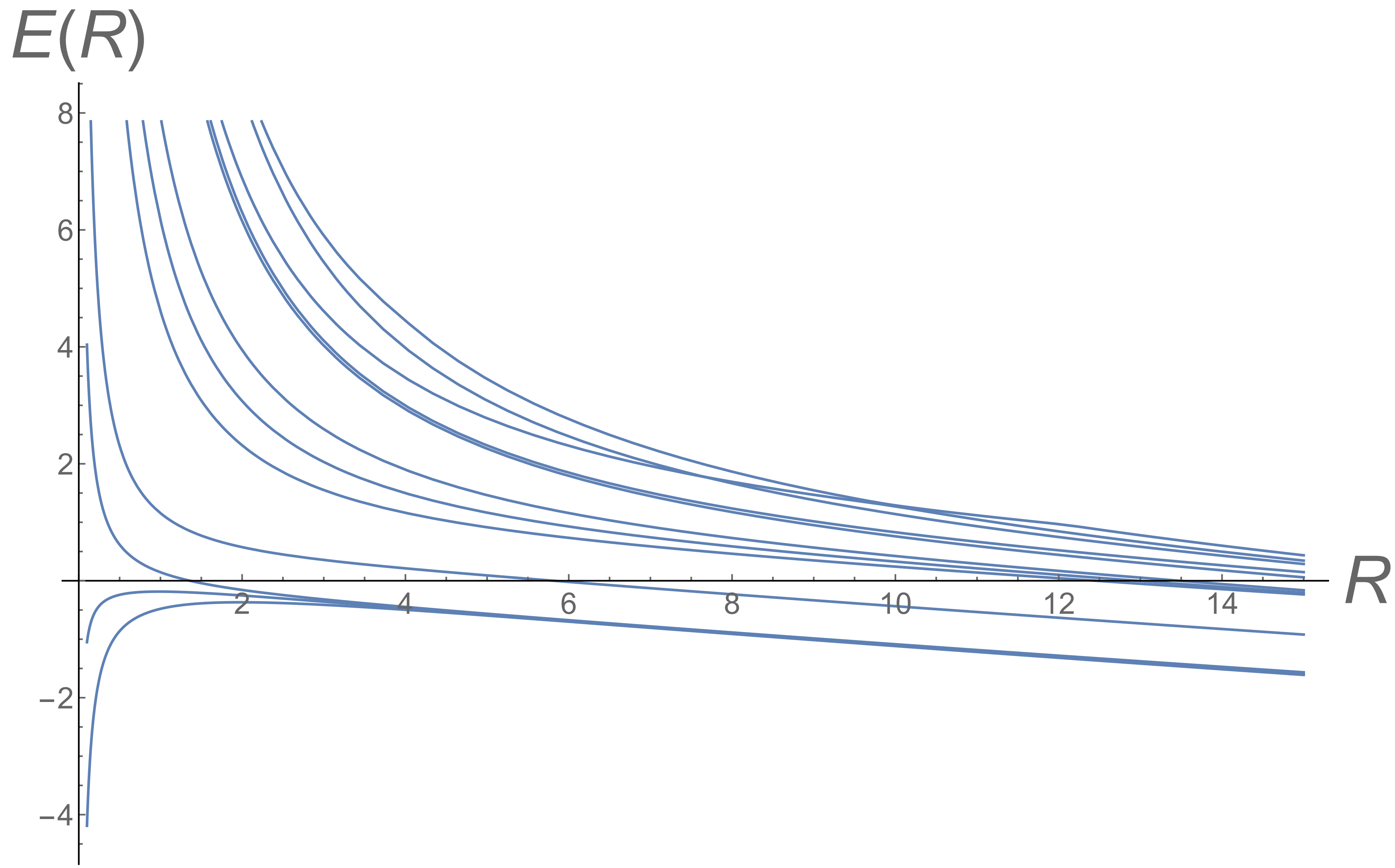
Example 1. $M(5,4) + \phi_{1,3}$ [2311.00746 (KK)]



The numerical result suggests GSD=3, consistent w/ math.

Classification: examples

Example 2. $M(6,5) + \phi_{2,1}$ [2311.00746 (KK)]



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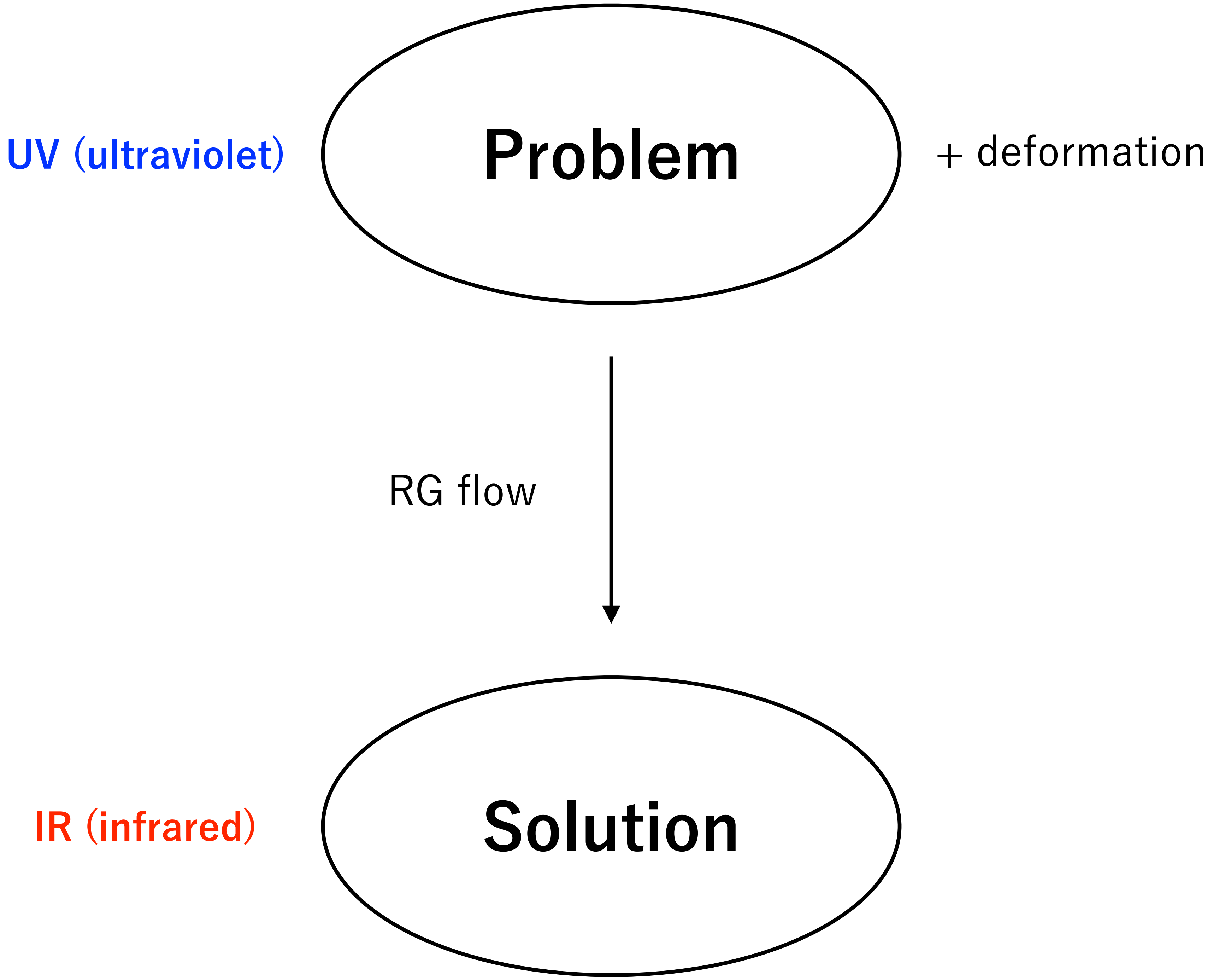
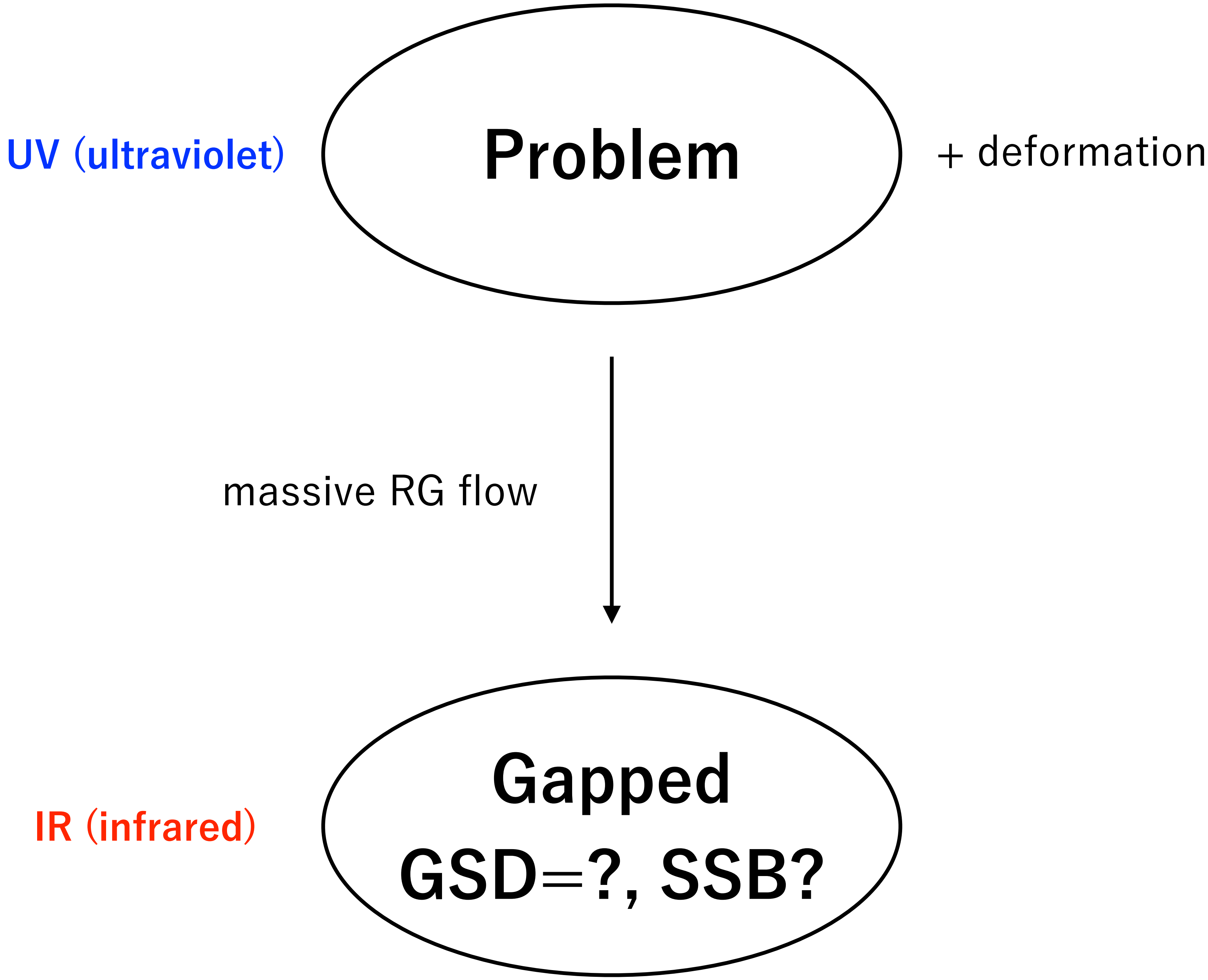
Summary


UV (ultraviolet)
Problem
+ deformation
RG flow
IR (infrared)
Solution

Summary


UV (ultraviolet)
Problem
+ deformation
massive RG flow
IR (infrared)
Gapped
GSD=?, SSB?

Summary

- (Categorical) symmetry constrains physics.
- $\{\mathcal{C}$-sym. 2d gapped phases$\}=\{\mathcal{C}$-module cats.$\}$ (GSD=rank).
- Developed method to classify module cats.,
  which constrain (or fix) GSDs and spont. $\mathcal{C}$ breaking.

[2311.00746 (KK)]

Summary

- We further proved

[2311.15631 (KK)]

Theorem. The following C symmetries are spontaneously broken in 2d gapped phases:

$$C = \begin{cases} \mathbb{Z}_2 \text{ w/ } (d, h) \neq (1,0), (-1, \frac{1}{2}), \\ \textbf{Fib}, \\ \mathbb{Z}_3 \text{ w/ } h = \pm \frac{1}{3}, \\ \textbf{Ising}, \\ \textbf{Non-symmetric Rep}(S_3), \\ \textbf{Symmetric Rep}(S_3) \text{ w/ } d = -1, \\ psu(2)_5 \,. \end{cases}$$

Many future directions

- w/ multiplicity
- More general C
- Higher dimensions
- etc.

Appendix

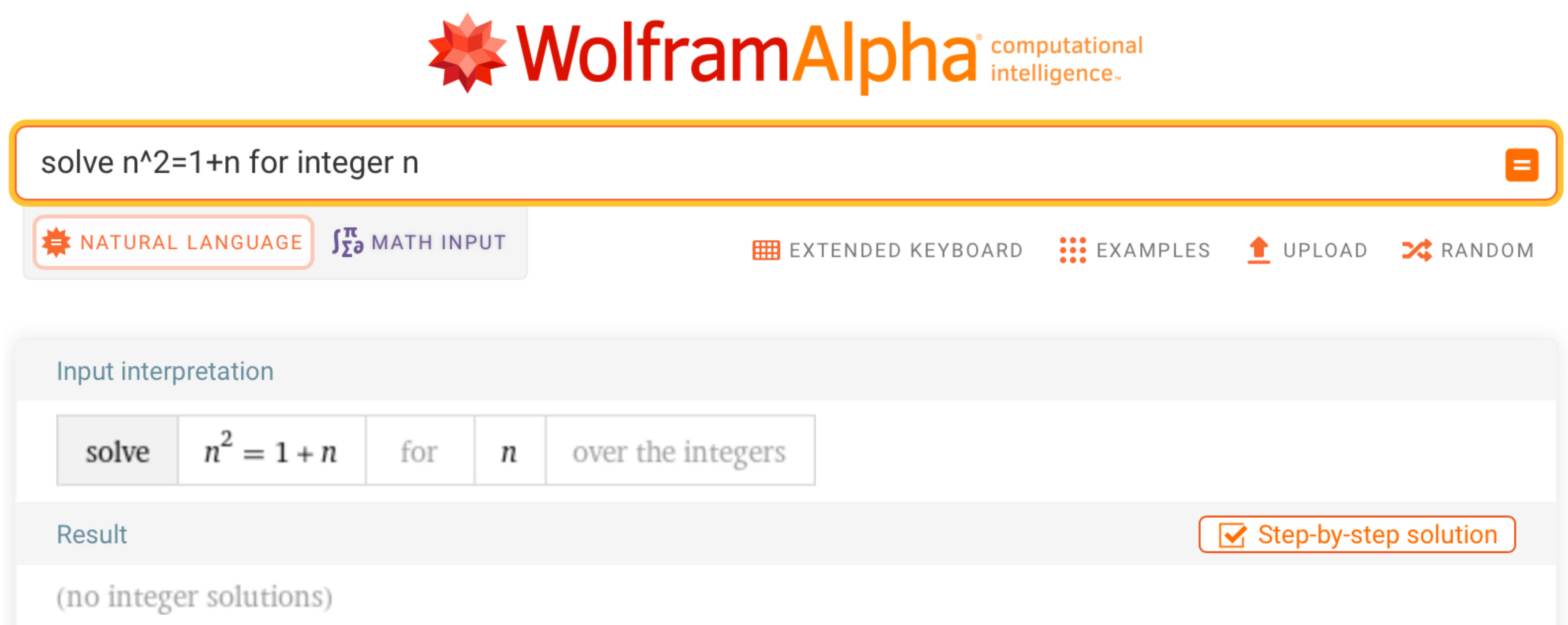
Rule out $FR^{1,0}_1$

Non-negative integer matrix (NIM) encodes module actions.

A rank 1 NIM should obey

$$N_W^2 = 1 + N_W.$$

One finds no 1×1 NIM (i.e., natural number) N_W:



Why GSD > 1 implies SSB

In TQFT,

SSB of C $\iff$ $\exists$charged operator under C.

Theorem. If C-module cat. M is indecomposable,

rank$(M) > 1 \Rightarrow$ SSB of C. [2311.00746 (KK)]

Proof. Assume C-preservation, i.e., C acts trivially.

Then, for $m \in M$, $\forall c \in C$, $c \triangleright m = m$, implying m is a rank=1 C-module category, contradicting M is indecomposable. $\square$