Correlation functions from homotopy algebras and the applications

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based on arXiv 2305.11634 **K.K.**, Y. Okawa
arXiv 2305.13103 **K.K.**
in preparation **K.K.**, Y. Okawa

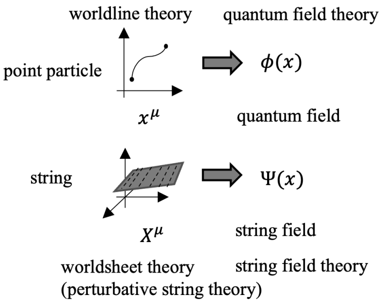
Motivations

We want to know the non-perturbative definition of string theory!

Candidates

- matrix models
- string field theory (SFT)

etc....



String Field Theory and Homotopy algebra

Homotopy algebras are effective tools to construct SFT.

- (quantum) A_∞ algebra $\to$ open SFT
- (quantum) L_∞ algebra $\to$ closed SFT
- open-closed homotopy algebra $\to$ open-closed SFT

etc....

Advantage

- Systematic
- Algebraic: We do not need path integral.
- Universal

etc....

Construct the formulas for QFT and extend them to SFT!!

Homotopy Algebra and Field Theory

There are many things we can do using homotopy algebras!

- relating covariant and light-cone string field theories
  [Erler and Matsunaga, arXiv:2012.09521]
- calculating scattering amplitudes
  [Kajiura, math/0306332] etc.
- integrating out fields
  [Sen, arXiv:1609.00459],
  [Erbin, Maccaferri, Schnabl and Vošmera, arXiv:2006.16270],
  [Koyama, Okawa and Suzuki, arXiv:2006.16710] etc.
- correlation functions
  scalar field [Okawa, arXiv:2203.05366],
  free scalar and Dirac field [K.K. and Okawa, arXiv:2305.11634],
  general scalar and Dirac field [K.K., arXiv:2305.13103] etc.

etc....

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Homotopy algebra and the universal property

Formula for on-shell amplitudes

$$\pi_1 \, \mathbf{P} \, \mathbf{Q} \, \mathbf{P} + \pi_1 \, \mathbf{P} \, \boldsymbol{m} \, \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \, \mathbf{P} \, ,$$

$\mathbf{I}$:identity on $T\mathcal{H}$
$\mathbf{P}$:on-shell projection
$\boldsymbol{h}, \boldsymbol{m}, \mathbf{U}$:determined by the theory

Homotopy algebra

Vector space

$$\mathcal{H} = \bigoplus_{i\in\mathbb{Z}} \mathcal{H}_i \,,$$

$$T\mathcal{H} = \bigoplus_{i=0}^{\infty} \mathcal{H}^{\otimes n} \,,$$

Action

$$S = -\frac{1}{2}\omega(\Phi, \pi_1 \mathbf{Q}\Phi) - \sum_{n=0}^{\infty} \frac{1}{n+1}\omega(\Phi, \pi_1 \boldsymbol{m}_n(\Phi \otimes \ldots \otimes \Phi)) \,,$$

where π_n is the projection from $T\mathcal{H}$ onto $\mathcal{H}^{\otimes n}$.

Gauge transformation

$$\delta_\lambda \Phi = \pi_1 \boldsymbol{Q}\lambda + \sum_{k=1}^{\infty}\sum_{l=0}^{k-1} \pi_1 \boldsymbol{m}_k(\underbrace{\Phi \otimes \ldots \otimes \Phi}_{l} \otimes \lambda \otimes \underbrace{\Phi \otimes \ldots \otimes \Phi}_{k-l-1}) \,,$$

Gauge invariance (quantum A_∞ relations)

$$\left(\mathbf{Q} + \sum_{n=0}^{\infty} \boldsymbol{m}_n + i\hbar\, \mathbf{U}\right)^2 = 0$$

Homotopy algebra and Correlation Function

In the theories without gauge symmetry, it is sufficient to consider two spaces:

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \,.$$

We take Φ to be

$$\Phi = \int d^d x \, \varphi(x) \, c(x) + \int d^d x \, (\, \overline{\theta}_\alpha(x) \, \Psi_\alpha(x) + \overline{\Psi}_{\alpha(x)} \, \theta_\alpha(x) \,) \,,$$

where $\varphi(x)$ is the scalar field and $\Psi(x)$ is the Dirac field.
Then, correlation functions are given by

$$\langle \, \Phi^{\otimes n} \, \rangle = \pi_n \, \boldsymbol{f} \, \mathbf{1} \,,$$

$$\begin{aligned} \boldsymbol{f} &= \frac{1}{\mathbf{I} + \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U}} \\ &= \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \, (\, \boldsymbol{h} \, \boldsymbol{m} + i\hbar \, \boldsymbol{h} \, \mathbf{U} \,)^n \,. \end{aligned}$$

Homotopy algebra and Correlation Function

For simplicity, we consider only the scalar part.
Then,

$$
\begin{aligned}
\langle\, \Phi^{\otimes n} \,\rangle &= \langle\, \underbrace{\Phi \otimes \Phi \otimes \ldots \otimes \Phi}_{n} \,\rangle \\
&= \int d^d x_1 d^d x_2 \ldots d^d x_n \,\langle\, \varphi(x_1)\, \varphi(x_2)\, \ldots\, \varphi(x_n) \,\rangle \\
&\qquad\qquad\qquad\qquad c(x_1) \otimes c(x_2) \otimes \ldots \otimes c(x_n)\,,
\end{aligned}
$$

and correlation functions are obtained as the coefficients of the bases of $\mathcal{H}^{\otimes n}$.

The information of Schwinger-Dyson equations are contained by

$$
\boldsymbol{f}^{-1} \boldsymbol{f} \mathbf{1} = \mathbf{1}\,.
$$

This fact confirms that our formula represents the correlation functions in the ordinary quantum field theory.

Nonperturbative extension

In zero-dimensional case, we can represent the homotopy operators by infinite dimensional matrices.
In the quartic Euclidean theory, it is given by

$$\boldsymbol{h}\,\mathbf{U} = \frac{1}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 2 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 3 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 4 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\boldsymbol{h}\,\boldsymbol{m}_3 = \frac{\lambda}{m^2}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 1 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Nonperturbative extension

Recall that the key operator $\mathbf{f}$ is defined perturbatively:

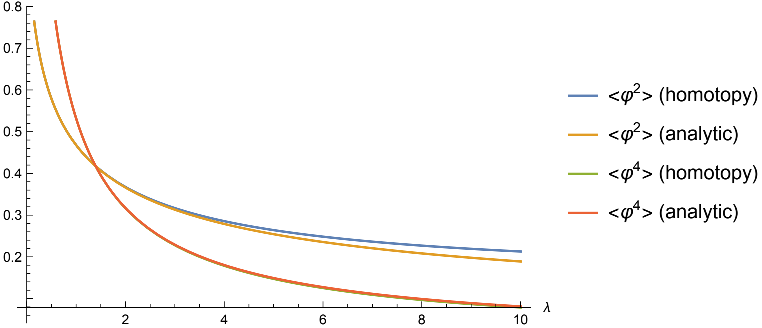
$$\boldsymbol{f} = \frac{1}{\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}} = \mathbf{I} + \sum_{n=1}^{\infty} (-1)^n \left(\boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U} \right)^n .$$

In this case, we can calculate exact inverse of $\mathbf{I} + \boldsymbol{h}\,\boldsymbol{m} + i\hbar\,\boldsymbol{h}\,\mathbf{U}$ by truncating infinite matrices to N by N matrices.

Nonperturbative extension

the quartic Euclidean theory

$$S = \frac{1}{2}\varphi^2 + \frac{1}{4}\lambda\varphi^4$$



Nonperturbative extension

Even when $\lambda = 0.2$, the nonperturbatibe effects dominate.

$$\langle \varphi^2 \rangle = 0.724059$$

and the estimated value from our formula for $N = 100$

$$\boldsymbol{f}_{20} = 0.724059\,.$$

the number of loops	perturbative values for $\langle\varphi^2\rangle$
1	0.4
2	1.36
3	-1.016
4	6.8176
5	-25.2714
6	131.548

Summary and Outlook

We proposed the formula for correlation functions involving the scalar fields and the Dirac field in homotopy algebras:

$$\langle\, \Phi^{\otimes n} \,\rangle = \pi_n\, \boldsymbol{f}\, \mathbf{1}\,.$$

Moreover, nonperturbative extension of the formula was discussed.

The future works are as follows:

- general description for nonperturbative correlation functions in homotopy algebras
- symmetries: K.K. and J.Yoshinaka *in preparation*
- extentions of perturbative formula to general gauge theory
  e.g.) Yang-Mills theory: *working in progress* with J.Yoshinaka
- extentions to string field theory