Gradient flow exact renormalization group: Illustration in the gauged NJL model

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- H. Sonoda (Kobe Univ.), H.S.,
  PTEP **2019**, no.3, 033B05 (2019) [arXiv:1901.05169 [hep-th]]
  PTEP **2021**, no.2, 023B05 (2021) [arXiv:2012.03568 [hep-th]]
  PTEP **2022**, no.5, 053B01 (2022) [arXiv:2201.04448 [hep-th]]
- Y. Miyakawa (Kyushu Univ.), H.S.,
  PTEP **2021**, no.8, 083B04 (2021) [arXiv:2106.11142 [hep-th]]
- Y. Miyakawa, H. Sonoda, H.S.,
  PTEP **2022**, no.2, 023B02 (2022) [arXiv:2111.15529 [hep-th]]
  PTEP **2023**, no.6, 063B03 (2023) [arXiv:2304.14753 [hep-th]]
- and ongoing works. . .

K. Wilson's Exact Renormalization Group (ERG)

- Change of effective interactions under the change of scale:

$$\langle \phi(x_1) \cdots \phi(x_n) \rangle_{S_\tau} \sim e^{n[(D-2)/2](\tau-\tau_0)} Z(\tau, \tau_0)^n \left\langle \phi(e^{\tau-\tau_0} x_1) \cdots \phi(e^{\tau-\tau_0} x_n) \right\rangle_{S_{\tau_0}}$$

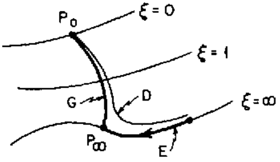


Fig. 12.6. Renormalization group trajector

- Continuum QFT. Correlation length $\xi = \xi_0 |K - K_c|^{-1/y_E}$:

$$\begin{aligned} &\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_g \\ &\equiv \lim_{\tau_0 \to -\infty} e^{n[(D-2)/2](\tau-\tau_0)} Z(\tau, \tau_0)^n \\ &\qquad \times \left\langle \phi(e^{\tau-\tau_0} x_1) \cdots \phi(e^{\tau-\tau_0} x_n) \right\rangle_{S_{\tau_0}, K=K_c - g e^{-y_E(\tau-\tau_0)}} \end{aligned}$$

- Non-perturbative fixed point, relevant to particle physics?
- Many-flavor gauge theories (Banks-Zaks fixed point)?
- Asymptotically-safe gravity?
- Gauge symmetry must be essential in these theories...

ERG in scalar field theory: Wilson-Polchinski (WP) equation

- Introduce a smooth momentum cutoff such as

$$K(p/\Lambda) = e^{-p^2/\Lambda^2}$$

- "Integrate out" momentum modes $|p| > \Lambda$ to yield the Wilson action $S_\Lambda[\phi]$
- $S_\Lambda[\phi]$: reaction under the change of UV cutoff Λ
- Make everything dimensionless by taking Λ as a unit
- $S_\tau[\phi]$ ($\tau \sim -\ln\Lambda$): reaction under the change of scale
- WP equation:

$$\frac{\partial}{\partial\tau} e^{S_\tau[\phi]} = \int d^D x \left(-2\partial^2 - \frac{D-2}{2} - \gamma_\tau - x \cdot \frac{\partial}{\partial x}\right) \phi(x) \cdot \frac{\delta}{\delta\phi(x)} e^{S_\tau[\phi]} + \int d^D x \, (-2\partial^2 + 1 - \gamma_\tau) \frac{\delta}{\delta\phi(x)} \cdot \frac{\delta}{\delta\phi(x)} e^{S_\tau[\phi]}$$

  (We have generalized as $K(p)[1-K(p)] \to p^2$ and the anomalous dimension is defined by $\gamma_\tau = \partial_\tau \ln Z(\tau, \tau_0)$)
- Huge application in critical phenomena...

ERG in gauge field theory

- Local gauge transformation mixes different momentum modes:

$$A_\mu^a(k) \to A_\mu^a(k) + ik_\mu \chi^a(k) - g \int_q f^{abc} \chi^b(q) A_\mu^c(k-q)$$

$$\psi(p) \to \psi(p) - g \int_q \chi^a(q) T^a \psi(p-q)$$

- ERG with momentum cutoff cannot keep a manifest gauge invariance
- ERG keeps a modified gauge invariance (Becchi, Ellwanger, Bonini-D'Attanasio-Marchesini, Reuter-Wetterich, Higashi-Itou-Kugo, Igarashi-Itoh-Sonoda), but its precise form depends on the Wilson action itself!
- This prevents us to set a gauge-invariant ansatz or truncation for the Wilson action. . .
- . . . critical exponents can depend on the gauge fixing parameter. . .
- ERG with manifest gauge invariance is highly desired
- How to do that?

Field diffusion and the Wilson action in scalar theory

- We note an "integral representation" of the Wilson action:

$$e^{S_\tau[\phi]}$$
$$= \hat{s} \int [d\phi'] \prod_x \delta\left(\phi(x) - e^{\int_{\tau_0}^{\tau} d\tau' \, [(D-2)/2+\gamma_{\tau'}]} \phi'(t - t_0, e^{\tau-\tau_0} x)\right) (\hat{s}')^{-1} e^{S_{\tau_0}[\phi']}$$

Here, $\phi'(t, x)$ is the solution to the diffusion equation:

$$\partial_t \phi'(t, x) = \partial^2 \phi'(t, x), \qquad \phi'(0, x) = \phi'(x),$$

where the diffusion time is given by

$$t - t_0 = e^{2(\tau-\tau_0)} - 1$$

Scrambler $\hat{s}$:

$$\hat{s} = \exp\left[\frac{1}{2} \int d^D x \, \frac{\delta^2}{\delta\phi(x)\delta\phi(x)}\right]$$

- ERG and the field diffusion: Abe-Fukuma, Carosso-Hasenfratz-Neil, Matsumoto-Tanaka-Tsuchiya

Replace it by a gauge-covariant diffusion?

- Yang-Mills gradient flow (Narayanan-Neuberger, Lüscher):

$$\partial_t A'^a_\mu(t,x) = D'_\nu F'^a_{\nu\mu}(t,x) = \partial^2 A'^a_\mu(t,x) + \cdots, \qquad A'^a_\mu(0,x) = A'^a_\mu(x)$$

- For fermion (Lüscher):

$$\partial_t \psi'(t,x) = D'_\mu D'_\mu \psi'(t,x) \qquad \psi'(0,x) = \psi'(x)$$

$$\partial_t \bar{\psi}'(t,x) = \bar{\psi}'(t,x) \overleftarrow{D}'_\mu \overleftarrow{D}'_\mu \qquad \bar{\psi}'(0,x) = \bar{\psi}'(x)$$

- Imitating the scalar theory,

$$e^{S_\tau[A,\psi,\bar{\psi}]}$$

$$= \hat{s} \int [dA' d\psi' d\bar{\psi}']$$

$$\times \prod_{x,\mu,a} \delta\left(A^a_\mu(x) - e^{\int_{\tau_0}^{\tau} d\tau' \, [(D-2)/2+\gamma_{\tau'}]} A'^a_\mu(t-t_0, e^{\tau-\tau_0}x)\right)$$

$$\times \prod_x \delta\left(\psi(x) - e^{\int_{\tau_0}^{\tau} d\tau' \, [(D-1)/2+\gamma_{F\tau'}]} \psi'(t-t_0, e^{\tau-\tau_0}x)\right)$$

$$\times \prod_x \delta\left(\bar{\psi}(x) - e^{\int_{\tau_0}^{\tau} d\tau' \, [(D-1)/2+\gamma_{F\tau'}]} \bar{\psi}'(t-t_0, e^{\tau-\tau_0}x)\right) (\hat{s}')^{-1} e^{S_{\tau_0}[A',\psi',\bar{\psi}']}$$

$$\hat{s} = \exp\left[\frac{1}{2}\int d^Dx \, \frac{\delta^2}{\delta A^a_\mu(x)\delta A^a_\mu(x)}\right] \exp\left[-i\int d^Dx \, \frac{\overrightarrow{\delta}}{\delta\psi(x)} \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\right]$$

Good points of GFERG

- GFERG keeps a manifest gauge invariance: If S_{τ_0} is invariant under ($g_\tau \equiv e^{-\int^\tau d\tau' [(D-4)/2+\gamma_{\tau'}]}$)

$$A_\mu^a(x) \to A_\mu^a(x) + \partial_\mu \chi^a(x) + g_\tau f^{abc} A_\mu^b(x) \chi^c(x)$$
$$\psi(x) \to \psi(x) - g_\tau \chi^a(x) T^a \psi(x)$$
$$\bar\psi(x) \to \bar\psi(x) + g_\tau \chi^a(x) \bar\psi(x) T^a$$

  then S_τ is invariant too.

- GFERG keeps a modified exact chiral symmetry: If S_{τ_0} satisfies

$$\int d^D x \left\{ S_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)} \gamma_5 \psi(x) + \bar\psi(x)\gamma_5 \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} S_\tau + 2iS_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)} \gamma_5 \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} S_\tau - 2i \,\mathrm{tr}\left[\gamma_5 \frac{\overrightarrow{\delta}}{\delta\bar\psi(x)} S_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)} \right] \right\} = 0$$

  then S_τ does too. This is a generalized Ginsparg-Wilson (GW) relation

Wilson-Polchinski equation in GFERG

- To diffuse gauge modes too, we introduce the Zwanziger[1] like term,

$$\partial_t A'^a_\mu(t,x) = D'_\nu F'^a_{\nu\mu}(t,x) + \alpha_0 D'_\mu \partial_\nu A'^a_\nu(t,x) \qquad \text{etc.}$$

  (The gauge invariance holds even with this term)

- Taking the τ derivative of the integral representation,

$$\frac{\partial}{\partial\tau} e^{S_\tau[A,\psi,\bar\psi]}$$
$$= \int d^D x \, \frac{\delta}{\delta A^a_\mu(x)} \Bigg[-2D_\nu F^a_{\nu\mu}(x) - 2\alpha_0 D_\mu \partial_\nu A^a_\nu(x) - \left(\frac{D-2}{2} + \gamma_\tau + x \cdot \frac{\partial}{\partial x} \right) A^a_\mu(x) \Bigg] \Bigg|_{A \to A + \delta/\delta A} e^{S_\tau[A,\psi,\bar\psi]}$$
$$+ \text{(fermion)}$$

- Seemingly simple, but this contains functional derivatives up to 4th order! (conventional ERG contains only up to 2nd order)
- Price of the manifest gauge symmetry. . .

[1] More precisely, Nakagoshi-Namiki-Ohba-Okano ('83)

1PI action Γ_τ

- Conventionally, the so-called 1PI action Γ_τ (Nicoll-Chang, Wetterich, Morris, Bonini-D'Attanasio-Marchesini) is employed in non-perturbative study in ERG
- We can also define the Legendre transformation in GFERG:

$$\mathcal{A}_\mu(x) = A_\mu(x) + \frac{\delta S_\tau}{\delta A_\mu(x)}$$

$$\Psi(x) = \psi(x) + i\frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)} S_\tau, \qquad \bar{\Psi}(x) = \bar{\psi}(x) + iS_\tau \frac{\overleftarrow{\delta}}{\delta\psi(x)}$$

$$\begin{aligned}
\Gamma_\tau[\mathcal{A}_\mu, \Psi, \bar{\Psi}] &- \frac{1}{2}\int d^Dx\, \mathcal{A}_\mu(x)\mathcal{A}_\mu(x) + i\int d^Dx\, \bar{\Psi}(x)\Psi(x) \\
&= S_\tau[A_\mu, \psi, \bar{\psi}] + \frac{1}{2}\int d^Dx\, A_\mu(x)A_\mu(x) - i\int d^Dx\, \bar{\psi}(x)\psi(x) \\
&\quad - \int d^Dx\, \mathcal{A}_\mu(x)A_\mu(x) + i\int d^Dx\, \left[\bar{\Psi}(x)\psi(x) + \bar{\psi}(x)\Psi(x)\right]
\end{aligned}$$

- Manifest gauge invariance and the modified chiral symmetry are kept, although GFERG equation tends to be quite involved. . .

Finiteness issue

- I said as if GFERG is perfect, but there is a concern
- In GFERG, UV cutoff is implemented effectively by the diffusion, not by an explicit cutoff
- Moreover, the diffusion contains interactions (for the gauge invariance)
- So, it is not clear if GFERG defines a UV finite framework
- Unfortunately, a perturbative analysis shows that it does not if the Wilson action has no gauge fixing (we will see it later)
- Our original objective was opposite; we wanted to understand the “finiteness” of the gradient flow (Lüscher-Weisz) from ERG

GFERG with gauge fixing?

- Introduce Faddeev-Popov (FP) ghost-anti-ghost and Nakanishi-Lautrup (NL) field
- It is easy to make diffusion equations to commute with the conventional BRST,

$$\begin{aligned}
\delta A^a_\mu(x) &= \partial_\mu c^a(x) + g_\tau f^{abc} A^b_\mu(x) c^c(x) \\
\delta c^a(x) &= -\frac{1}{2} g_\tau f^{abc} c^b(x) c^c(x) \\
\delta \bar{c}^a(x) &= B^a(x) \\
\delta B^a(x) &= 0
\end{aligned}$$

- However, a simple choice of the scrambler,

$$\begin{aligned}
\hat{s} = \exp &\left[\int d^D x \, \frac{1}{2} \frac{\delta^2}{\delta A^a_\mu(x) \delta A^a_\mu(x)} \right] \\
&\times \exp \left[- \int d^D x \, \frac{\delta}{\delta c^a(x)} \frac{\delta}{\delta \bar{c}^a(x)} \right] \exp \left[- \int d^D x \, \frac{1}{2} \frac{\delta^2}{\delta B^a(x) \delta B^a(x)} \right]
\end{aligned}$$

is not invariant under BRST; we go back to the modified BRST symmetry. . .

In abelian gauge theory, at least, we can circumvent the difficulty

- FP ghost sector completely decouples and solvable:

$$S_{\mathrm{ghost}} = \int_k \bar{c}(-k) \frac{-k^2}{E(e^{-2\tau}k^2)e^{-2k^2} + k^2} c(k)$$

- Gauge fixing term turns to be (note: $\xi \to \infty$ is no gauge fixing)

$$S_{\text{gauge fixing}} = -\frac{1}{2} \int_k A_\mu(k) A_\nu(-k) \frac{k_\mu k_\nu}{\xi_\tau E(e^{-2\tau}k^2)e^{-2k^2} + k^2}$$

- BRST symmetry reduces to the Ward-Takahashi (WT) identity:

$$ik_\mu \frac{\delta S_\tau}{\delta A_\mu(k)} + \frac{k^2}{\xi_\tau E(e^{-2\tau}k^2)e^{-2k^2}} ik_\mu \left[A_\mu(-k) + \frac{\delta S_\tau}{\delta A_\mu(k)} \right]$$
$$- ie_\tau \int_p \bar{\psi}(-p-k) \frac{\overrightarrow{\delta}}{\delta \bar{\psi}(-p)} S_\tau + ie_\tau \int_p S_\tau \frac{\overleftarrow{\delta}}{\delta \psi(p+k)} \psi(p) = 0$$

  This is linear in the Wilson action

- In the conventional ERG, WT identity is infinite order in the Wilson action
- Running of the gauge fixing parameter is controlled by the beta function:

$$\partial_\tau \xi_\tau = 2\gamma_\tau \xi_\tau, \qquad \partial_\tau e_\tau = \left(\frac{4-D}{2} - \gamma_\tau \right) e_\tau$$

WT identity in the conventional ERG

- In the conventional ERG,

$$\frac{\xi e^{-2k^2}+k^2}{\xi e^{-k^2}} k_\mu \frac{\delta S_I}{\delta A_\mu(k)}$$
$$= e e^{-S}\int_p e^{-(p+k)^2+p^2}\,\mathrm{Tr}\left\{\left[\psi(p)+i\frac{\overrightarrow{\delta}}{\delta\bar\psi(-p)}\right]e^S\right\}\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}$$
$$- e e^{-S}\int_p e^{-p^2+(p+k)^2}\,\mathrm{Tr}\,\frac{\overrightarrow{\delta}}{\delta\bar\psi(-p)}\left\{e^S\left[\bar\psi(-p-k)+i\frac{\overleftarrow{\delta}}{\delta\psi(p+k)}\right]\right\},$$

  where $S_I = S - S^{(0)}$
- This is infinite order in the Wilson action S. Any ad-hoc ansatz would not able to fulfill this relation exactly

Illustration in $U(1)$ gauged NJL model in 4D (ongoing with Sonoda)

- Let us try some non-perturbative study in abelian gauge theory
- GFERG for the 1PI action Γ:

$$
\begin{aligned}
&\partial_\tau \Gamma + \int d^D x \left(\frac{D-2}{2} + \gamma + x\cdot\partial + 2\partial^2 \right) \mathcal{A}_\mu(x) \cdot \frac{\delta \Gamma}{\delta \mathcal{A}_\mu(x)} \\
&\quad + \int d^D x\, \Gamma \frac{\overleftarrow{\delta}}{\delta \Psi(x)} \left[\frac{D-1}{2} + \gamma_F + x\cdot\partial + 2\partial^2 - 4ie\mathcal{A}(x)\cdot\partial - 2e^2 \mathcal{A}(x)^2 \right] \Psi(x) \\
&\quad + \int d^D x \left[\frac{D-1}{2} + \gamma_F + x\cdot\partial + 2\partial^2 + 4ie\mathcal{A}(x)\cdot\partial - 2e^2 \mathcal{A}(x)^2 \right] \bar{\Psi}(x) \cdot \frac{\overrightarrow{\delta}}{\delta \bar{\Psi}(x)} \Gamma \\
&= \int d^D x \Bigg(- \left(2\partial^2_{x'} - 1 + \gamma\right) \frac{\delta \mathcal{A}_\mu(x')}{\delta A_\mu(x)} + \mathrm{tr} \left(2\partial^2_{x'} - \frac{1}{2} + \gamma_F \right) \Psi(x') \frac{\overleftarrow{\delta}}{\delta \psi(x)} \\
&\quad + 4ie\Gamma \frac{\overleftarrow{\delta}}{\delta \Psi(x)} \frac{\delta}{\delta A_\mu(x')} \partial_\mu \Psi(x) - 4ie\, \mathrm{tr} \left[\mathcal{A}_\mu(x')\partial_\mu \Psi(x') + \frac{\delta}{\delta A_\mu(x'')} \partial_\mu \Psi(x') \right] \frac{\overleftarrow{\delta}}{\delta \psi(x)} \\
&\quad + 2e^2 \Gamma \frac{\overleftarrow{\delta}}{\delta \Psi(x)} \Bigg\{ \mathcal{A}_\mu(x) \frac{\delta}{\delta A_\mu(x')} \Psi(x) \\
&\qquad\qquad + \frac{\delta}{\delta A_\mu(x')} \left[\mathcal{A}_\mu(x'')\Psi(x) \right] + \frac{\delta^2}{\delta A_\mu(x')\delta A_\mu(x'')} \Psi(x) \Bigg\} \\
&\quad - 2e^2\, \mathrm{tr} \Bigg\{ \mathcal{A}_\mu(x'')\mathcal{A}_\mu(x'')\Psi(x') + \mathcal{A}_\mu(x'') \frac{\delta}{\delta A_\mu(x'')} \Psi(x') \\
&\qquad\qquad + \frac{\delta}{\delta A_\mu(x'')} \left[\mathcal{A}_\mu(x''')\Psi(x') \right] + \frac{\delta^2}{\delta A_\mu(x'')\delta A_\mu(x''')} \Psi(x') \Bigg\} \frac{\overleftarrow{\delta}}{\delta \psi(x)} + (\psi \leftrightarrow \bar{\psi}) \Bigg)
\end{aligned}
$$

- Red containing the gauge coupling e are peculiar to GFERG; gauge invariance requires them

Illustration in $U(1)$ gauged NJL model in 4D

- Under GFERG, 1PI action $\mathit{\Gamma}$ (besides the gauge fixing term) remains invariant under the conventional gauge transformation:

$$\delta\mathcal{A}_\mu(x) = \partial_\mu\chi(x), \qquad \delta\Psi(x) = ie\chi(x)\Psi(x), \qquad \delta\bar{\Psi}(x) = -ie\chi(x)\bar{\Psi}(x)$$

- The naive gauge invariance in a naive ansatz, such as

$$\begin{aligned}\mathit{\Gamma} = &-\frac{1}{4}\int d^Dx\,[\partial_\mu\mathcal{A}_{\nu,-1}(x) - \partial_\nu\mathcal{A}_{\mu,-1}(x)]^2 - \frac{1}{2\xi}\int d^Dx\,[\partial_\mu\mathcal{A}_{\mu,-1}(x)]^2 \\ &+ i\int d^Dx\,\bar{\Psi}_{-1}(x)\,[\not{\partial} - ie\not{\mathcal{A}}_{-1}(x) - m]\,\Psi_{-1}(x) \\ &- \int d^Dx\,\left\{G_V\left[\bar{\Psi}_{-1}(x)\gamma_\mu\Psi_{-1}(x)\right]^2 + G_A\left[\bar{\Psi}_{-1}(x)\gamma_\mu\gamma_5\Psi_{-1}(x)\right]^2\right\},\end{aligned}$$

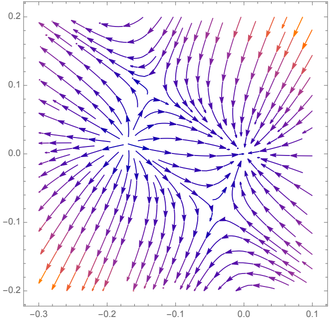
is therefore preserved under GFERG!

- In this ansatz, -1 variables are used; these are defined from the original ones through diffusion equations
- We also (approximately) impose the chiral symmetry by the GW relation:

$$\begin{aligned}&\int d^Dx\left[\mathit{\Gamma}\frac{\overleftarrow{\delta}}{\delta\Psi_{-1}(x)}\gamma_5\Psi_{-1}(x) + \bar{\Psi}_{-1}(x)\gamma_5\frac{\overrightarrow{\delta}}{\delta\bar{\Psi}_{-1}(x)}\mathit{\Gamma}\right] \\ &\quad - \int d^Dx\,\mathrm{tr}\left[\gamma_5\Psi(x')\frac{\overleftarrow{\delta}}{\delta\psi(x)} + \frac{\overrightarrow{\delta}}{\delta\bar{\psi}(x)}\bar{\Psi}(x')\gamma_5\right] = 0\end{aligned}$$

Illustration in $U(1)$ gauged NJL model in 4D

- With the ansatz and truncation in powers of fields, we may do fully non-perturbative study
- This is laboriously very hard. So here let us be content with the sub-regions:
  1. $e = 0$, non-perturbative in m, $G_{V,A}$; this is the same as the conventional ERG
  2. To $O(e^2)$, corresponding to the 1-loop QED
  3. [At the moment, we are working on $O(G_{V,A}e^2)$ terms]
- (GF)ERG flow of $(G_V/(16\pi^2), G_A/(16\pi^2))$ for $e = 0$



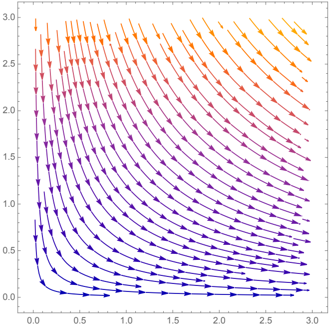
- We have 4 fixed points in this $e = 0$ subspace: 1 of them possesses 2 relevant directions, while 2 have a single relevant direction

Illustration in $U(1)$ gauged NJL model in 4D

- Still, GFERG possesses an advantage over ERG
- For e small, GFERG yields (these receive no correction from $G_{V,A}$)

$$\partial_\tau e^2 = -2\frac{1}{(4\pi)^2}\frac{8}{3}e^4, \qquad \partial_\tau \xi = 2\frac{1}{(4\pi)^2}\frac{8}{3}e^2\xi,$$

- e^2 is marginally irrelevant and $\xi \to \infty$ in IR
- GFERG flow in (ξ, e^2) plane:



- The direction of the gauge coupling is irrelevant (at least for $e \ll 1$)

Illustration in $U(1)$ gauged NJL model in 4D

- Anomalous dimensions related to the fermion (in $m \to 0$):

$$\gamma_F = \frac{3}{(4\pi)^2} e^2, \qquad \partial_\tau m = \left[1 + \frac{6}{(4\pi)^2} e^2\right] m$$

  These are independent of ξ and physical

- The perturbative computation of these, however, involves the integral

$$\int_k \frac{\partial}{\partial k_\rho} \left[k_\rho \frac{k_\mu k_\nu}{k^2} \frac{\xi e^{-2k^2}}{\xi e^{-2k^2} + k^2} e^{2k^2} \tilde{\mathcal{V}}_{\mu\nu}(-p, -k, k, p) \right]$$

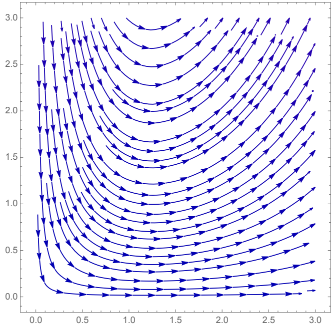
$$= \int_k \frac{\partial}{\partial k_\rho} \begin{cases} k_\rho O(e^{-2k^2}) & \text{if } \xi \text{ is finite before the integration} \\ k_\rho O(1) & \text{if } \xi \to \infty \text{ before the integration} \end{cases}$$

$$= \begin{cases} 0 & \text{independent of } \xi \\ \infty & \text{if } \xi \to \infty \text{ before the integration} \end{cases}$$

- This is the aforementioned finiteness problem without gauge fixing, $\xi = \infty \ldots$

Illustration in $U(1)$ gauged NJL model in 4D

- On the other hand, a naive application of ERG yields the flow in (ξ, e^2) plane:



- e^2 appears to be marginally relevant and one would think that the fixed points on the $e = 0$ plane possess another relevant direction (this must be wrong)
- This illustrates the danger of the gauge non-invariant conventional ERG

Summary

- We formulated GFERG that keeps a manifest gauge invariance and a modified chiral symmetry, at least formally
- The finiteness issue, however, appears much serious than we expected before
- This requires the gauge fixing at least in perturbation theory
- A manifest BRST symmetry in FP-NL sector is difficult to realize in a simple form
- We can circumvent this in abelian gauge theory; we want to pursue the study of non-trivial fixed points in QED (Aoki-Morikawa-Sumi-Terao-Tomoyose, Gies-Jaeckel, Igarashi-Itoh-Pawlowski, Gies-Ziebell, . . .)
- We need some breakthrough in non-Abelian theory; gauge fixing without FP ghost as in the stochastic quantization?
- Here, we have stuck to the continuum framework, i.e., not lattice), having a generalization to gravity in mind. . .

Backup: Modified correlation function (Sonoda 2015)

- The conventional correlation function

$$\langle \phi(p_1) \cdots \phi(p_n) \rangle_{S_\tau} = \int [d\phi]\, e^{S_\tau[\phi]} \phi(p_1) \cdots \phi(p_n)$$

  does not exhibit a simple scaling relation
- However, for the modified correlation function,

$$\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\tau} = \int [d\phi]\, e^{S_\tau[\phi]} \hat{s}^{-1} e^{p_1^2} \phi(p_1) \cdots e^{p_n^2} \phi(p_n),$$

  where

$$\hat{s}^{-1} = \exp\left[-\frac{1}{2} \int_p \frac{\delta^2}{\delta\phi(p)\delta\phi(-p)} \right],$$

  one finds the exact scaling

$$\begin{aligned} &\langle\langle \phi(p_1) \cdots \phi(p_n) \rangle\rangle_{S_\tau} \\ &= e^{-n[(D+2)/2](\tau-\tau_0)} Z(\tau, \tau_0)^n \left\langle\left\langle \phi(e^{-(\tau-\tau_0)} p_1) \cdots \phi(e^{-(\tau-\tau_0)} p_n) \right\rangle\right\rangle_{S_{\tau_0}} \end{aligned}$$

- Quite often people say "integrating out", but actually nothing is lost under ERG flow; we can go back and forth between IR and UV!

Backup: Critical exponents in the NJL model

- In $e = 0$ subspace:

$G_V/(16\pi^2)$	$G_A/(16\pi^2)$	m	exponents
0	0	0	-2, -2
-0.1545276	$+0.0122904$	-0.279097	$+1.82664$, $+1.66678$
-0.0506273	-0.0995112	$+0.120272$	-2.15652, $+1.41817$
-0.1317989	$+0.0875545$	-0.342978	-1.66246, $+1.46644$

The mass parameter m is determined by approximately solving the GW relation (with the operator truncation)