Fractional topological charge in lattice non-Abelian gauge theory

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KEK Theory Workshop 2023@ KEK Tsukuba Camp.
2023/11/30

Topology of SU(N) lattice gauge theories coupled with $\mathbb{Z}_N$ 2-form gauge fields

M. Abe, O. Morikawa, S. Onoda, H. Suzuki and Y. Tanizaki

arXiv:2303.10977[hep-th]

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PTEP 2023 (2023) 2, 023B03 [arXiv:2210.12967[hep-th]]

Symmetry and Anomaly Ⅰ

- Classical Theory：Symmetry ⟷ Conservation low （Noether Theorem）
- Quantum Theory：The conservation low may be broken (Anomaly).
  - ➢ Focus on the Partition function,

$$Z[A] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}.$$

  - ➢ How to distinguish the anomaly : Whether the Z is invariant or not under a transformation

$$Z' \overset{?}{=} Z$$

$$\to Z'[A+\partial\theta] = \int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A+\partial\theta]}$$

$$= \underbrace{e^{\mathcal{A}[\theta,A]}}_{\text{'t Hooft anomaly}} \underbrace{\int [\mathcal{D}(\text{field})]\ e^{S[\text{field},A]}}_{=Z}.$$

Symmetry and Anomaly Ⅱ

- ➢ We can predict the low energy dynamics of the gauge theory.

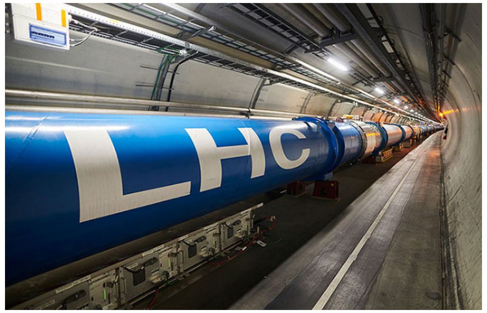
  ※ Gauge theory : A theory which describes the Standard Model of particles

- ✓ e. g., we decided the theory for the strong interaction is the SU(3) gauge theory because the theory and the experiment are well matched.

Particle Theory

Predict

Particle Experiment



https://www.icepp.s.u-tokyo.ac.jp/information/20220426.html

High — Low

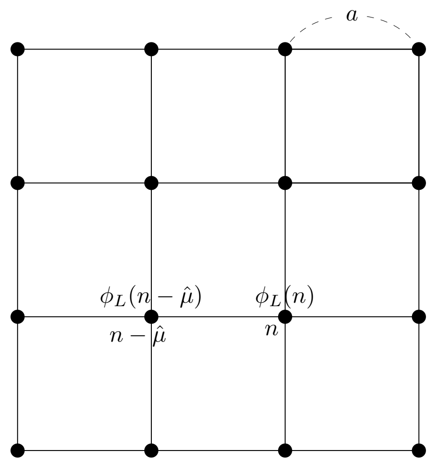
Energy

Recent Developments in Anomalies

- Recently, Gaiotto et al. has extended the concept of symmetry. : Higher Form Symmetry
  (Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148[hep-th])
  - ➢ By anomalies with higher form (and discrete) symmetries, the low energy dynamics of gauge theories has been discussed. (Gaiotto, Kapustin, Komargodski, Seiberg, arXiv:1703.00501[hep-th])
  - ➢ Many new anomalies have been discovered and related studies has been done.
    - ✓ Yamaguchi, arXiv:1811.09390[hep-th]
    - ✓ Hidaka, Hirono, Nitta, Tanizaki, Yokokura, arXiv:1903.06389[hep-th]
    - ✓ Honda, Tanizaki, arXiv:2009.10183[hep-th]
    - ✓ etc.

☆Motivation : Understand these anomalies in the lattice field theory where we treat the regularization well.

Lattice Gauge Theory



Anomaly of the $SU(N)$ gauge theory with θ term

- The $SU(N)$ gauge theory with the θ term has the time reversal ($\mathcal{T}$) symmetry at $\theta = \pi$.

$$Z = \int \mathcal{D}a\ e^{S[a]} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\theta Q[a]}, \quad Q \in \mathbb{Z}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(-Q[a])} = \int \mathcal{D}a\ e^{S_{SU(N)}[a]} e^{i\pi(+Q[a])} \underbrace{e^{-i2\pi Q[a]}}_{=1} = Z$$

- Then, we construct the $SU(N)$ gauge theory with the higher form symmetry ($\mathbb{Z}_N$ 1-form gauge symmetry). This means we couple $\mathbb{Z}_N$ 2-form gauge field to the theory.
    - The topological charge (TC) becomes fractional, so it is not invariant under the $\mathcal{T}$ transformation.

Important!!

$$e^{-i2\pi Q} \neq 1$$

    - This theory at $\theta = \pi$ has the mixed anomaly between the $\mathbb{Z}_N$ 1-form gauge and $\mathcal{T}$ symmetry.

Topological Charge on the Lattice

- How to calculate the topological charge Q,

$$Q = -\frac{1}{24\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{f(n,\mu)} \mathrm{d}^3x \,\mathrm{tr}\left[(v_{n,\mu}^{-1}\partial_\nu v_{n,\mu})(v_{n,\mu}^{-1}\partial_\rho v_{n,\mu})(v_{n,\mu}^{-1}\partial_\sigma v_{n,\mu})\right]$$
$$-\frac{1}{8\pi^2}\sum_n \sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} \int_{p(n,\mu,\nu)} \mathrm{d}^2x \,\mathrm{tr}\left[(v_{n,\mu}\partial_\rho v_{n,\mu}^{-1})(v_{n-\hat{\mu},\nu}^{-1}\partial_\sigma v_{n-\hat{\mu},\nu})\right].$$

  - $v_{n,\mu}(x)$ is the gauge translation function (transition function).

Admissibility condition

$$\|\mathbb{1} - U_p(n)\| < \varepsilon.$$

- On the lattice, topological values are ill-defined.
  - Restricting the size of plaquette (Admissibility condition), Lüscher constructed integral TC on the lattice (Lüscher, Commun. Math. Phys. 85 (1982)).
  - We aim to construct the fractional TC on the $SU(N)$ lattice by extended the Lüscher's topological charge.
    - ✓ Itou, arXiv:1811.05708[hep-th]
    - ✓ Anosova, Gattringer, Göschl, Sulejmanpasic, Török, arXiv:1912.11685 [hep-lat]

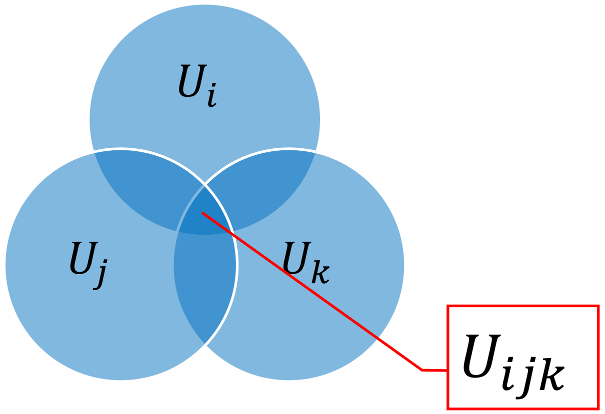
Fiber Bundle

- The fiber bundle describes the gauge theory.
  - ➢ Covering a manifold M by patches U_i, each patch has the gauge field a_i and the matter field ϕ_i with the irreducible representation ρ.
- Gauge fields at $U_{ij} = U_i \cap U_j$ are connected by the gauge transformation function g_{ij}.

$$a_j = g_{ij}{}^{-1} a_i g_{ij} - i g_{ij}{}^{-1} \mathrm{d} g_{ij},$$
$$\phi_j = \rho(g_{ij}{}^{-1}) \phi_i.$$

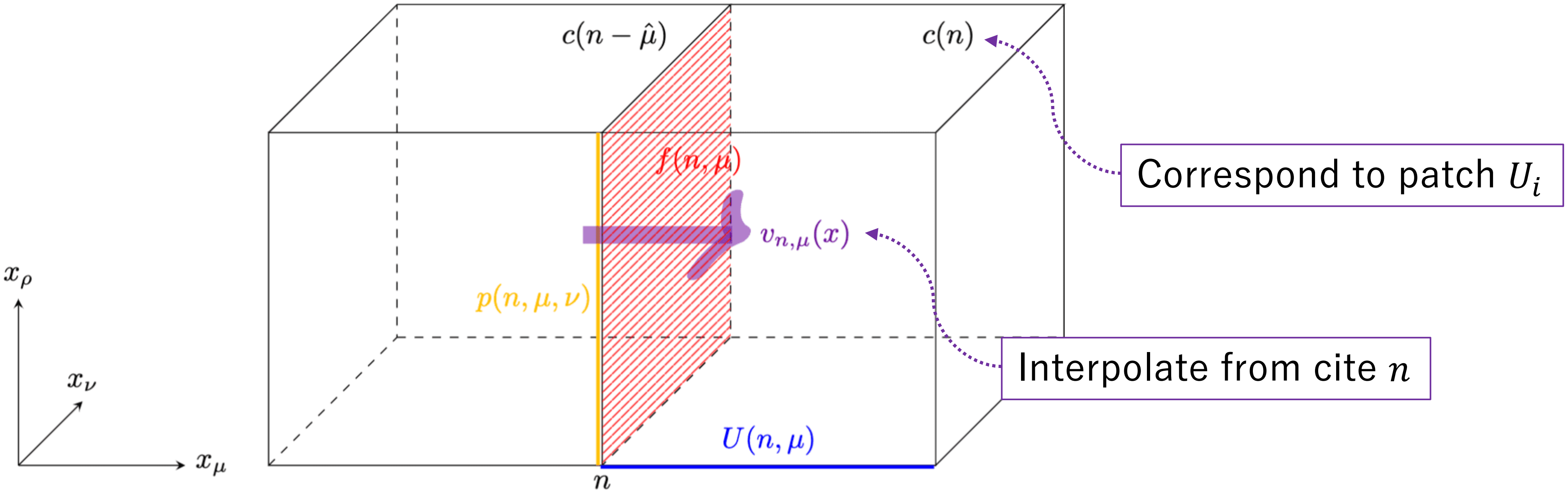
- At $U_{ijk} = U_i \cap U_j \cap U_k$, the cocycle condition is satisfied,

$$g_{ij} g_{jk} g_{ki} = 1.$$



Fiber Bundle on the Lattice

- The manifold is divided by hyper cubes $c(n)$.
- e. g., in the 3d,



Transition Function for Fractional TC

- Coupling $\mathbb{Z}_N$ 2-form field to the theory, the structure of fiber bundle becomes rich.

$$\tilde{v}_{n-\hat{\nu},\mu}(n)\tilde{v}_{n,\nu}(n)\tilde{v}_{n,\mu}(n)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(n)^{-1} = e^{\frac{2\pi i}{N}B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\mathbb{1}.$$

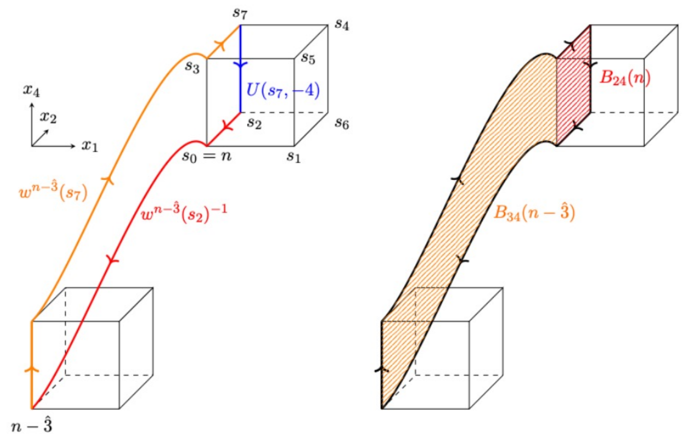
- We find that the $\mathbb{Z}_N$ 1-form gauge invariance plays the center role.

➢ Admissibility condition

$$\|\mathbb{1} - \tilde{U}_{\mu\nu}(n)\| < \varepsilon,$$

$$\tilde{U}_{\mu\nu}(n) \equiv e^{-\frac{2\pi i}{N}B_{\mu\nu}(n)} \times U(n,\mu)U(n+\hat{\mu},\nu)U(n+\hat{\nu},\mu)^{-1}U(n,\nu)^{-1}.$$

➢ Components of transition function



Construction of $u^{n-\hat{3}}_{s_7 s_2}$ Construction of $\tilde{u}^{n-\hat{3}}_{s_7 s_2}$

Fractional TC

- By the $\mathbb{Z}_N$ 1-form invariant transition function, we calculate TC,

$$z_{\mu\nu} = \sum_{p\in(T^2)_{\mu\nu}} B_p \qquad \mathrm{mod}\ N,$$

$$Q_{\mathrm{top}} = -\frac{1}{N}\int_{T^4}\frac{1}{2}P_2(B_p)\ \mathrm{mod}\ 1 \in -\frac{1}{N}\frac{\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}{8} + \mathbb{Z},$$

$$P_2(B_p) = B_p \cup B_p + B_p \cup_1 \mathrm{d}B_p\,.$$

- In the $U(1)$ lattice gauge theory, we make sure that (cf. Abe, Morikawa, Suzuki, arXiv:2210.12967[hep-th])

$$Q_{\mathrm{top}} = \frac{1}{32\pi^2}\sum_{n\in\Lambda}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\tilde{F}_{\mu\nu}(n)\tilde{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu}) \in \frac{1}{N^2}\mathbb{Z}+\mathbb{Z}.$$

Anomaly Ⅰ

- Again, the action on the lattice is

$$S[U_l, B_p] \equiv -S_W[U_l, B_p] + i\theta Q_{\text{top}}[U_l, B_p].$$

- The topological charge is

$$Q_{\text{top}} = -\frac{1}{N}\int_{T^4} \frac{1}{2} P_2(B_p) + \mathbb{Z} \equiv \text{frac}[B_p] + \text{int}[U_l, B_p].$$

  - ✧ Manifestly invariant under the $\mathbb{Z}_N$ one-form gauge transformation

➢ We discuss the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and θ shift.

Anomaly II

- At $\theta = \pi$, the partition function is, under $\mathcal{T}$ transformation,

$$Z[B_p] = \int \mathcal{D}U_l \; e^{S[U_l,B_p]} = \int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\theta Q_{\mathrm{top}}[U_l,B_p]}$$

$$\xrightarrow[\theta=\pi,\ \theta \text{ shift.}]{} Z'[B_p] = \int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\pi(-Q_{\mathrm{top}}[U_l,B_p])} = \int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\pi Q_{\mathrm{top}}[U_l,B_p]} \underbrace{e^{-i2\pi Q_{\mathrm{top}}[U_l,B_p]}}_{=e^{-i2\pi \mathrm{int}[U_l,B_p]} e^{-i2\pi \mathrm{frac}[B_p]}}$$

$$= e^{-i2\pi \mathrm{frac}[B_p]} \underbrace{\int \mathcal{D}U_l \; e^{-S_W[U_l,B_p]} e^{i\pi Q_{\mathrm{top}}[U_l,B_p]}}_{=Z} \neq Z[B_p]$$

- This means that there is the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and θ shift.

Conclusion and Future Work

☆Conclusion

- We construct the fractional topological charge on the $SU(N)$ lattice gauge theory.
- By this topological charge, we construct the anomaly at $\theta = \pi$ between the $\mathbb{Z}_N$ 1-form gauge and θ shift on the lattice.

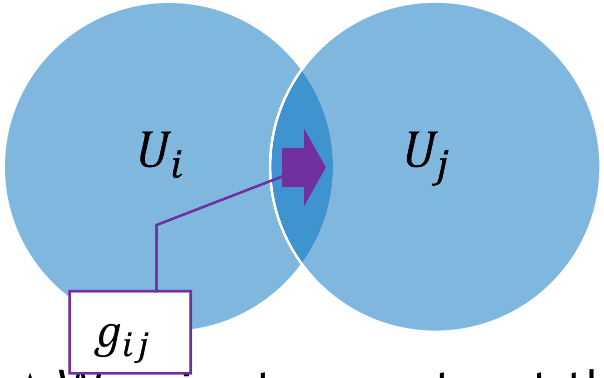
☆Future work

- Construct the magnetic operator under the admissibility condition on the lattice
  - ✓ cf. Abe, Morikawa, Onoda, Suzuki, Tanizaki, arXiv:2304.14815 [hep-lat]
  - ✓ Talk by Onoda (today, 14:30~)
- Construct non-invertible symmetries on the lattice

U(1) Part

Fiber Bundle and Fractional Topological Charge

- We construct the fiber bundle which makes the topological charge fractional.
  ('t Hooft, Nucl. Phys. B 153 (1979), van Baal, Commun. Math. Phys. 85 (1982))



cocycle condition: $g_{ij}g_{jk}g_{ki} = \underbrace{\exp\left(\frac{2\pi i}{N} n_{ijk}\right)}_{\in \mathbb{Z}_N}$

$$\text{(non-trivial transition function)} \sim \omega_\mu \times (SU(N) \text{ transition function})$$

factor of fractionality

☆We aim to construct the fractional topological charge on the lattice.

- ➢ We utilize the formulation for the integer topological charge on the $SU(N)$ lattice gauge theory.
  (Lüscher, Commun. Math. Phys. 85 (1982))

➢We pay attention to the $\mathbb{Z}_N$ one form invariance.

New Transition Function on the Lattice

transition function with the factor of fractionality in the continuum theory

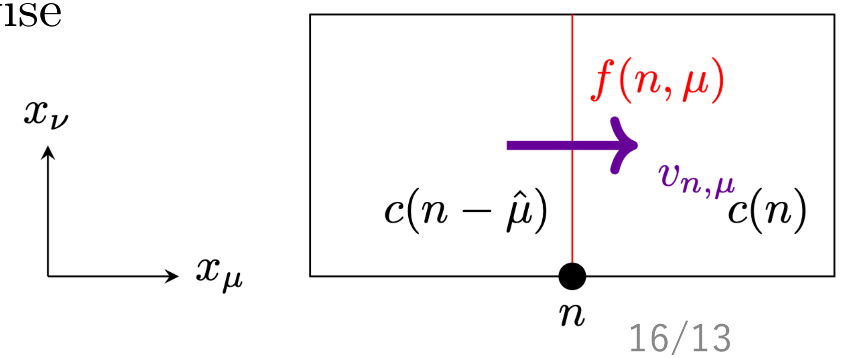
$$(\text{non-trivial transition function}) \sim \omega_\mu \times (SU(N) \text{ transition function})$$

- We construct the transition function $v_{n,\mu}$ at $x \in f(n,\mu)$ in the $U(1)/\mathbb{Z}_q$ lattice gauge theory.
  - ➢ ω_μ is the factor of fractionality on the lattice.

$$v_{n,\mu}(x) = \omega_\mu(x)\check{v}_{n,\mu}(x)$$

$$\omega_\mu(x) \equiv \begin{cases} \exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right) & \text{for } x_\mu = 0 \bmod L \\ 1 & \text{otherwise} \end{cases}$$

  - ➢ $z_{\mu\nu} \in \mathbb{Z}$ and $z_{\mu\nu} = -z_{\nu\mu}$



Topological Charge on the Lattice

- We calculate the topological charge by the new transition function.

$$Q=-\frac{1}{8\pi^2}\sum_n\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\int_{p(n,\mu,\nu)}\mathrm{d}^2x\,[v_{n,\mu}(x)\partial_\rho v_{n,\mu}(x)^{-1}][v_{n-\hat{\mu},\nu}(x)^{-1}\partial_\sigma v_{n-\hat{\mu},\nu}(x)]$$

- By the new transition function $v_{n,\mu}(x)=\omega_\mu(x)\check{v}_{n,\mu}(x)$

factor of fractionality

$$\omega_\mu(x)\sim\exp\left(\frac{\pi i}{q}\sum_{\nu\neq\mu}\frac{z_{\mu\nu}x_\nu}{L}\right)$$

$$Q=\underbrace{\frac{1}{8q^2}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}z_{\rho\sigma}}_{\text{Fractional!!}}+\underbrace{\frac{1}{8\pi q}\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}z_{\mu\nu}\sum_{n_\mu=0}\check{F}_{\rho\sigma}(n)}_{\text{cross term}}$$

$$+\underbrace{\frac{1}{32\pi^2}\sum_n\sum_{\mu,\nu,\rho,\sigma}\varepsilon_{\mu\nu\rho\sigma}\check{F}_{\mu\nu}(n)\check{F}_{\rho\sigma}(n+\hat{\mu}+\hat{\nu})}_{\text{integer}}$$

Anomaly

- The action on the lattice is

$$S \equiv \overbrace{\frac{1}{4g_0^2}\sum_n \sum_{\mu,\nu} \check{F}_{\mu\nu}(n)\check{F}_{\mu\nu}(n) + S_{\text{matter}}}^{=S_0} \underbrace{-iq\theta Q}$$

By the Witten effect(Honda, Tanizaki, arXiv:2009.10183)

- The topological charge is

$$qQ = \frac{1}{8q}\sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma} + \mathbb{Z} \equiv \text{frac}[z] + \text{int}[a, z]$$

✧ Invariant under the $\mathbb{Z}_q$ one-form gauge transformation

✧ Odd under the $\mathcal{T}$ transformation on the lattice, $qQ \xrightarrow{\mathcal{T}} -qQ$

➢ We discuss the anomaly at $\theta = \pi$ between the $\mathbb{Z}_q$-one form gauge and the $\mathcal{T}$ symmetry.

Anomaly

- Adding the local counter term, at $\theta = \pi$ the partition function is, under $\mathcal{T}$ transformaion,

$$Z[z] = \int \mathcal{D}a\ e^{S[a,z]} = \int \mathcal{D}a\ e^{S_0[a,z]} e^{i\theta q Q[a,z]}$$

$$\xrightarrow[\theta=\pi,\ \mathcal{T}\ \text{trans.}]{} Z' = \int \mathcal{D}a\ e^{S_0[a,z]} e^{i\pi(-qQ[a,z])} = \int \mathcal{D}a\ e^{S_0[a,z]} e^{i\pi q Q[a,z]} \underbrace{e^{-i2\pi q Q[a,z]}}_{=e^{-i2\pi \mathrm{int}[a,z]} e^{-i2\pi \mathrm{frac}[z]}}$$

$$= e^{-i2\pi \mathrm{frac}[z]} \underbrace{\int \mathcal{D}a\ e^{S_0[a,z]} e^{i\pi q Q[a,z]}}_{=Z} \neq Z$$

$$\xrightarrow[\text{including counter term}]{} \exp\left[-\frac{2\pi i (2k+1)}{8q} \underbrace{\sum_{\mu,\nu,\rho,\sigma} \varepsilon_{\mu\nu\rho\sigma} z_{\mu\nu} z_{\rho\sigma}}_{=0,\pm 8, \pm 16, \cdots,} \right] Z$$

For $q \in 2\mathbb{Z}$, the anomaly exists!

Back Up

Higher Form Symmetry Ⅰ

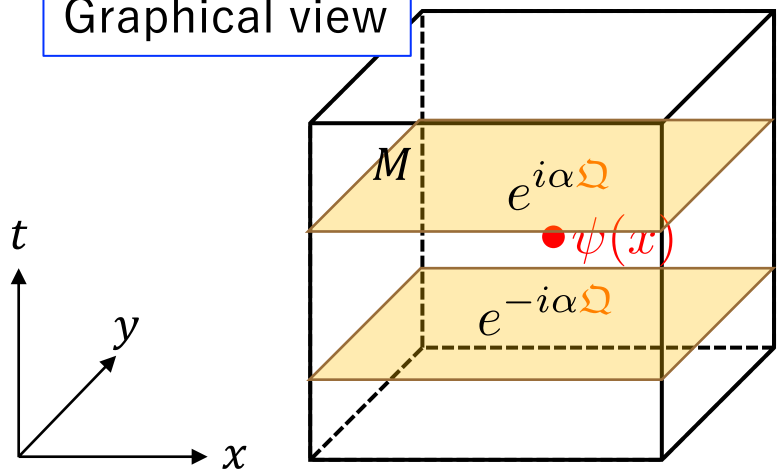
Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148[hep-th]

- Traditional symmetry (0-form symmetry) : transform the point $\psi(x)$.
  - ✓ e.g., global $U(1)$ symmetry $\psi(x) \to e^{i\alpha}\psi(x)$.

➢ In (2+1)d, look this $\psi(x)$'s transformation by the symmetry operator,

$$e^{i\alpha\mathfrak{Q}}\psi(x)e^{-i\alpha\mathfrak{Q}} = e^{-i\alpha}\psi(x), \quad \mathfrak{Q} = \int_M \mathrm{d}^2x\, j^0(x), \quad j^\mu(x) = i\bar{\psi}(x)\gamma^\mu\psi(x).$$

Graphical view



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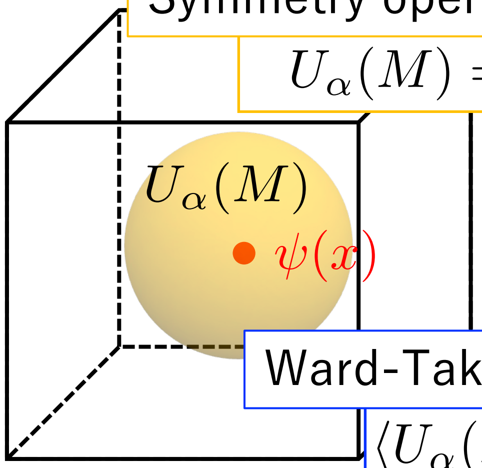

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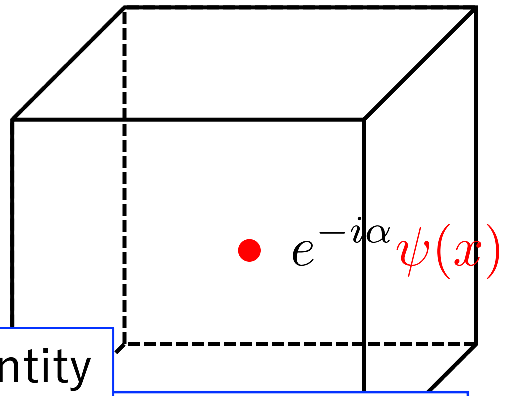


Ward-Takahashi identity

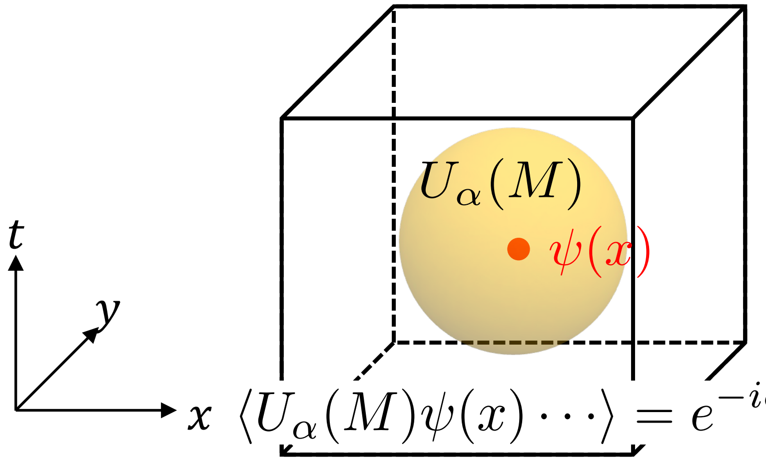
$$\langle U_\alpha(M)\psi(x)\cdots\rangle = e^{-i\alpha}\langle\psi(x)\cdots\rangle$$

Higher Form Symmetry Ⅱ

Gaiotto, Kapustin, Seiberg, Willett, arXiv:1412.5148[hep-th]

- Traditional symmetry (0-form symmetry) : transform the point $\psi(x)$.
  - ✓ e.g., global $U(1)$ symmetry $\psi(x) \to e^{i\alpha}\psi(x)$.
- ➢ Extend the point to 2d, 3d,⋯ objects

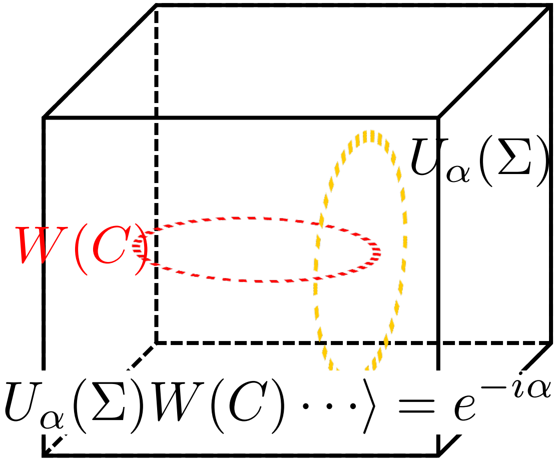
- 0-form symmetry
  - ➢ Transform point $\psi(x)$



$$\langle U_\alpha(M)\psi(x)\cdots\rangle = e^{-i\alpha}\langle\psi(x)\cdots\rangle$$

Extension

- 1-form symmetry
  - ➢ Transform loop $W(C)$



$$\langle U_\alpha(\Sigma)W(C)\cdots\rangle = e^{-i\alpha}\langle W(C)\cdots\rangle$$

$\mathbb{Z}_N$ 1-form Global Transformation on the Lattice

- Lattice $SU(N)$ gauge theory, the action is

$$S_W[U_l] \equiv \sum_p \beta\left[\mathrm{tr}\left(\mathbb{1} - U_p\right) + \mathrm{c.c.}\right].$$

➢ Center transformation ($\mathbb{Z}_N$ 1-form global transformation) on the lattice acts on the link variables,

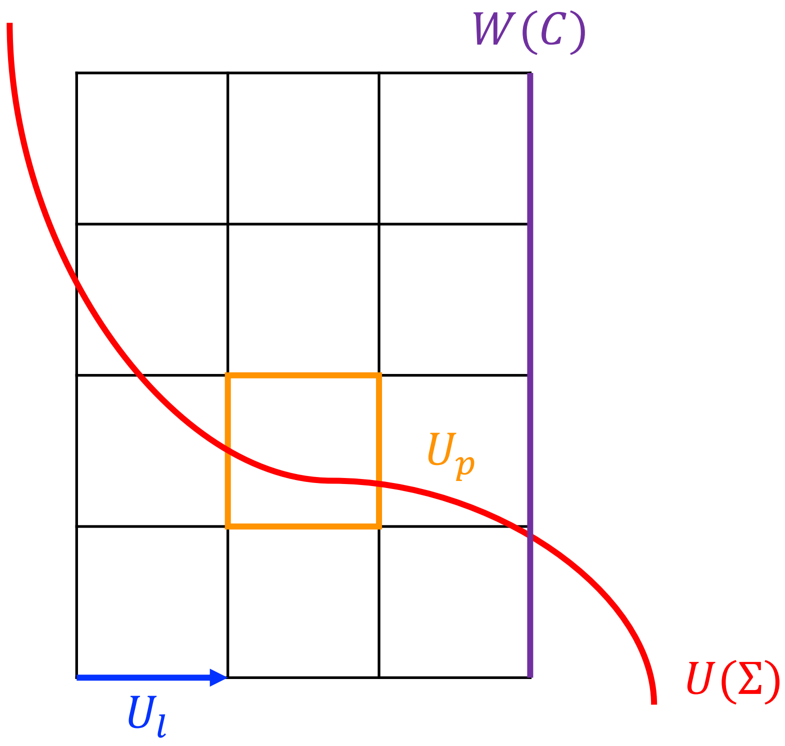
$$U_l \mapsto e^{\frac{2\pi i}{N}k} U_l, \quad W(C) \mapsto e^{\frac{2\pi i}{N}k} W(C).$$

※ Recall that under the 1-form global transformation in the continuum theory, the Wilson line changes,

$$\langle U_\alpha(\Sigma) W(C) \cdots \rangle = e^{-i\alpha} \langle W(C) \cdots \rangle$$

➢ The transition function satisfies the cocycle condition still.

$$v_{n-\hat{\mu},\nu}(x) v_{n,\mu}(x) v_{n,\nu}(x)^{-1} v_{n-\hat{\nu},\mu}(x)^{-1} = \mathbb{1}.$$



$\mathbb{Z}_N$ 1-form Gauge Transformation on the Lattice

- Gauging the center symmetry, the action becomes

$$S_{\mathrm{W}}[U_\ell, B_p] = \sum_p \beta \left[\mathrm{tr}\left(\mathbb{1} - e^{-\frac{2\pi i}{N}B_p} U_p\right) + \mathrm{c.c.}\right].$$

➢ Invariant under the $\mathbb{Z}_N$ 1-form gauge transformation,

$$U_l \mapsto e^{\frac{2\pi i}{N}\lambda_l} U_l, \quad B_p \mapsto B_p + (\mathrm{d}\lambda)_p.$$

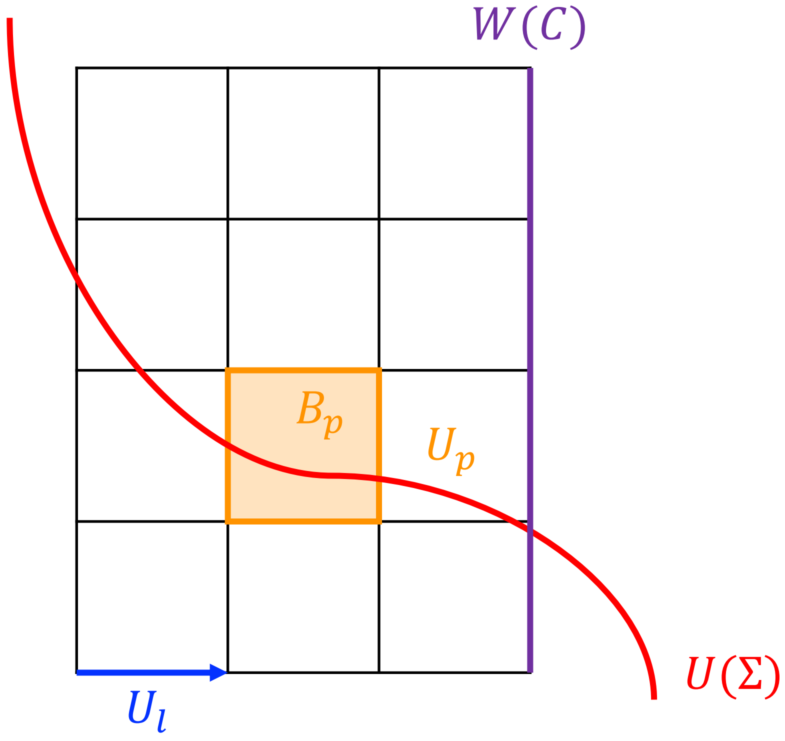
➢ The cocycle condition is relaxed,

$$\tilde{v}_{n-\hat{\nu},\mu}(n)\tilde{v}_{n,\nu}(n)\tilde{v}_{n,\mu}(n)^{-1}\tilde{v}_{n-\hat{\mu},\nu}(n)^{-1} = e^{\frac{2\pi i}{N}B_{\mu\nu}(n-\hat{\mu}-\hat{\nu})}\mathbb{1}.$$

※ ‘t Hooft twisted boundary condition

$$U(n+L\hat{\nu},\mu) = g_{n,\mu}^{-1} U(n,\mu) g_{n+\hat{\mu},\nu}$$

$$g_{n+L\hat{\nu},\mu}^{-1} g_{n,\nu}^{-1} g_{n,\mu} g_{n+L\hat{\mu},\nu} = e^{\frac{2\pi i}{N}z_{\mu\nu}}, \quad z_{\mu\nu} = \sum_p B_p \bmod N.$$



Review of Lüscher's construction

Lüscher, Commun. Math. Phys. 85 (1982)

- For the integral topological charge,

Step1

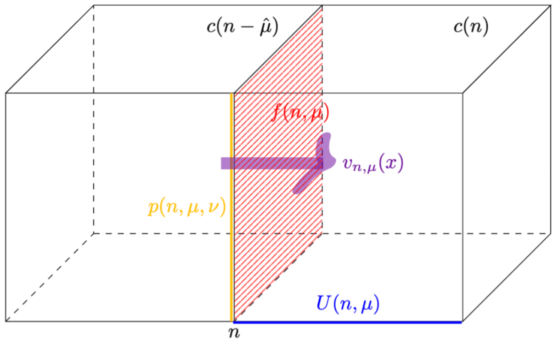
- Define $v_{n,\mu}(n)$ at the corner of $f(n,\mu)$ with complete axial gauge.
- ➢Define the parallel transporter $w^m(x)$ with complete axial gauge.

Step2

- Interpolate $v_{n,\mu}(n)$ to the x in the face $f(n,\mu)$.
- ➢Define the interpolate function $S^m_{n,\mu}(x)$.

Step3

- To define $S^m_{n,\mu}(x)$ correctly, we need the admissibility condition.



Step1: Transition Function

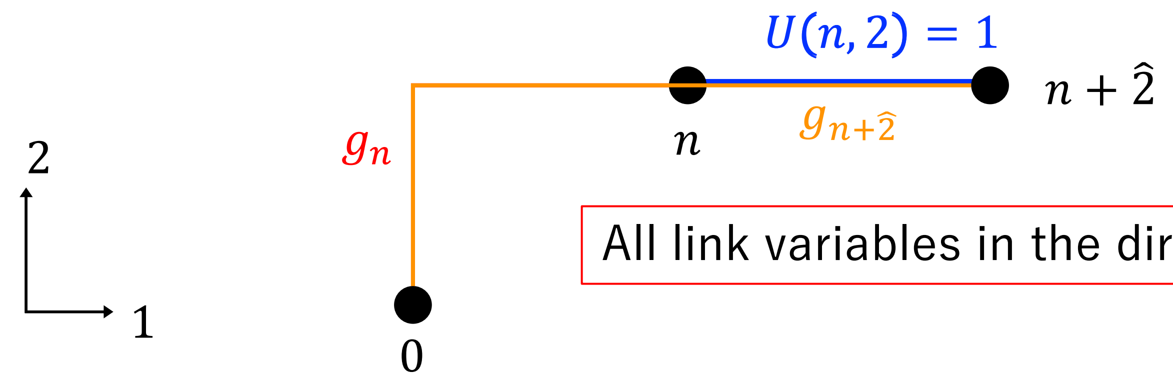
- Define the parallel transporter $(n \to x = n + \sum_\mu z_\mu \hat{\mu},\ z_\mu = \{0,1\})$,

$$w^n(x) := U(n,4)^{z_4} U(n + z_4\hat{4}, 3)^{z_3} U(n + z_4\hat{4} + z_3\hat{3}, 2)^{z_2} U(n + z_4\hat{4} + z_3\hat{3} + z_2\hat{2}, 1)^{z_1}.$$

➢ This selection means to take the complete axial gauge on the lattice.

➢ Gauge transformation on the lattice,

$$U(n,\mu) \mapsto {g_n}^{-1} U(n,\mu) g_{n+\hat{\mu}}.$$



All link variables in the direction 2 become 1.

Step1: Transition Function

- Define the parallel transporter $(n \to x = n + \sum_\mu z_\mu \hat{\mu},\ z_\mu = \{0,1\})$,

$$w^n(x) := U(n,4)^{z_4} U(n + z_4\hat{4}, 3)^{z_3} U(n + z_4\hat{4} + z_3\hat{3}, 2)^{z_2} U(n + z_4\hat{4} + z_3\hat{3} + z_2\hat{2}, 1)^{z_1}.$$

➢ This selection means to take the complete axial gauge on the lattice.

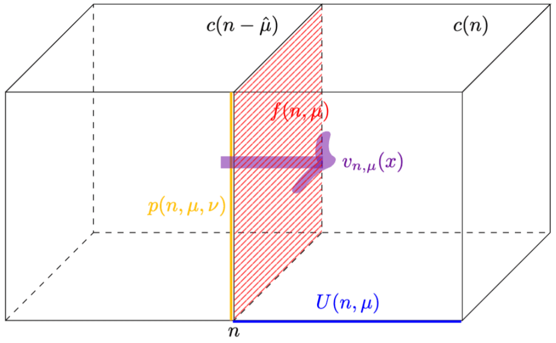
➢ Define link variables in complete axial gauge $(n \to x \to x + \hat{\mu} \to n)$,

$$u^n_{x,x+\hat{\mu}} := w^n(x) U(x,\mu) w^n(x+\hat{\mu})^{-1}.$$

➢ Define link variables in complete axial gauge in another way,

$$u^{n-\hat{\mu}}_{x,x+\hat{\mu}} = w^{n-\hat{\mu}}(x) \underbrace{w^n(x)^{-1} u^n_{x,x+\hat{\mu}} w^n(x+\hat{\mu})}_{\sim U(x,\mu)} w^{n-\hat{\mu}}(x+\hat{\mu})^{-1}$$

$$= v_{n,\mu}(x) u^n_{x,x+\hat{\mu}} v_{n,\mu}(x+\hat{\mu})^{-1}.$$



$$a_j \sim g_{ij}{}^{-1} a_i g_{ij}$$

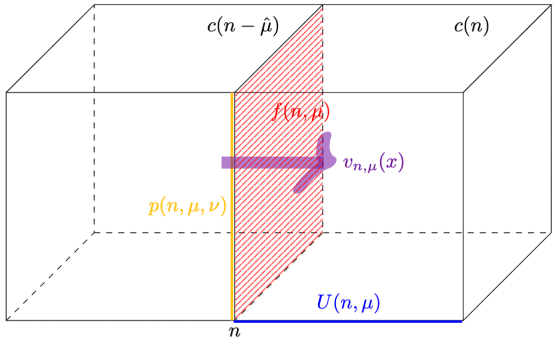
Transition function

$$v_{n,\mu}(x) := w^{n-\hat{\mu}}(x) w^n(x)^{-1}$$

Step2: To the Coordinate x

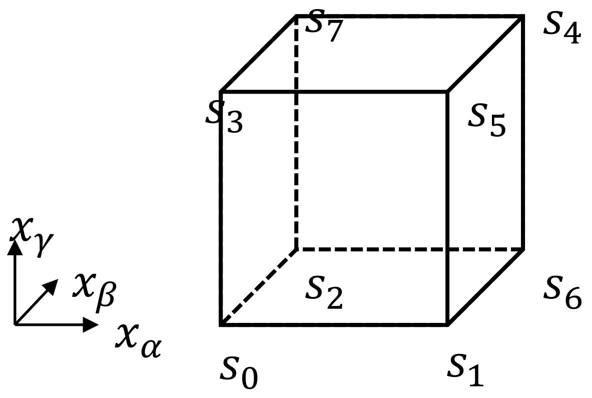
- Interpolate the transition function to the x in the face $f(n,\mu)$,

$$v_{n,\mu}(x) = S^{n-\hat{\mu}}_{n,\mu}(x)^{-1} v_{n,\mu}(n) S^{n}_{n,\mu}(x).$$



➢ Define the interpolate function $S^m_{n,\mu}(x)$.

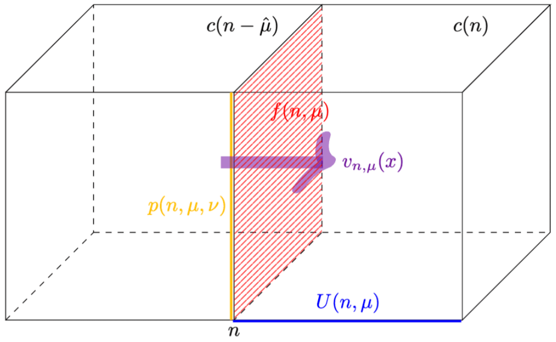
$$\begin{aligned}
f^m_{n,\mu}(x_\gamma) &= (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_7} u^m_{s_7 s_2} u^m_{s_2 s_0})^{y_\gamma} u^m_{s_0 s_2} (u^m_{s_2 s_7})^{y_\gamma}, \\
g^m_{n,\mu}(x_\gamma) &= (u^m_{s_5 s_1})^{y_\gamma} (u^m_{s_1 s_5} u^m_{s_5 s_4} u^m_{s_4 s_6} u^m_{s_6 s_1})^{y_\gamma} u^m_{s_1 s_6} (u^m_{s_6 s_4})^{y_\gamma}, \\
h^m_{n,\mu}(x_\gamma) &= (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_5} u^m_{s_5 s_1} u^m_{s_1 s_0})^{y_\gamma} u^m_{s_0 s_1} (u^m_{s_1 5})^{y_\gamma}, \\
k^m_{n,\mu}(x_\gamma) &= (u^m_{s_7 s_2})^{y_\gamma} (u^m_{s_2 s_7} u^m_{s_7 s_4} u^m_{s_4 s_6} u^m_{s_6 s_2})^{y_\gamma} u^m_{s_2 s_6} (u^m_{6 s_4})^{y_\gamma}, \\
l^m_{n,\mu}(x_\beta, x_\gamma) &= \left[f^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \left[f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \\
&\quad \cdot h^m_{n,\mu}(x_\gamma) \left[g^m_{n,\mu}(x_\gamma)\right]^{y_\beta}, \\
S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) &= (u^m_{s_0 s_3})^{y_\gamma} \left[f^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \left[l^m_{n,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha}.
\end{aligned}$$



Step2: To the Coordinate x

- Interpolate the transition function to the x in the face $f(n,\mu)$,

$$v_{n,\mu}(x) = S_{n,\mu}^{n-\hat{\mu}}(x)^{-1} v_{n,\mu}(n) S_{n,\mu}^{n}(x).$$



- e.g., in 2d, $(0 \le y_1 \le 1)$

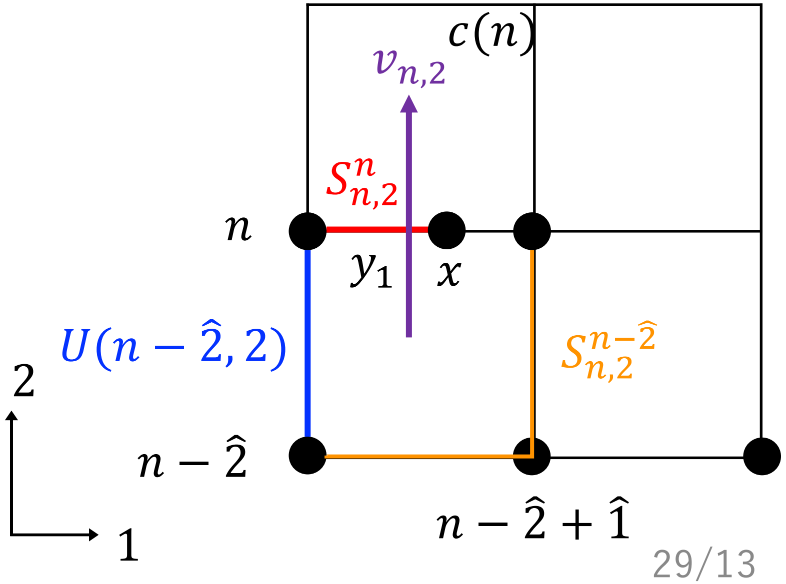
$$S_{n,2}^{n-\hat{2}}(x) = U(n-\hat{2},1)^{y_1} U(n-\hat{2}+\hat{1},2)^{y_1} U(n-\hat{2},2)^{-y_1},$$

$$S_{n,2}^{n}(x) = U(n,1)^{y_1},$$

$$v_{n,2}(n) = U(n-\hat{2},2),$$

$$v_{n,2}(x) = S_{n,2}^{n-\hat{2}}(x)^{-1} v_{n,2}(n) S_{n,2}^{n}(x)$$

$$= U(n-\hat{2},2)\exp\left[iy_1 F_{12}(n-\hat{2})\right].$$



Step3: Admissibility condition

- We take the compact $SU(N)$ lattice gauge theory.

➢ For simplicity, in the compact $U(1)$ lattice gauge theory,

$$U_\mu(n) = e^{ia_\mu(n)}, \quad (-\pi \leq a_\mu(n) \leq \pi)$$

$$F_{\mu\nu}(n) = \frac{1}{i} \ln U_\mu(n) U_\nu(n+\hat{\mu}) U_\mu(n+\hat{\nu})^{-1} U_\nu(n)^{-1} := \frac{1}{i} \ln U_p. \quad (-\pi \leq F_{\mu\nu}(n) \leq \pi)$$

➢ When $|F_{\mu\nu}| = \pi$, the component, $e^{iyF_{\mu\nu}} = (-1)^y$, becomes ambiguous.

➢ Require the gauge field smooth (admissibility condition),

$$\|\mathbb{1} - U_p\| < \varepsilon.$$

Step3: Admissibility condition

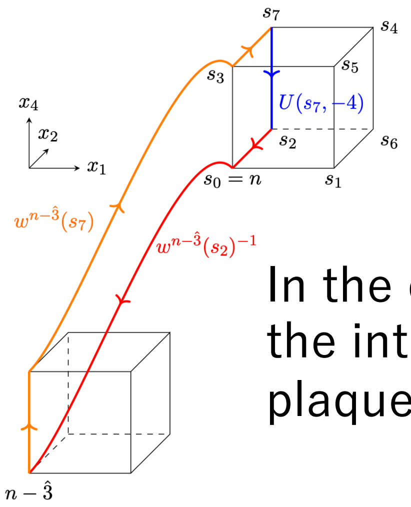
- We take the compact $SU(N)$ lattice gauge theory.

➢ Considering the interpolate function, we select the value of ε.

$$\|\mathbb{1} - U_p\| < \varepsilon.$$

➢ Recall the link variables with the complete axial gauge and interpolate function.

➢ e.g.,



Construction of $u^{n-\hat{3}}_{s_7 s_2}$

In the component of the interpolate function, plaquette is appeared.

$$
\begin{aligned}
f^m_{n,\mu}(x_\gamma) &= (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_7} u^m_{s_7 s_2} u^m_{s_2 s_0})^{y_\gamma} u^m_{s_0 s_2} (u^m_{s_2 s_7})^{y_\gamma}, \\
g^m_{n,\mu}(x_\gamma) &= (u^m_{s_5 s_1})^{y_\gamma} (u^m_{s_1 s_5} u^m_{s_5 s_4} u^m_{s_4 s_6} u^m_{s_6 s_1})^{y_\gamma} u^m_{s_1 s_6} (u^m_{s_6 s_4})^{y_\gamma}, \\
h^m_{n,\mu}(x_\gamma) &= (u^m_{s_3 s_0})^{y_\gamma} (u^m_{s_0 s_3} u^m_{s_3 s_5} u^m_{s_5 s_1} u^m_{s_1 s_0})^{y_\gamma} u^m_{s_0 s_1} (u^m_{s_1 5})^{y_\gamma}, \\
k^m_{n,\mu}(x_\gamma) &= (u^m_{s_7 s_2})^{y_\gamma} (u^m_{s_2 s_7} u^m_{s_7 s_4} u^m_{s_4 s_6} u^m_{s_6 s_2})^{y_\gamma} u^m_{s_2 s_6} (u^m_{6 s_4})^{y_\gamma}, \\
l^m_{n,\mu}(x_\beta, x_\gamma) &= \left[f^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \left[f^m_{n,\mu}(x_\gamma) k^m_{n,\mu}(x_\gamma) g^m_{n,\mu}(x_\gamma)^{-1} h^m_{n,\mu}(x_\gamma)^{-1}\right]^{y_\beta} \\
&\quad \cdot h^m_{n,\mu}(x_\gamma) \left[g^m_{n,\mu}(x_\gamma)\right]^{y_\beta}, \\
S^m_{n,\mu}(x_\alpha, x_\beta, x_\gamma) &= (u^m_{s_0 s_3})^{y_\gamma} \left[f^m_{n,\mu}(x_\gamma)\right]^{y_\beta} \left[l^m_{n,\mu}(x_\beta, x_\gamma)\right]^{y_\alpha}.
\end{aligned}
$$

Summary of Lüscher's construction

- Define the transition function at the coordinate $x \in f(n,\mu)$,

$$v_{n,\mu}(x) = S^{n-\hat{\mu}}_{n,\mu}(x)^{-1} v_{n,\mu}(n) S^{n}_{n,\mu}(x).$$

✓ Satisfy the cocycle condition,

$$v_{n-\hat{\mu},\nu}(x) v_{n,\mu}(x) v_{n,\nu}(x)^{-1} v_{n-\hat{\nu},\mu}(x)^{-1} = \mathbb{1}.$$

- Substituting $v_{n,\mu}(x)$, calculate integral topological charge.
- Define the admissibility condition,

$$\|\mathbb{1} - U_p\| < \varepsilon.$$

➢ Extend these to the fractional topological charge.