

Lattice study of RG fixed point based on gradient flow in 3D $O(N)$ sigma model

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Introduction

- Renormalization group (RG) is characterized by scaling relation
- Beyond perturbation theory, nonperturbative construction of quantum field theory
 - ▶ Nontrivial fixed point, anomalous dimensions

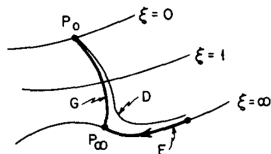
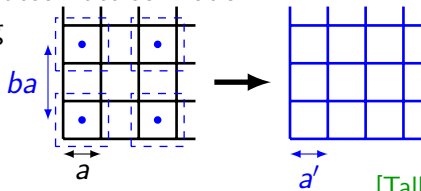


Fig. 12.6. Renormalization group trajectories near two fixed points. The curve D begins close to the critical surface. As the origin of D approaches the critical surface, the trajectory D approaches the trajectories E and G . For this example G is $S(\infty)$.

- Condensed matter lattice model: Spin blocking



[Wilson-Kogut '74]

- Nonperturbative field theory: Requires a momentum cutoff. . . Crucial problem because it is incompatible with gauge symmetry

[Talks by Haruna, Suzuki]

Gradient flow and block spin transformation?

- **Gradient flow** [Narayanan–Neuberger, Lüscher]:
an evolution of physical quantities along a *fictitious* time
- It looks like gradient flow acts as “coarse-graining” [Lüscher ...]
- E.g., Yang–Mills gradient flow equation

$$\partial_t B_\mu(t, x) = - \left. \frac{\delta S_{\text{YM}}}{\delta A_\mu} \right|_{A_\mu \rightarrow B_\mu} = D_\nu G_{\nu\mu}(t, x), \quad B_\mu(0, x) = A_\mu(x)$$

where $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu]$

- Diffusion eq. $\partial_t \phi(t, x) = \partial_\mu^2 \phi(t, x)$ w/ diffusion length $x \sim \sqrt{8t}$
- Blocked field is given as $\varphi_b(x_b) \sim \sum_{\epsilon} \varphi(x + \epsilon)$
 - ▶ $b \sim \sqrt{8t}$?
- Solution of $\partial_t \phi(t, x) = \partial_\mu^2 \phi(t, x)$
$$\phi(t, x) \sim \int_y e^{-(x-y)^2/4t} \varphi(y)$$

- Can we define **gauge-invariant** Wilson's RG by gradient flow?

Wilsonian RG flow [Makino–OM–Suzuki '18]

- Scaling relation under Wilsonian RG (running coupling $g_i(\xi)$)

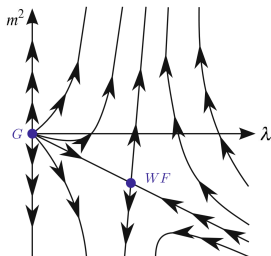
$$\begin{aligned} & \langle \mathcal{O}_1(e^{2\xi} t_1, e^\xi x_1) \dots \mathcal{O}_N(e^{2\xi} t_N, e^\xi x_N) \rangle_{\{g_i\}} \\ &= Z(\xi) \langle \mathcal{O}_1(t_1, x_1) \dots \mathcal{O}_N(t_N, x_N) \rangle_{\{g_i(\xi)\}} \end{aligned}$$

- Define new scheme as $g_{\text{GF}}^2(\mu) = \langle \mathcal{O}(t) \rangle_{\mu=1/\sqrt{8t}} \stackrel{\text{pert.}}{\sim} g^2 + O(g^4)$
 - cf. Numerical determination of α_S in QCD [ALPHA '17]
- Assumption as one-to-one map $g_i(\xi) \mapsto \langle \mathcal{O}_i(t) \rangle = \mathcal{R}_i[\{g_j(\xi)\}]$

$$\text{Gradient flow} \quad \langle \mathcal{O}_j(e^{2\xi} t) \rangle_{\{g_i\}} = \langle \mathcal{O}_j(t) \rangle_{\{g_i(\xi)\}} \quad \text{Running}$$

[Makino–OM–Suzuki '18]

- Gradient flow is **gauge covariant!**
A gauge-invariant construction of RG flow
 - Banks–Zaks fixed point in (2-loop approx) N_f -flavor QCD
 - Wilson–Fisher fixed point in Large N 3D $O(N)$ linear sigma model



3D $O(N)$ linear sigma model and large N

- Euclidean action ($i = 1, \dots, N$)

$$S_E = \int d^3x \left\{ \frac{1}{2} [\partial_\mu \phi_i(x)]^2 + \frac{1}{2} m_0^2 \phi_i^2(x) + \frac{1}{8N} \lambda_0 [\phi_i(x)^2]^2 \right\}$$

→ **Wilson-Fisher IR fixed point** [72]

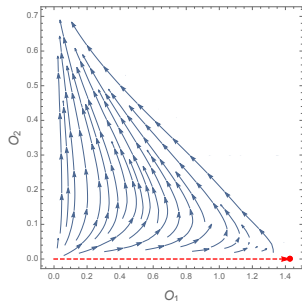
- Solution is well known at $N \rightarrow \infty$
- Flow eq. $\partial_t \varphi_i(t, x) = \partial_\mu^2 \varphi_i(t, x)$
- Dimensionless flowed operators:

$$\mathcal{O}_1(t, x) \equiv - \frac{N [\varphi_i(t, x)^2]^2}{\langle \varphi_j(t, x)^2 \rangle^2} + (N+2) \stackrel{t \rightarrow 0}{\sim} \lambda_0 t^{1/2}$$

$$\mathcal{O}_2(t, x) \equiv \frac{4t \partial_\mu \varphi_i(t, x)^2}{(2\pi)^{1/2} \langle \varphi_j(t, x)^2 \rangle} - \frac{1}{(2\pi)^{1/2}} \stackrel{t \rightarrow 0}{\sim} M t^{1/2}$$

M : physical mass determined by gap equation

- **IR fixed point** at $M/\lambda_0 \rightarrow 0$
 - ▶ cf. [Aoki-Balog-Onogi-Weisz '16]

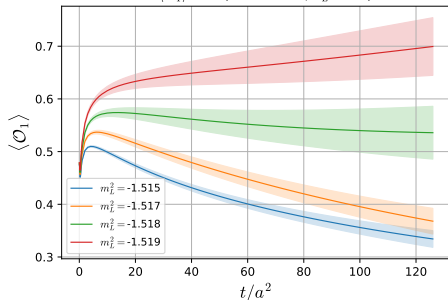


[Makino-OM-Suzuki '18]

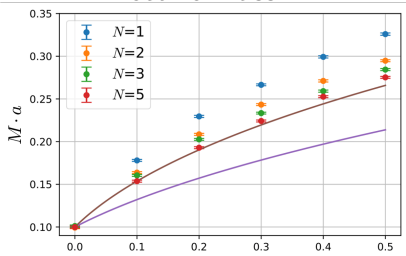
Lattice setup with finite N

- Parameter set
 - ▶ Lattice $64^3, 128^3$
 - ▶ $N = 1$ (partially 2, 3, 5)
 - ▶ Tuning $\lambda_0 a, m_0 a$ for small M
- Numerical simulation
 - ▶ Overrelaxed heatbath
 - ▶ 4th order Runge–Kutta method for gradient flow

$t - \langle \mathcal{O}_1 \rangle$ Plot (size = $128^3, \lambda_L = 5.0$)

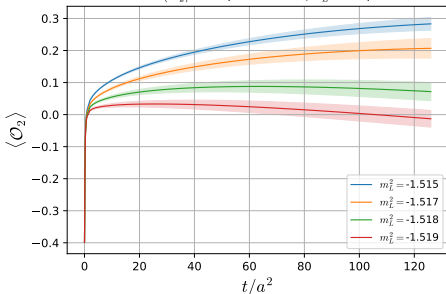


Effective mass M



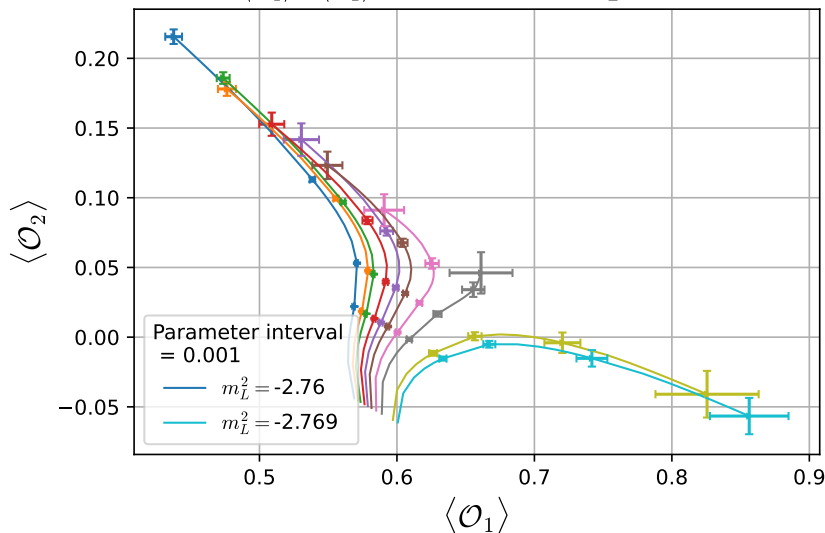
Two curves given by gap eq. at $N \rightarrow \infty$
with $\Lambda = \pi/2a$ or π/a

$t - \langle \mathcal{O}_2 \rangle$ Plot (size = $128^3, \lambda_L = 5.0$)



Lattice simulation of RG flow near criticality

$\langle \mathcal{O}_1 \rangle - \langle \mathcal{O}_2 \rangle$ Plot (size = 128^3 , $\lambda_L = 10.0$)

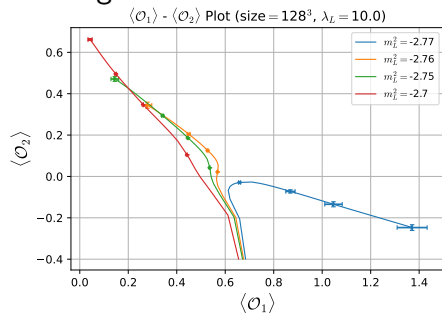


- Discussion: Finite size effect, Gaussian fixed point (following slides)

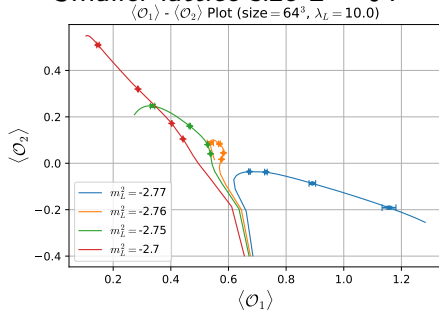
Finite lattice size effect

- Some trajectories in RG flow are intersecting near IR.
- This is because of a finite size effect as follows:

Larger lattice size $L = 128$

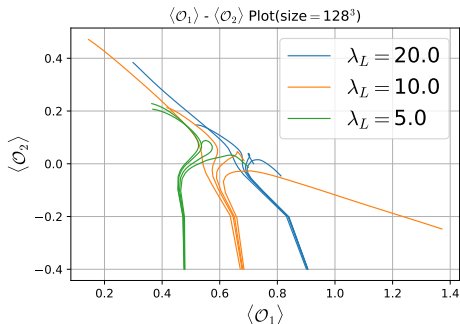


Smaller lattice size $L = 64$



From different “lattice models” to WF fixed point

- Starts from different points at UV (not Gaussian)
 - ▶ Not continuum limit (i.e., Gaussian) but just a lattice model.
 - ▶ Critical behaviors for each models should be same near WF.



- Small λ : close to Gaussian and far from WF
 - ▶ Much bigger box size and reducing errors at large t
- Large λ : far from Gaussian but hopefully close to WF
 - ▶ More computational costs when generating configurations

Summary

- Numerical simulation of 3D $O(N)$ linear sigma model
- Depicted RG flow near criticality by using operators evolved by gradient flow
 - ▶ Observation of Wilson–Fisher fixed point at IR ($t \rightarrow \infty$)
- More understanding on Gradient flow and RG in **gauge theory!**
 - ▶ Manifestly gauge-invariant construction of QFT
 - ▶ See Gradient-flow Exact RG
 - [Talks by Haruna, Suzuki (Plenary tomorrow)]
 - ▶ Application to lattice simulation of gauge theory?
- Gravity? (asymptotic safety and Ricci flow)
- Wilson's philosophy: every quantum system *sub specie aeternitatis* is clarified as a continuum theory near criticality (with infinite distances).