Curved domain-wall fermion and its applications

Naoto Kan (Osaka University)

Based on collaborations with Shoto Aoki, Hidenori Fukaya, Mikito Koshino, Yoshiyuki Matsuki (Osaka U.), arXiv:2304.13954 [cond-mat.mes-hall].

What will happen to a magnetic monopole when it is put inside a topological insulator?

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EFT of topological insulator and Witten effect

A topological insulator: the bulk is the insulator (gapped), but the edge is the gapless. The effective theory of the T-symmetric topological insulator is described by the $\theta = \pi$ vacuum. In the presence of the magnetic monopole, the θ-term is given by

$$L_\theta = \frac{\theta}{8\pi^2} \int d^3x \, \boldsymbol{E} \cdot \boldsymbol{B} = -\frac{q_m}{2} \int d^3x \, A^0 \delta^{(3)}(\boldsymbol{r}).$$

This implies that there is a particle with electric charge $q_e = -q_m/2$ which is coupled to the A^0 potential.

The monopole with $q_m = 1$ obtains the electric charge $q_e = -1/2$.

The effective theory description above is quite simple, but can't answer to the following questions:

(1) what is the origin of the electric charge? (must be electrons)

(2) if the origin is the electrons, why is it bound to monopole?

(3) why is the electric charge fractional?

Our idea: we introduce the Wilson term to the ordinary Dirac Hamiltonian.

Regularized Dirac equation

We consider the "regularized" Dirac Hamiltonian:

$$H_{\mathrm{reg}} = \gamma_0 \left(\gamma^i D_i + m + \frac{D_i^\dagger D^i}{M_{\mathrm{PV}}} \right).$$

We can interpret the additional "Wilson term" as a reg. eff., since

$$\begin{aligned} Z &= \det \left(\frac{D + m}{D + M_{\mathrm{PV}}} \right), \\ &= \det \Bigg[\frac{1}{M_{\mathrm{PV}}} \Bigg(D + m + \frac{1}{M_{\mathrm{PV}}} D_\mu^\dagger D^\mu \\ &\qquad + \mathcal{O}(1/M_{\mathrm{PV}}^2, m/M_{\mathrm{PV}}, F_{\mu\nu}/M_{\mathrm{PV}}) \Bigg) \Bigg]. \end{aligned}$$

in the Pauli–Villars reg., where $M_{\mathrm{PV}} (> 0)$ is the PV mass.

Since the Laplacian $D_i^\dagger D^i$ is always positive, the mass shift due to the Wilson term is always positive when we take $M_{\rm PV}$ positive.

For $m < 0$ (or inside topological insulators), it is possible to locally flip the sign of the "effective" mass

$$m < 0 \quad \rightarrow \quad m_{\rm eff} = m + \frac{D_i^\dagger D^i}{M_{\rm PV}} \sim m + \frac{1}{M_{\rm PV} r_1^2} > 0,$$

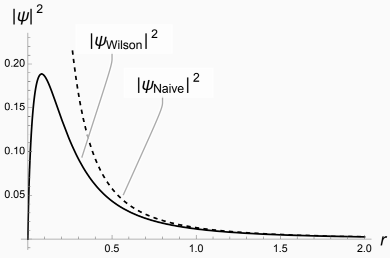
when the magnetic flux is concentrated in the region $r < r_1$.

It's implies that the inside region $r < r_1$ becomes a normal insulator, and the spherical domain-wall is dynamically created and the chiral edge-mode appears on it! (It doesn't happen in the normal insulator with $m > 0$.)

The analytical solution of the zero-mode for $r_1 \to 0$ is given by

$$\psi_{j,j_3}^{\mathrm{mono}} = \frac{Be^{-M_{\mathrm{PV}} r/2}}{\sqrt{r}} I_\nu(\kappa r) \begin{pmatrix} 1 \\ -\mathrm{sign}(n) \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi),$$

where $\nu = \sqrt{2|n|+1}/2$, $\kappa = M_{\mathrm{PV}} \sqrt{1 + 4m/M_{\mathrm{PV}}}/2$, and $n \in \mathbb{Z}$ is the magnetic charge.

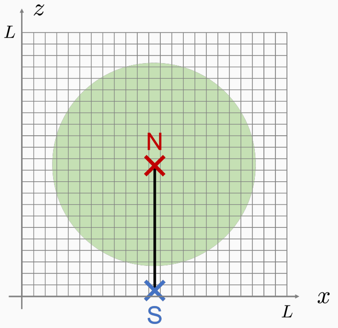


The plot with $n = 1$, $m = -0.1$, $M_{\mathrm{PV}} = 10$.

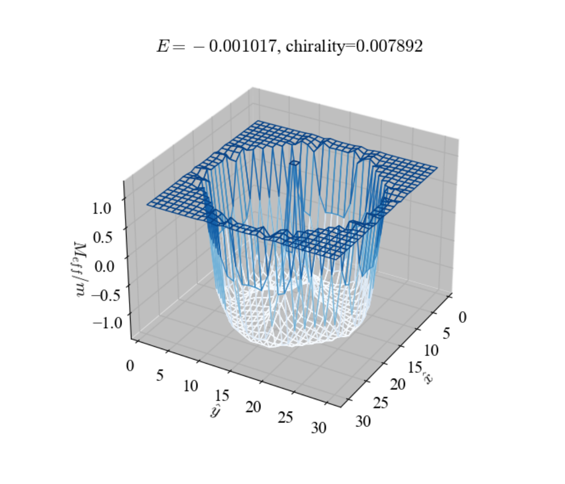
Numerical analysis

On a three-dimensional hyper-cubic lattice with size L with open boundary conditions, we put a monopole at $\boldsymbol{x}_m = (L/2, L/2, L/2)$ with a magnetic charge $n/2$. We also put an antimonopole at $\boldsymbol{x}_a = (L/2, L/2, 1/2)$ with the opposite charge $-n/2$.

For the fermion
field, we assign a position-dependent mass term to be $m(\boldsymbol{x}) = -m_0 (< 0)$ for $\sqrt{|\boldsymbol{x} - \boldsymbol{x}_m|} < 3L/8$,
and $m(\boldsymbol{x}) = +m_0$ otherwise.



The position-dependent effective mass of the (nearest) zeromode with $n = 1$ on $z = 16$ $(L = 32)$ slice:



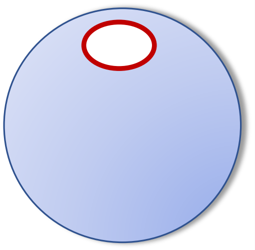
The Atiyah–Singer index theorem and the half-integral charge

So far, we considered a $\mathbb{R}^3$ space, but in order to discuss topological feature of the fermion zero mode, we also need an IR regularization, such as the one-point compactification, S^3.

Then the topological insulator region (with $m_{\text{eff}} < 0$) has topology of a disk with a small S^2 boundary at $r = r_1$.

However, due to the cobordism invariance of the AS index, the disk is not possible:

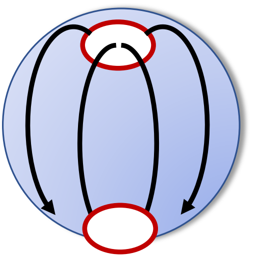
$$\int_{\partial M} F = \int_M dF = n \neq 0,$$



A resolution is: to create another domain-wall at, say, $r = r_0$, outside of the topological insulator.

Another zero mode is localized at the outside domain-wall, and the index is kept trivial.

$$0 = \int_M dF = \int_{\Sigma_{\mathrm{mono}}} F + \int_{\Sigma_{\mathrm{out}}} F,$$



where $\partial M = \Sigma_{\mathrm{mono}} \cup \Sigma_{\mathrm{out}}$.

This implies that the outside of the topological insulator must be a normal insulator (laboratory).

The edge states on the outside domain-wall

The regularized Hamiltonian around the outside domain-wall is

$$H = \gamma_0 \left(\gamma^i D_i + |m|\epsilon(r - r_0) + \frac{D_i^\dagger D^i}{M_{\text{PV}}} \right).$$

The edge-localized state is obtained as

$$\psi_{j,j_3}^{\text{DW}} = \begin{cases} \frac{\exp\left(\frac{M_{\text{PV}} r}{2}\right)}{\sqrt{r}} \left(B' K_\nu(\kappa_- r) + C' I_\nu(\kappa_- r) \right) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi) & (r < r_0), \\ \frac{D' \exp\left(\frac{M_{\text{PV}} r}{2}\right)}{\sqrt{r}} K_\nu(\kappa_+ r) \begin{pmatrix} 1 \\ s \end{pmatrix} \otimes \chi_{j,j_3,0}(\theta, \phi) & (r > r_0), \end{cases}$$

where $\kappa_\pm = \frac{M_{\text{PV}}}{2}\sqrt{1 \pm 4|m|/M_{\text{PV}}}$, and $s = \text{sign}(n)$.

At finite r_0, the paired zero-modes near the monopole at $r = r_1$ and the domain-wall at $r = r_0$ are mixed:

$$\psi = \alpha\psi^{\mathrm{mono}}_{j,j_3} + \beta\psi^{\mathrm{DW}}_{j,j_3}.$$

Because of $\{\bar{\gamma}, H\} = 0$,

$$\left(\psi^{\mathrm{mono}}_{j,j_3}\right)^\dagger H\psi^{\mathrm{mono}}_{j,j_3} = \left(\psi^{\mathrm{DW}}_{j,j_3}\right)^\dagger H\psi^{\mathrm{DW}}_{j,j_3} = 0,$$

and

$$\left(\psi^{\mathrm{mono}}_{j,j_3}\right)^\dagger H\psi^{\mathrm{DW}}_{j,j_3} = \left(\psi^{\mathrm{DW}}_{j,j_3}\right)^\dagger H\psi^{\mathrm{mono}}_{j,j_3} =: \Delta \sim \exp(-|m|r_0).$$

Then we can show $E = \pm\Delta$ and $\alpha = \pm\beta$.

Therefore,

$$\psi \sim \frac{1}{\sqrt{2}} \left(\psi_{j,j_3}^{\text{mono}} \pm \psi_{j,j_3}^{\text{DW}} \right).$$

The 50% amplitude is located around the monopole at $r = r_1$.

We conclude that (the expectation value of) the electric charge around the monopole is $q_e = -1/2$!

Summary

We discussed a microscopic description of the Witten effect with the Wilson term.

The spherical domain-wall is created around the monopole due to the effect of the Wilson term, which happens only in the topological insulator.

The chiral zero-modes localized at the domain-wall is topologically protected, so that the zero-modes are bound to the monopole.

The dressed electric charge of the monopole (dyon) by the Witten effect becomes $-1/2$, since the 50% of the wavefunction is localized around the monopole.