Phases in the fundamental Kazakov-Migdal model on the graph

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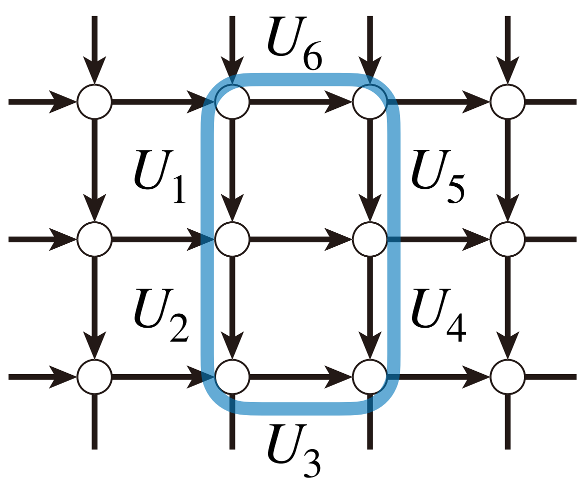
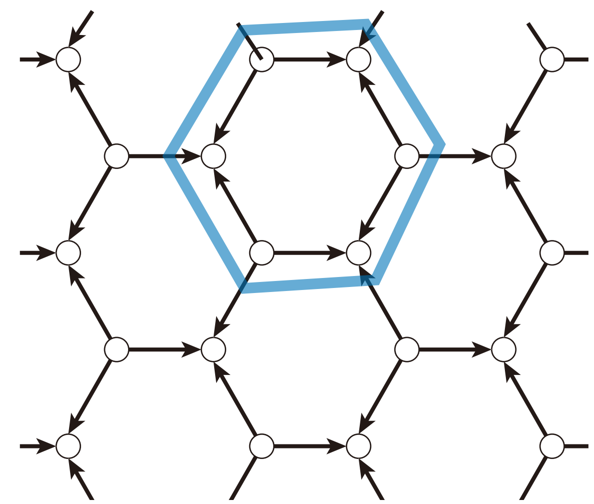
JHEP **2022**, 178 [arXiv: 2204.06424]
PTEP **2022**, 123B03 [arXiv:2208.14032]
Phys. Rev. D 108, 054504 [arXiv:2303.03692]
+ work in progress

“KEK Theory Workshop 2023”, 2023/11/30

Introduction

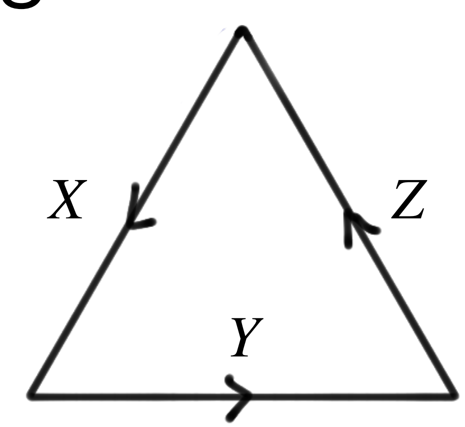
- Cycles on the *graph* play an important role in gauge and string theory
  - Wilson loops in lattice gauge theory:





$$W_C = \mathrm{Tr} U_1 U_2 U_3 U_4^\dagger U_5^\dagger U_6^\dagger \qquad W_C = \mathrm{Tr} U_1 U_2^\dagger U_3 U_4^\dagger U_5 U_6^\dagger$$

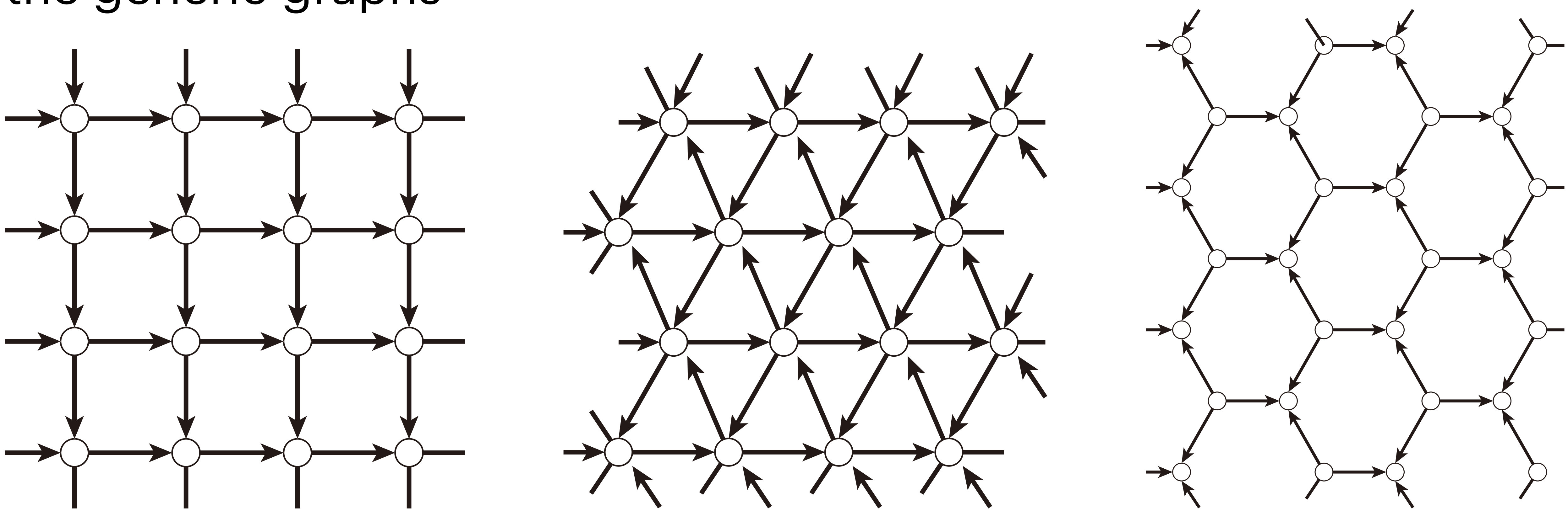
  - Gauge invariant operators in quiver gauge theory:



$$\mathcal{O} = \mathrm{Tr}\, XYZ$$

Introduction

- We propose a modification of the Kazakov-Migdal model defined on the generic graphs

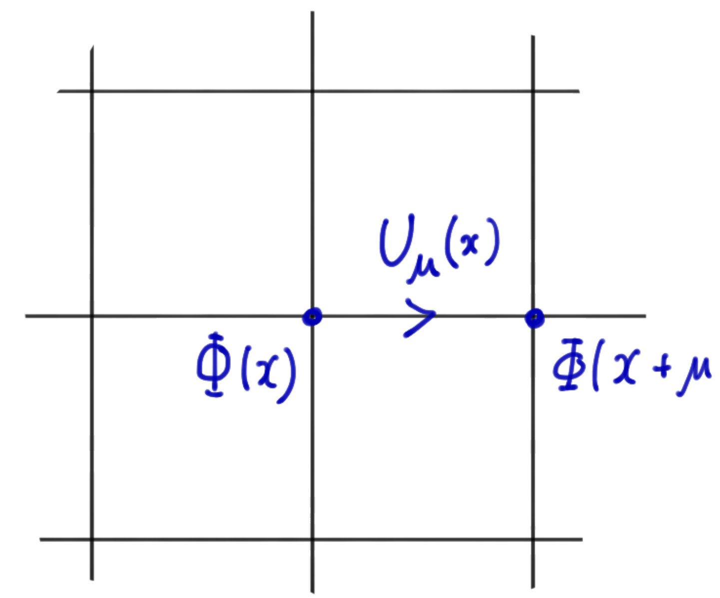


- This model has an interesting phase structure depending on the graph in the large *N* limit
- We also show numerical simulations to support our analytical results

Kazakov-Migdal Model

- Kazakov-Migdal model is defined by unitary matrices $U_\mu(x)$ on links (edges) and hermite matrices $\Phi(x)$ on sites (vertices) as D-dimensional lattice gauge theory [Kazakov and Migdal (1992)]:

$$S = \sum_x N \mathrm{Tr}\left(m_0^2 \Phi(x)^2 - \sum_{\mu=1,2,\cdots D} \Phi(x) U_\mu(x) \Phi(x+\mu) U_\mu^\dagger(x) \right)$$



- After eliminating $\Phi(x)$, we get

$$\int DUD\Phi\, e^{-S[U,\Phi]} \propto \int DU\, e^{-S_{\mathrm{ind}}[U]}$$

where S_{ind} is a induced action given by

$$S_{\mathrm{ind}}[U] = \frac{1}{2} \mathrm{Tr} \log \left(\delta_{x,y} - m_0^{-2} \sum_\mu U_\mu(x) \otimes U_\mu^\dagger(x) \delta_{x+\mu,y} \right)$$

$\Rightarrow$ local $U(1)$ symmetry

Fundamental Kazakov-Migdal (FKM) Model

- We replace the adjoint scalar field $\Phi(x)$ by a scalar field in the fundamental representation, which violates the local $U(1)$ symmetry

- We also generalize the KM model to one on the generic graph

$$S = \sum_{v \in V} m_v^2 \Phi_v^{\dagger I} \Phi_{vI} - q \sum_{e \in E} \left(\Phi_{s(e)}^{\dagger I} U_e \Phi_{t(e)I} + \Phi_{t(e)}^{\dagger I} U_e^{\dagger} \Phi_{s(e)I} \right)$$

where $I = 1, \cdots, N_f$, V and E are a set of the vertices and edges, $s(e)$ and $t(e)$ are vertices at source and target of the edge, respectively

$$s(e) \xrightarrow{\;e\;} t(e)$$

- If we tune the “mass” by

$$m_v^2 = 1 + q^2(\deg v - 1)$$

the partition function is expressed in terms of the graph zeta function (Ihara zeta function)

$$Z_G = \left(\frac{2\pi}{\beta} \right)^{N_f N_c n_V} (1 - q^2)^{N_f N_c (n_E - n_V)} \int \prod_{e \in E} dU_e \; \zeta_G(q; U)^{N_f}$$

where β is a overall coupling constant, N_c is a rank of the gauge group, $n_V = |V|$, $n_E = |E|$ and $\zeta_G(q; U)$ is the unitary matrix weighted Ihara zeta function

Ihara Zeta Function

- The (unitary matrix weighted) graph zeta function is defined by

$$\zeta_G(q;U) \equiv \frac{1}{(1-q^2)^{Nc(n_E-n_V)} \det\left(\mathbf{1}_{N_c n_V} - qA_U + q^2(D-\mathbf{1}_{N_c n_V})\right)}$$

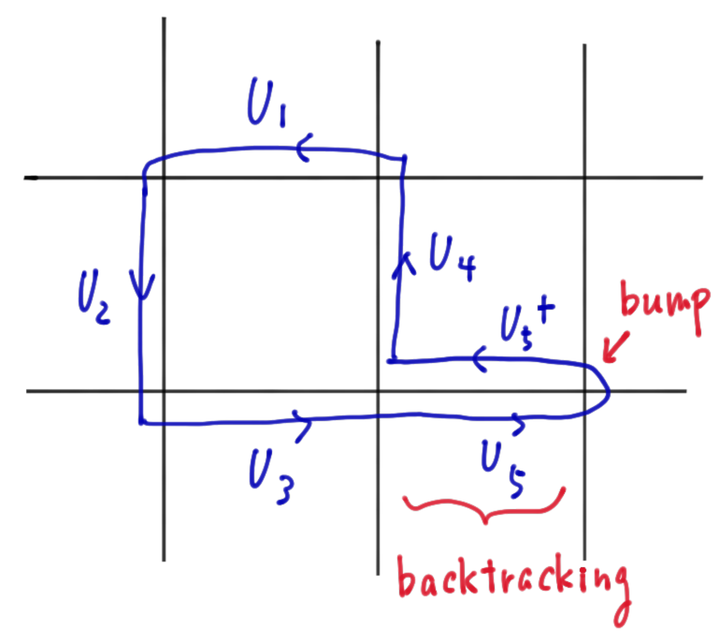
q-deformed graph Laplacian

where A_U is a unitary matrix weighted adjacency matrix and D is a diagonal degree matrix

- This graph zeta function has the following Euler product like expression

$$\zeta_G(q;U) = \prod_{C:\text{ prime cycles}} \frac{1}{\det\left(\mathbf{1}_{N_c} - q^{|C|}W_C\right)}$$

where W_C is a Wilson loop along a "prime" cycle C



- Using this Euler product expression, we can see the graph zeta function is a generating function of the possible Wilson loops on the graph without backtrackings nor bumps

- Recall the Euler product expression of the Riemann zeta function

$$\zeta(s) = \prod_{p:\text{ prime numbers}} \frac{1}{1-p^{-s}}$$

Induced Action

- Using the Euler product expression of the graph zeta function, we obtain the following induced action

$$S_{\rm ind} = \gamma N_c \sum_C \sum_{n=1}^{\infty} \frac{1}{n} q^{n|C|} \left(\mathrm{Tr}\, W_C^n + \mathrm{Tr}\, W_C^{\dagger n} \right)$$

where $\gamma = N_f / N_c$

- This induced action reduces to the Wilson action in the limit of $q \to 0$ and $\gamma \to \infty$ with $\lambda \equiv 1/\gamma q^l$ fixed
(l is a shortest length of the cycles (plaquette))

- By definition, the Wilson loops appearing in the action do not contain the backtracking nor bump

Strong/Weak Duality

- If we define the coupling q by $q \equiv \omega^{-s}$, where ω is the inverse of the largest convergence circle of $\zeta_G(q;U)$, we can see the following functional equation

$$\zeta_G(s;U) \sim \zeta_G(1-s;U)$$

  for the regular graphs G, which is an analog of the functional equation of the Riemann zeta function

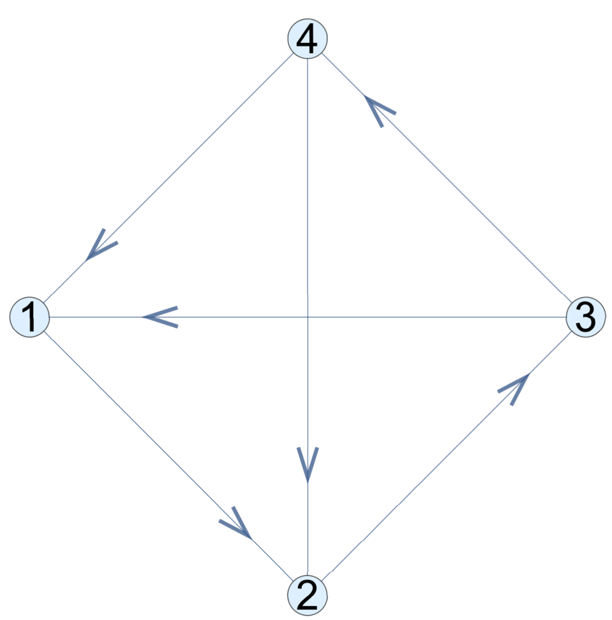
- This functional equation means that there exists a duality between q and $1/\omega q$

- For the general graphs, the approximate duality still holds

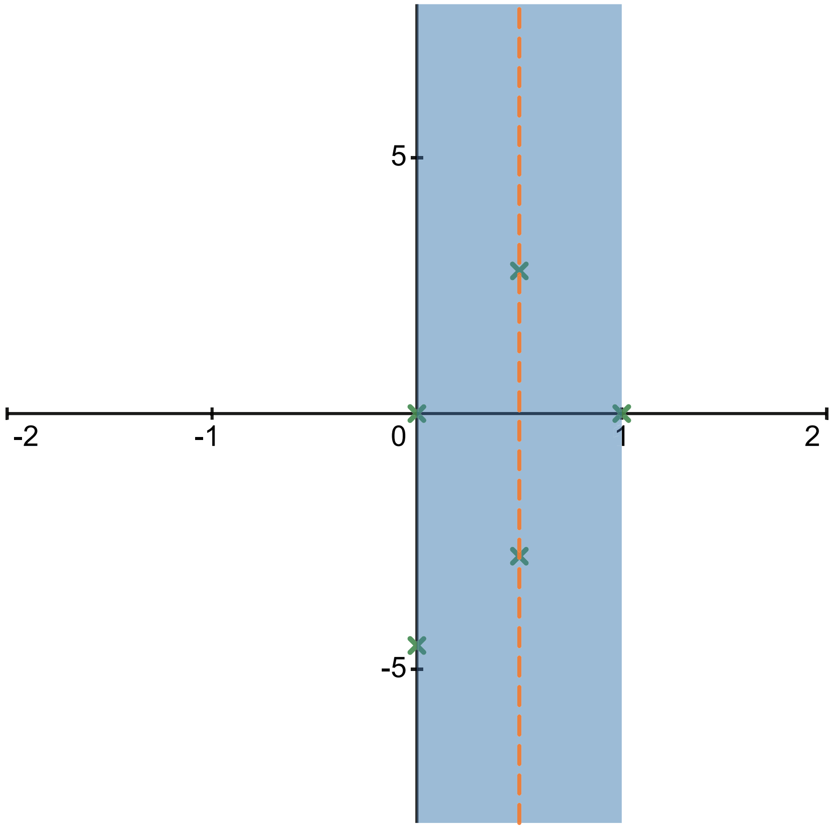
Instability

- From the functional equation, we can find that all the poles of the graph zeta function exist in the critical strip region $0 \leq \mathrm{Re}\, s \leq 1$

- For the regular graph, all non-trivial poles are on the critical line $\mathrm{Re}\, s = 1/2$
  $\Rightarrow$ Riemann hypothesis



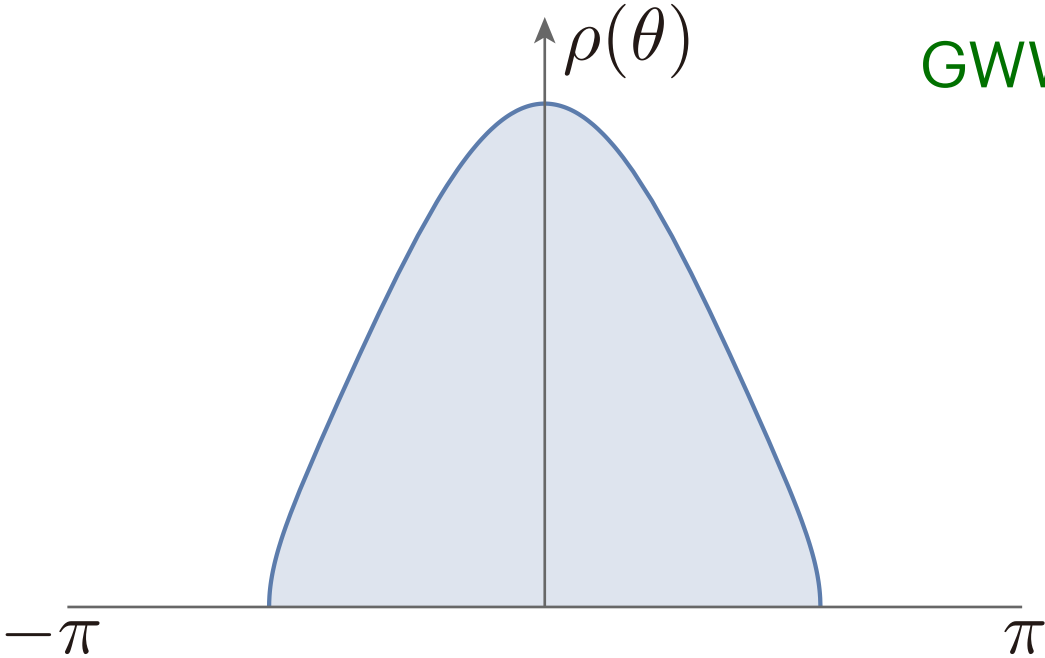
K_4 graph (tetrahedron)



- The FKM model becomes unstable in the critical strip, since the q-deformed graph Laplacian
  $$\Delta_q = 1 - qA + q^2(D-1)$$
  could contain the negative eigenvalues

Gross-Witten-Wadia Phase Transition

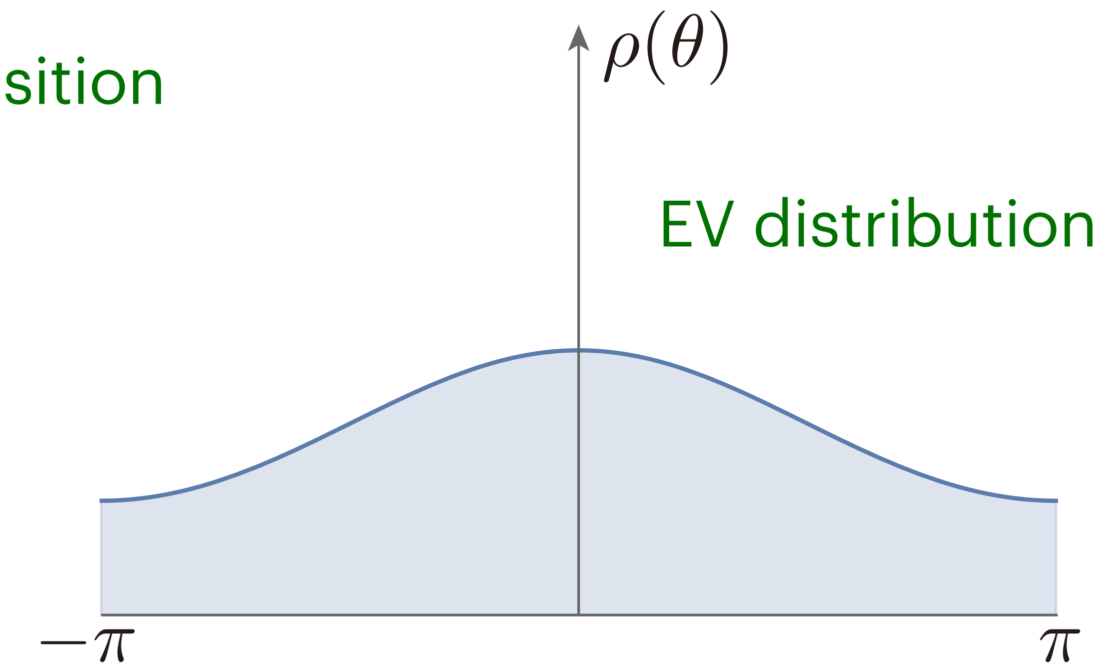
- In the FKM model, the GWW phase transition occurs for each cycle (Wilson loop)
- We can solve analytically by using the large *N* decomposition and saddle point approximation



GWW phase transition
(3rd order)

$\lambda = 2$

EV distribution



$$\left\langle \frac{1}{N_c} \mathrm{Tr}\, W_C \right\rangle \sim 1 - \frac{\lambda}{4}$$

Wilson loop

$$\left\langle \frac{1}{N_c} \mathrm{Tr}\, W_C \right\rangle \sim \frac{1}{\lambda}$$

$$C \sim 1/2$$

Specific heat

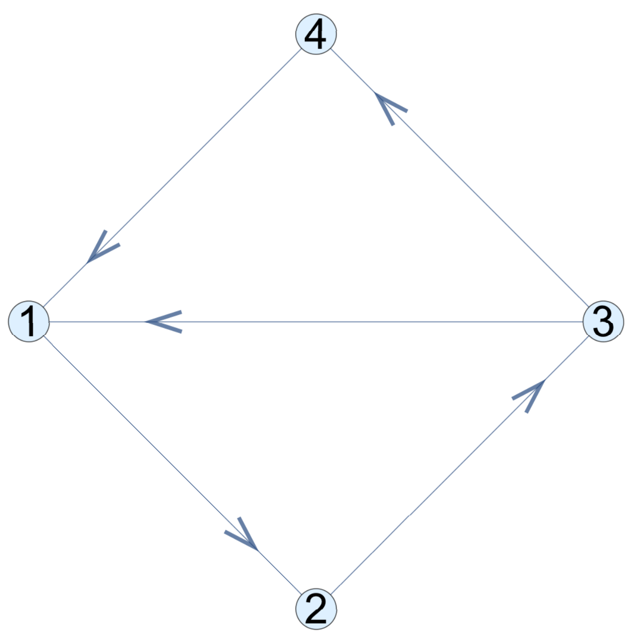
$$C \sim 2/\lambda^2$$

Weak coupling
(Deconfinement)

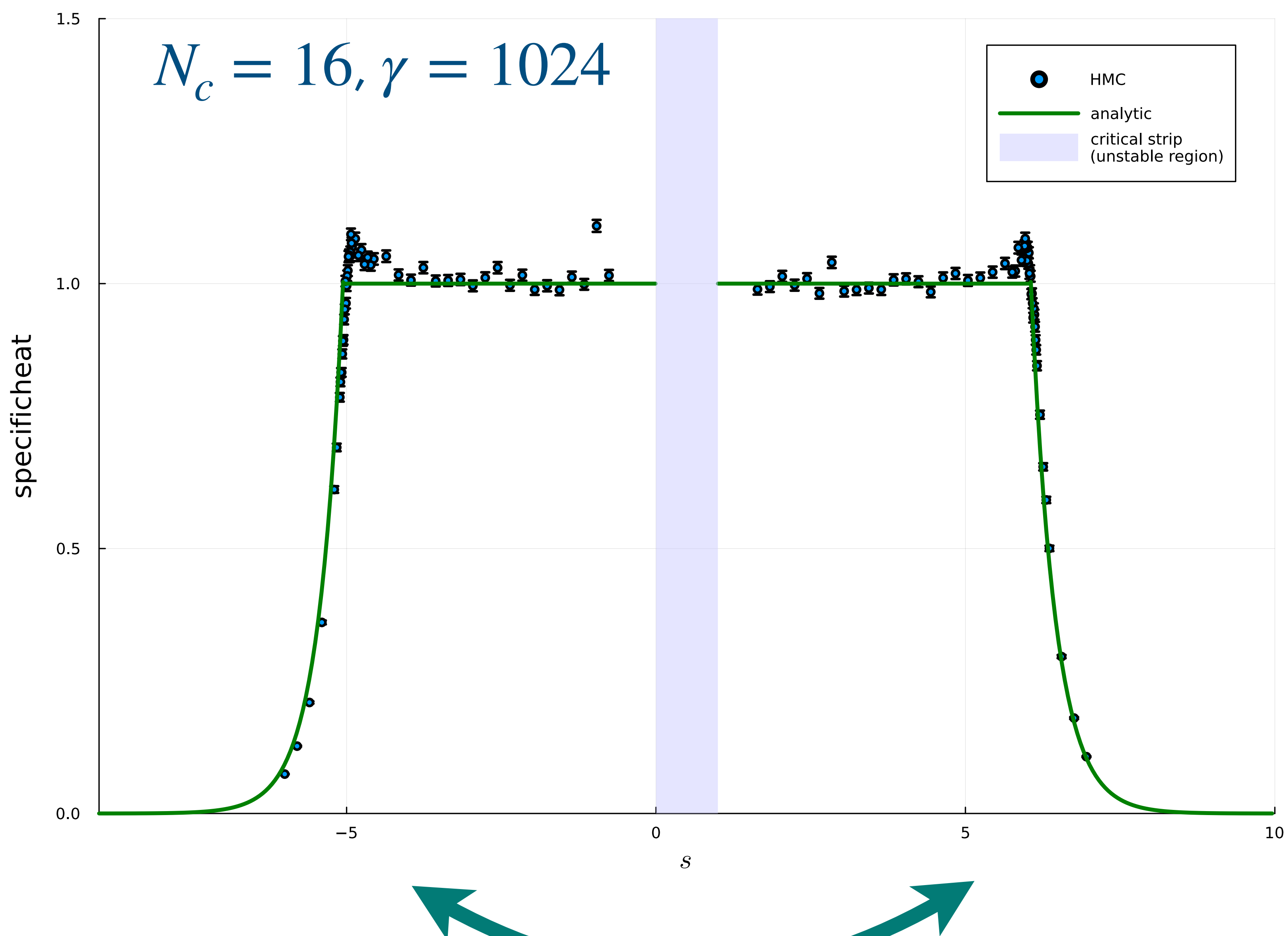
Strong coupling
(Cofinement)

Example 1: Double Triangle ($K_4 - e$)

- The double triangle graph is obtained by removing one edge from the tetrahedron (K_4)



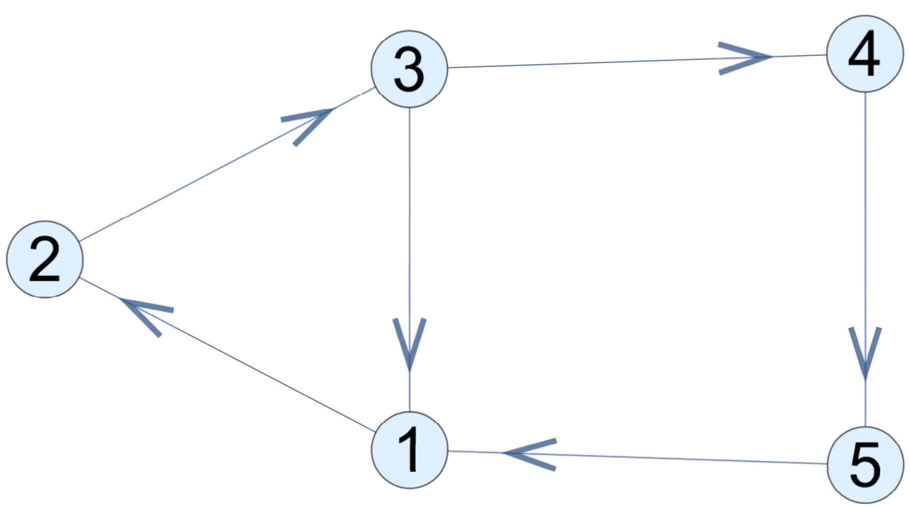
Double Triangle



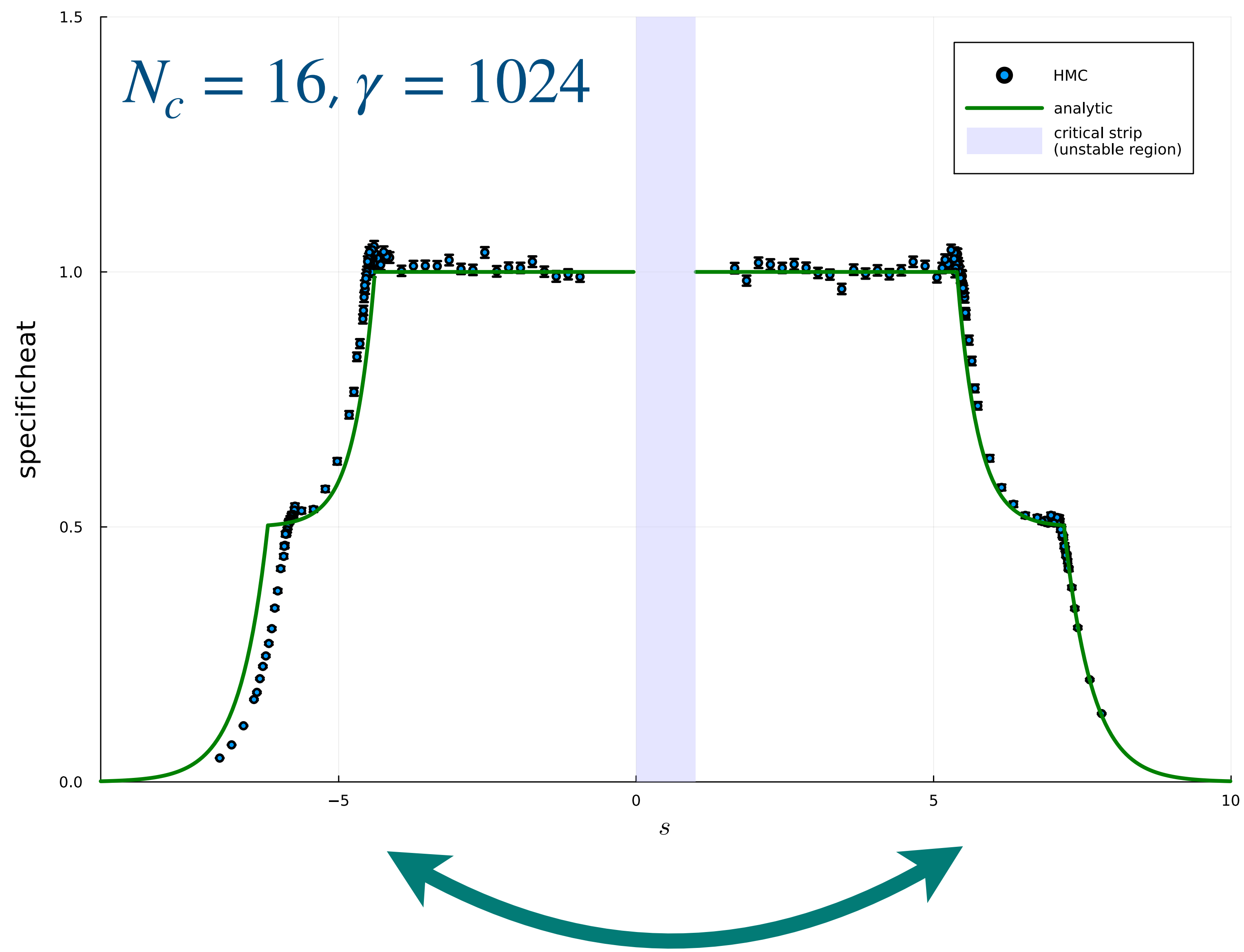
Duality

Example 2: Triangle-Square

- The triangle-square graph contains length 3 and 4 fundamental cycles
- There is an intermediate phase



Triangle-Square



Almost dual

Conclusion and Discussions

- We proposed a generalization of the Kazakov-Migdal model on the graph, which reproduces the weighted Ihara zeta function
- The graph Kazakov-Migdal model generates the countable Wilson loops systematically
- We can see the interesting “physics” like GWW phase transition in the graph zeta function model
- We are also interested in the continuous limit of the graph (grid graph), which is closely related to mathematics like L-function or Selberg’s trace formula