Non-invertible Duality Defect and Non-Commutative Fusion Algebra

Soichiro Shimamori

Osaka U.

Joint work with Yuta Nagoya [arXiv: 2309.05294, to appear on JHEP]

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◆ Outline

1. Non-invertible symmetry

2. Non-invertible symmetries in $c=2$ torus CFT

3. Summary and future directions

1. Non-invertible symmetry

Global symmetry = Existence of topological defect

$\mathcal{D}(M^{d-1})$: non-invertible symmetry defect

1. Topological property $\mathcal{D}(M^{d-1}) = \mathcal{D}(M'^{d-1})$

2. Non-group fusion rule $\mathcal{D}(M^{d-1}) \times \mathcal{D}^{\dagger}(M^{d-1}) \neq \mathbf{1}$

◆ Half-space gauging

[Y. Choi, C. Cordova, P. Hsin, H.Lam, S. Shao, 2021]
[J. Kaidi, K. Ohmori, Y. Zheng, 2021]

Suppose there is a global discrete symmetry $H^{[0]}$ in theory $\mathcal{T}_g$.

(g : coupling)

If we nicely tune the coupling $g = g^*$,

theory $\mathcal{T}_{g=g^*}$ is invariant under gauging .

$$Z[\mathcal{T}_{g^*}/H^{[0]}] = Z[\mathcal{T}_{g^*}] \qquad (\mathcal{T}_{g^*}/H^{[0]} \cong \mathcal{T}_{g^*})$$

Non-invertible symmetry

Ex. $c = 1$ compact boson model

$$S = \frac{R^2}{4\pi} \int_{M_2} d\phi \wedge \star d\phi \qquad , \qquad \phi \sim \phi + 2\pi$$

➢ Global symmetry $\mathbb{Z}_N^{\text{shift}} : \ \phi \mapsto \phi + \frac{2\pi}{N}$

$$R = \sqrt{N} \ : \ \mathcal{T}/\mathbb{Z}_N^{\text{shift}} \cong \mathcal{T}$$

(Rational CFT !)

Our Work

[Y. Nagoya, S.S , arXiv 2309.05294]

◆ Motivation

Q. What is the distribution of non-invertible symmetries from gauging in more generic two-dimensional CFT ?

$c = 1$ compact boson model $\Longrightarrow$ RCFT (See also [Y. Choi et.al, 2023])

Other CFTs? $\Longrightarrow$ $??$

2. Non-invertible symmetries in $c=2$ torus CFT

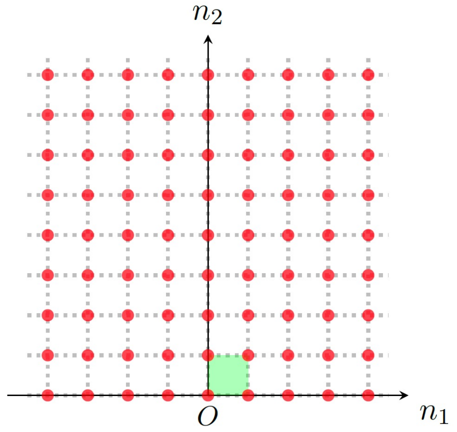
$c=2$ bosonic torus CFT

$$S[\phi^1,\phi^2]=\frac{1}{4\pi}\int_{M_2} G_{IJ}\, d\phi^I \wedge \star d\phi^J \quad , \qquad I,J=1,2$$

- We apply half-space gauging and explore non-invertible symmetries !

- Global symmetry : $\mathrm{U}(1)_1^{\mathrm{shift}} \times \mathrm{U}(1)_2^{\mathrm{shift}}$

- Charged object : $V_{\vec{n}} \equiv e^{\mathrm{i}\vec{n}\cdot\vec{\phi}}$ $\quad \vec{n}=(n_1,n_2)$

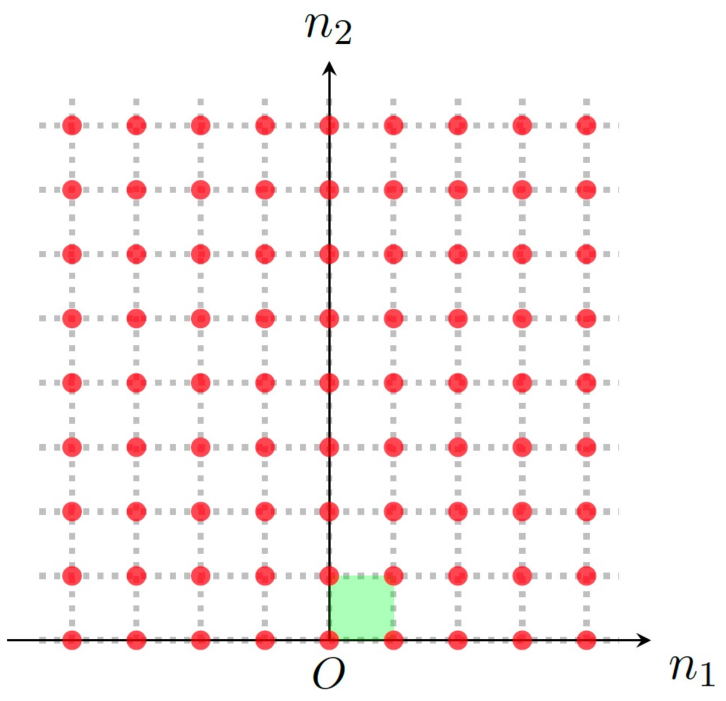


- Gauging discrete group $H \subset \mathrm{U}(1)_1^{\mathrm{shift}} \times \mathrm{U}(1)_2^{\mathrm{shift}}$

many choices

- We choose

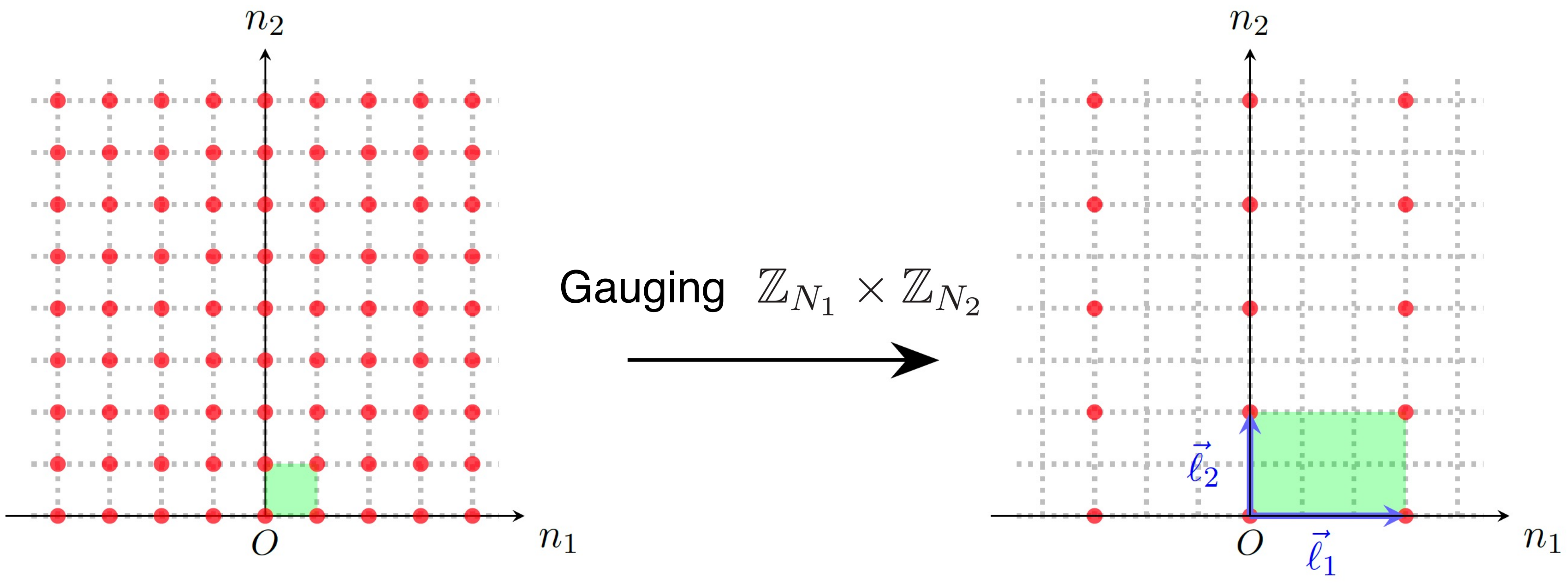
$$H = \begin{cases} \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} & : \; V_{\vec{n}} \mapsto e^{\mathrm{i}2\pi \frac{n_j}{N_j}} V_{\vec{n}} \\ (\mathbb{Z}_{2N})_{\mathrm{diag}} & : \; V_{\vec{n}} \mapsto e^{\mathrm{i}2\pi \frac{n_1+n_2}{2N}} V_{\vec{n}} \end{cases}$$



As a result of gauging H ,

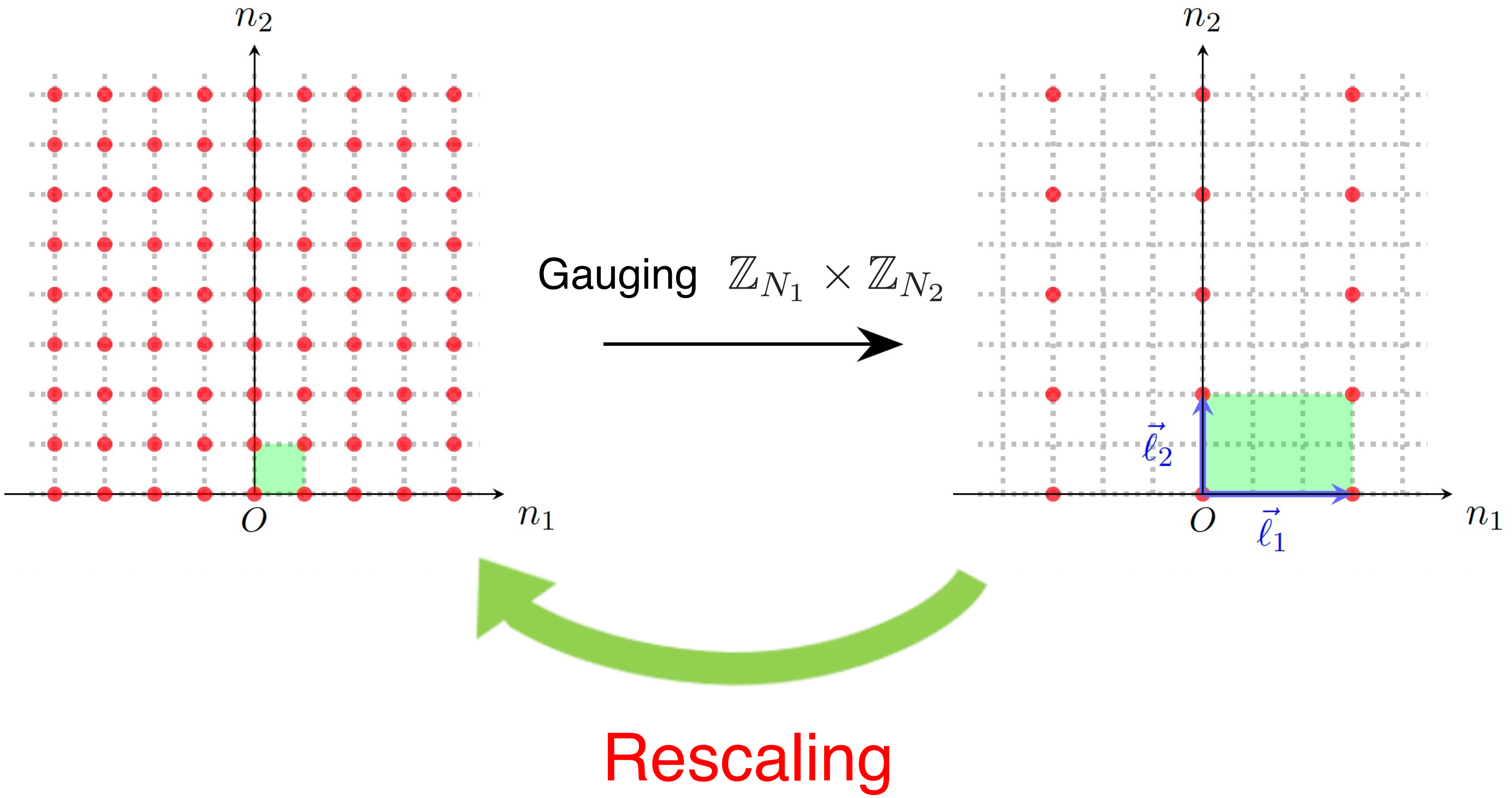
charge lattice is projected out.

$$H = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \;:\; V_{\vec{n}} \mapsto e^{\mathrm{i}2\pi \frac{n_j}{N_j}} V_{\vec{n}}$$

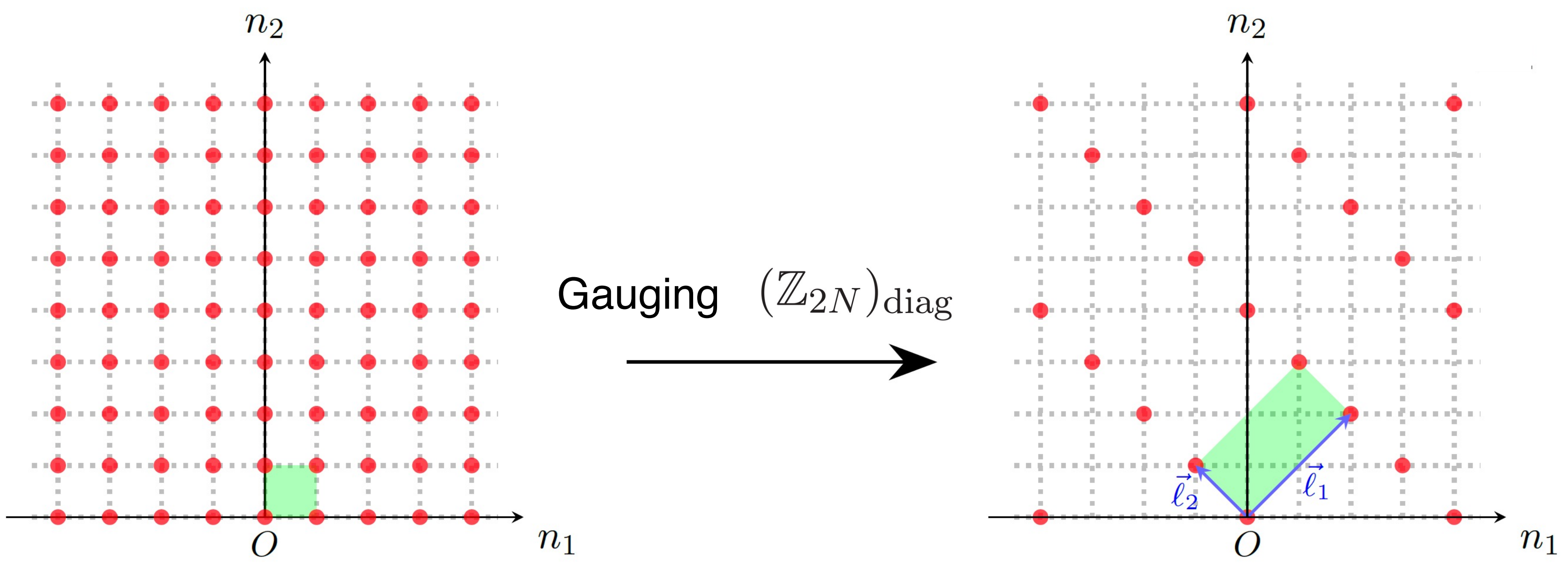


$$K = (\vec{\ell}_1, \vec{\ell}_2) = \begin{pmatrix} N_1 & 0 \\ 0 & N_2 \end{pmatrix}$$

To come back to original theory, we need only rescaling :

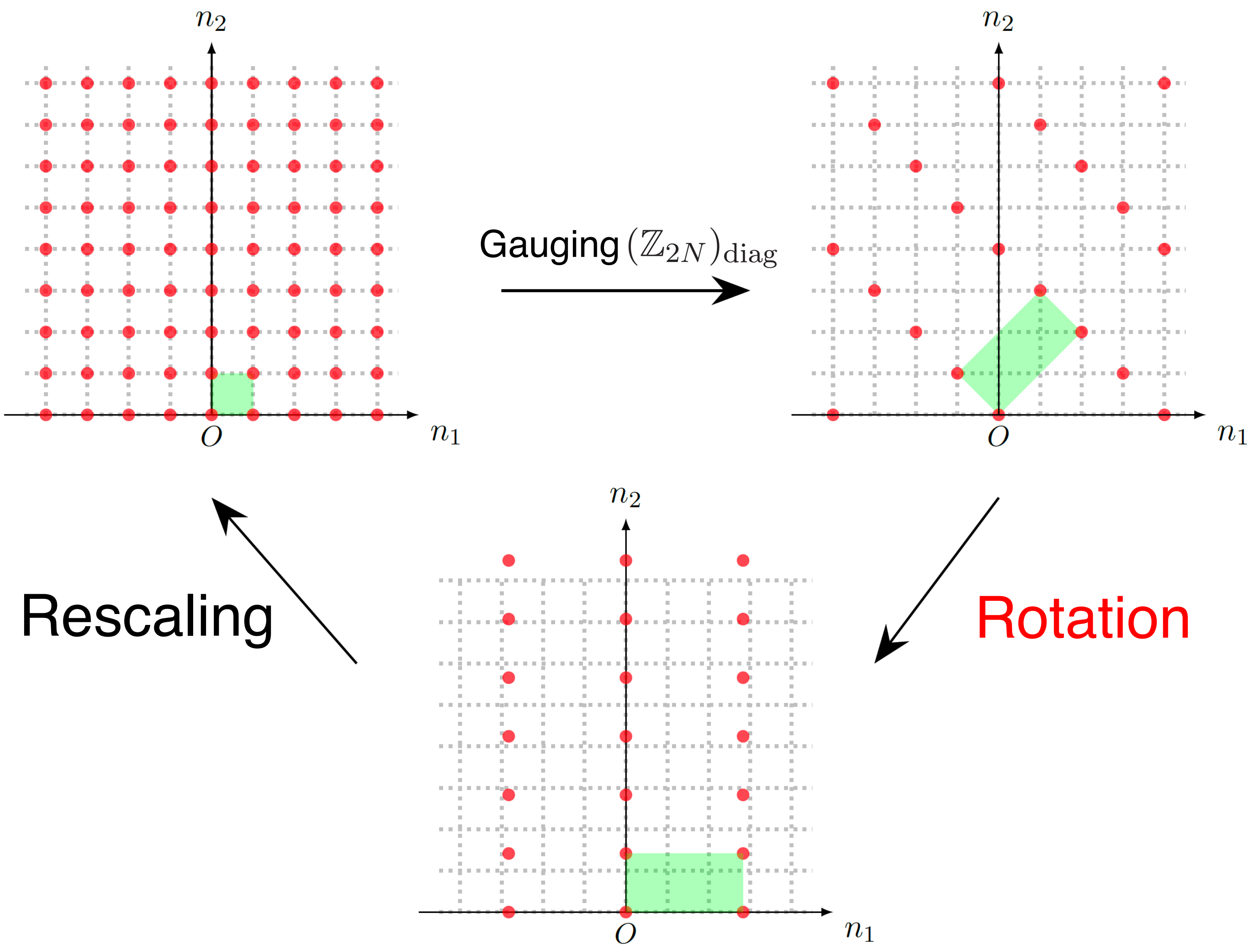


$$H = (\mathbb{Z}_{2N})_{\text{diag}} \ : \ V_{\vec{n}} \mapsto e^{\mathrm{i}2\pi \frac{n_1+n_2}{2N}} V_{\vec{n}}$$



$$K = (\vec{\ell}_1, \vec{\ell}_2) = \begin{pmatrix} N & -1 \\ N & 1 \end{pmatrix}$$

In this case, we must perform not only rescaling but rotation !



$$S[\phi^1, \phi^2] = \frac{1}{4\pi} \int_{M_2} G_{IJ}\, d\phi^I \wedge \star\, d\phi^J \qquad , \qquad I, J = 1\,, 2$$

- Theory is invariant under gauging H

  when the coupling G satisfies

Self-duality condition : $K^{\mathrm{T}}\, G^{-1}\, K = G$

◆ Classification of self-dual solutions

We can exactly solve the self-duality condition :

$$G_* = \sqrt{-\frac{\det K}{D}} \begin{pmatrix} 2K_{11} & K_{12}+K_{21} \\ K_{12}+K_{21} & 2K_{22} \end{pmatrix}$$

$$D = (K_{12}+K_{21})^2 - 4K_{11}K_{22}$$

Rational CFT or Irrational CFT ??

To classify them, we introduce two moduli :

$$\tau \equiv \frac{G_{12}}{G_{22}} + \mathrm{i}\frac{\sqrt{\det G}}{G_{22}} \quad , \quad \rho \equiv \mathrm{i}\sqrt{\det G}$$

We can easily check τ^* satisfies the following quadratic eq.

$$K_{22}\,(\tau^*)^2 - (K_{12} + K_{21})\tau^* + K_{11} = 0 \; .$$

$$\Longrightarrow \quad \boxed{\tau^* \in \mathbb{Q}(\sqrt{D})}$$ Quadratic imaginary field

$$D = (K_{12} + K_{21})^2 - 4K_{11}K_{22}$$

Gukov-Vafa's argument [Gukov, Vafa; 2004]

$\tau^* \in \mathbb{Q}(\sqrt{D})$ and $\rho^* \in \mathbb{Q}(\sqrt{D})$ $\Longleftrightarrow$ RCFT

By using this, we can show

- $K^{\mathrm{T}} = K$ $\Longleftrightarrow$ RCFT Ex. $H = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$
- $K^{\mathrm{T}} \neq K$ $\Longleftrightarrow$ Irrational CFT Ex. $H = (\mathbb{Z}_{2N})_{\mathrm{diag}}$

3. Summary

- We explore non-invertible symmetries in $c=2$ bosonic torus CFT by using half-space gauging.

- In particular, we discover non-invertible symmetry not only rational but irrational CFT.

- Fusion algebra is

commutative : RCFT

non-commutative : irrational CFT

◆ Future directions

- Deepen understanding of the fusion rules at irrational CFT

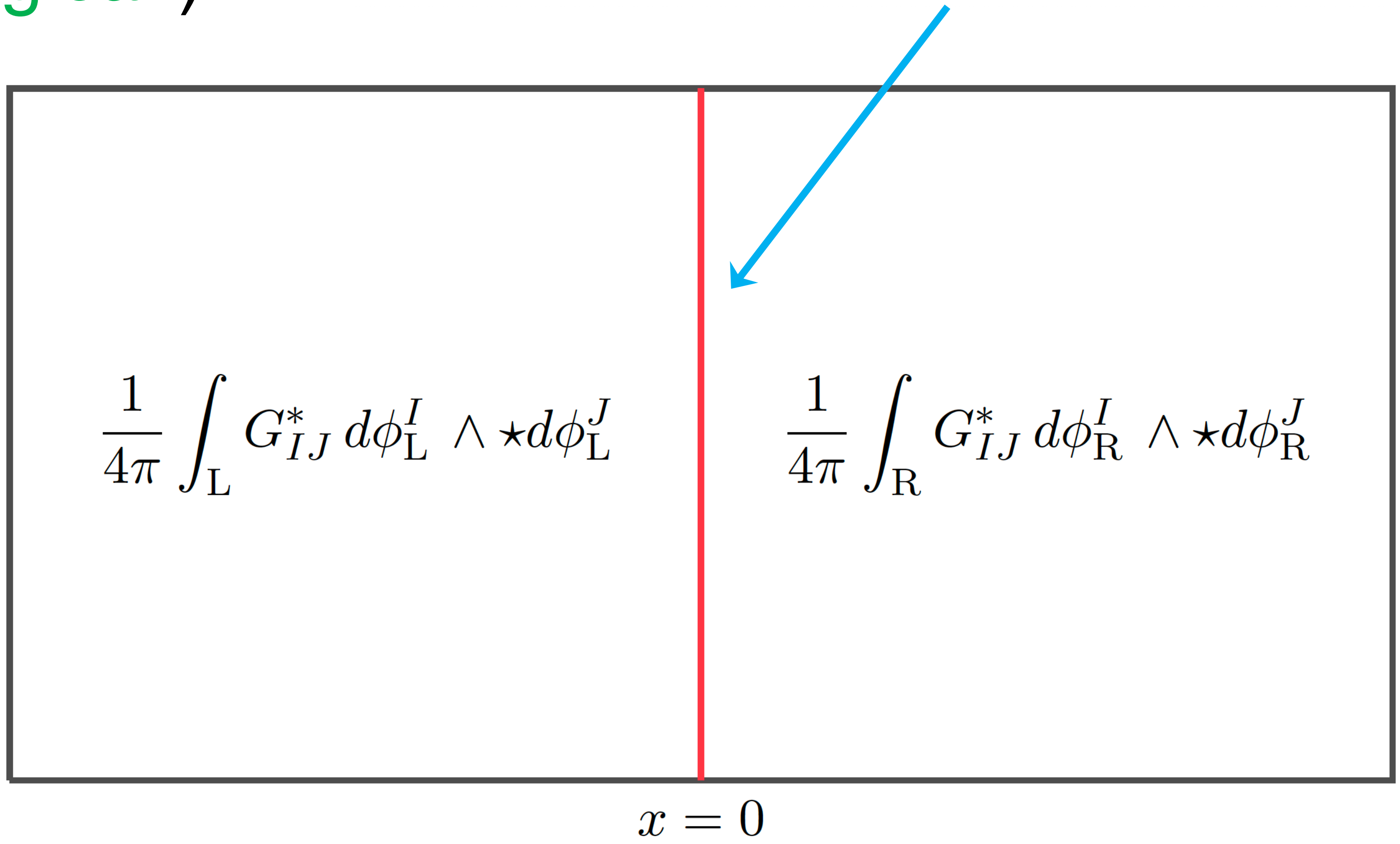
- More generic T-duality defect

- Application to string theory

Backup

Non-invertible defect $\mathcal{D}$: $\frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi^I_{\mathrm{L}}\,d\phi^J_{\mathrm{R}}$

(topological)



5. Non-commutative fusion algebra

In this talk, I will explain the fusion rules concerning

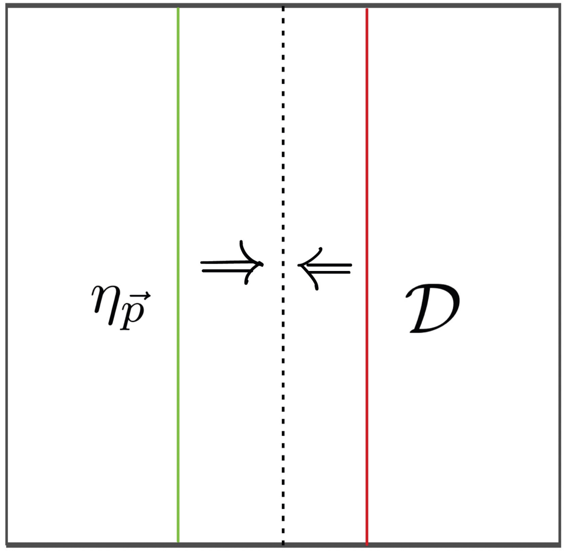
- Non-invertible defect $\mathcal{D} \ : \ \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ}\, \phi_{\mathrm{L}}^{I}\, d\phi_{\mathrm{R}}^{J}$
- H symmetry defect $\eta_{\vec{p}} \ : \ \frac{p^I}{N} \int_{\gamma} G_{IJ} \star d\phi^J$

$$p^I K_{IJ} = 0 \mod N$$

Ex. $\vec{p} = (1,1)$: $H = (\mathbb{Z}_N)_{\mathrm{diag}}$

◆ $\eta_{\vec{p}} \times \mathcal{D}$

$$S_{\text{Bulk}}[\phi_{\text{L}}, \phi_{\text{R}}] + \frac{p^I}{N}\int_{x=0} G_{IJ} \star d\phi_{\text{L}}^J + \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi_{\text{L}}^I\, d\phi_{\text{R}}^J$$



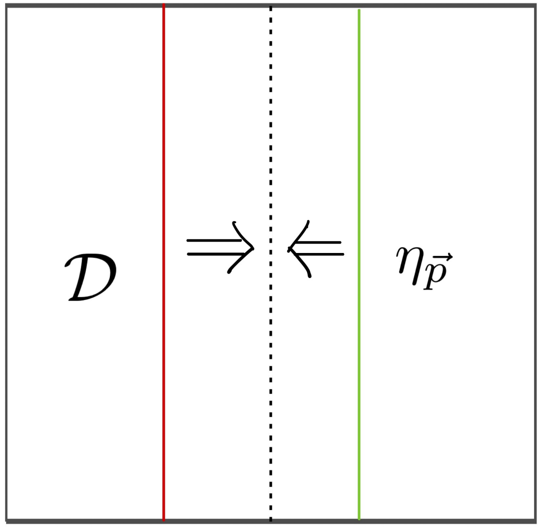
By changing integral variable

$$\Longrightarrow\quad S_{\text{Bulk}}[\phi'_{\text{L}}, \phi_{\text{R}}] + \underbrace{\frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi'^I_{\text{L}}\, d\phi_{\text{R}}^J}_{\mathcal{D}} - \frac{\mathrm{i}p^I K_{IJ}}{N}\int_{x=0} d\phi_{\text{R}}^J\,.$$

trivial

$$\Longrightarrow\quad \boxed{\eta_{\vec{p}} \times \mathcal{D} = \mathcal{D}}$$

$\mathcal{D} \times \eta_{\vec{p}}$



$$S_{\text{Bulk}}[\phi_{\text{L}}, \phi_{\text{R}}] + \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ}\, \phi^I_{\text{L}}\, d\phi^J_{\text{R}} + \frac{p^I}{N} \int_{x=0} G_{IJ} \star d\phi^J_{\text{R}}$$

$$\Longrightarrow S_{\text{Bulk}}[\phi_{\text{L}}, \phi'_{\text{R}}] + \frac{\mathrm{i}}{2\pi} \int_{x=0} K_{IJ}\, \phi^I_{\text{L}}\, d\phi'^J_{\text{R}} \quad \mathcal{D}$$

$$- \frac{\mathrm{i} p^J K_{IJ}}{N} \int_{x=0} d\phi^I_{\text{L}}$$

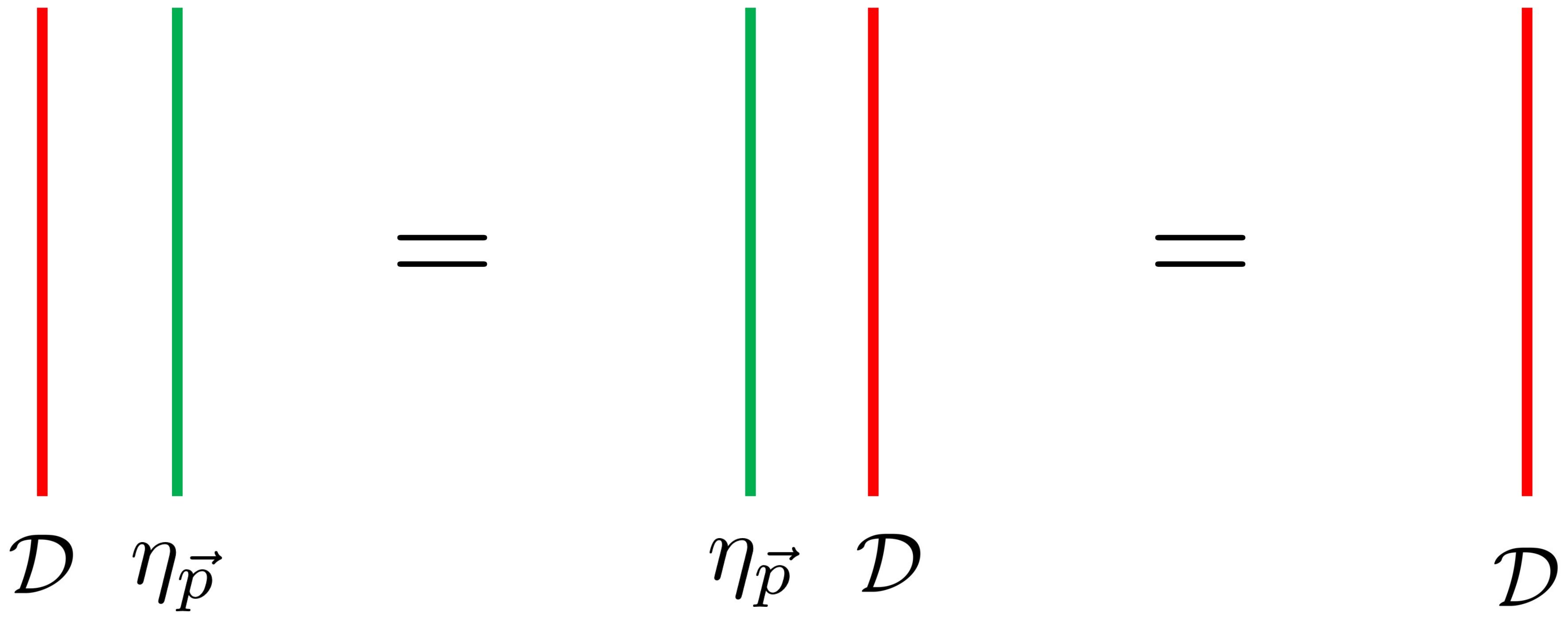
$K^{\text{T}} = K$: trivial

$K^{\text{T}} \neq K$: non-trivial $\mathbb{Z}_N$ winding

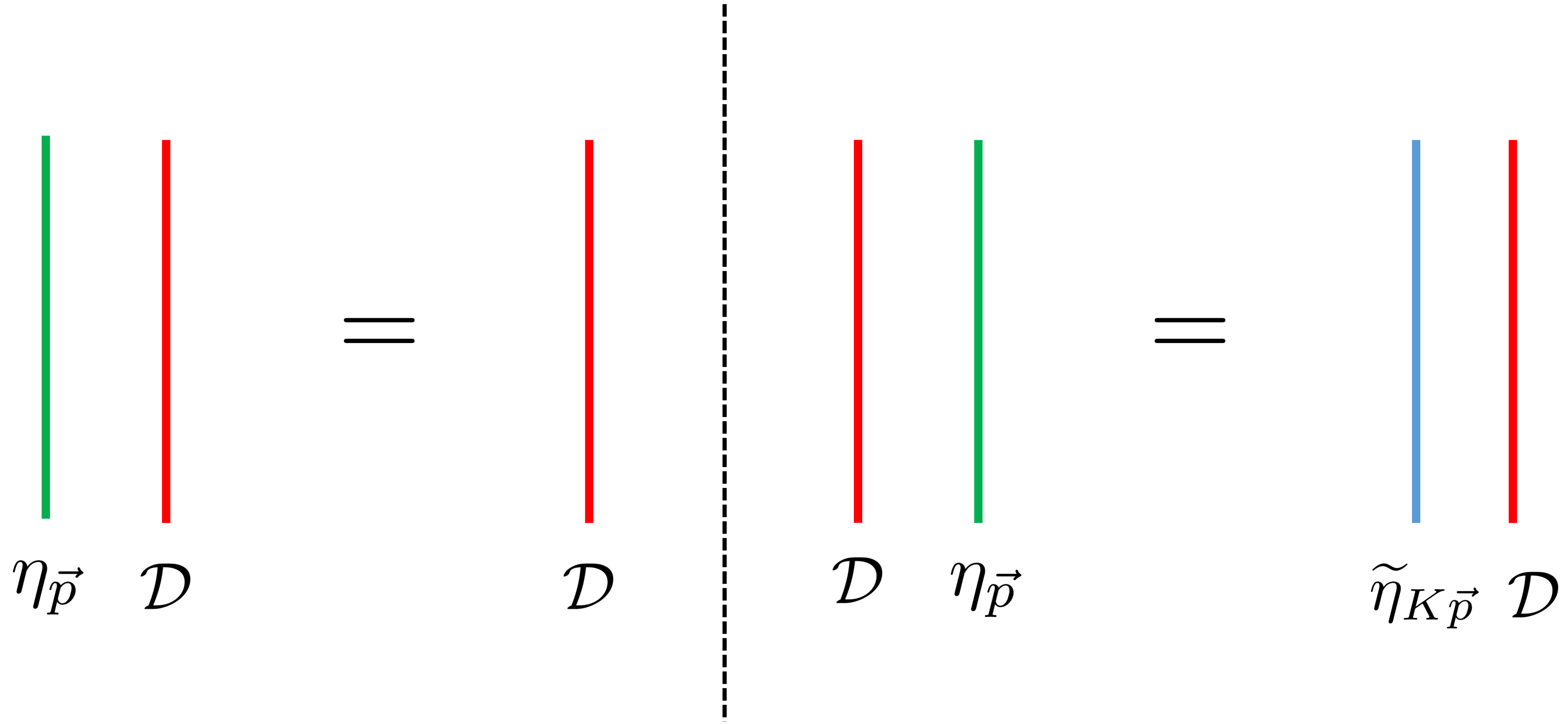
$\widetilde{\eta}_{K\vec{p}}$

We obtain the following fusion rules :

- $K^{\mathrm{T}} = K$ (RCFT) : Tambara-Yamagami category



➤ $K^{\mathrm{T}} \neq K$ (Irrational CFT) : non-commutative algebra



If we moreover close $\widetilde{\eta}_{K\vec{p}}$ to $\mathcal{D}$ from the right ,

what happens?

$$\mathcal{D} \times \widetilde{\eta}_{K\vec{p}} \ : \ \frac{\mathrm{i}}{2\pi}\int_{x=0} K_{IJ}\,\phi_{\mathrm{L}}^{I}\,d\phi_{\mathrm{R}}^{J} + \frac{(\vec{p}^{\mathrm{T}}\,K^{\mathrm{T}}\,K^{-1})^{I}}{N}\int_{x=0} G_{IJ}\,\star d\phi_{\mathrm{L}}^{I}\,.$$

$\mathcal{D}$ $\mathbb{Z}_{N^2}$ shift sym. generator

$(\because \det K = N)$

$$\mathcal{D} \times (\ \mathbb{Z}_N \text{ wind sym.}) = (\ \mathbb{Z}_{N^2} \text{ shift sym.}\ ) \times \mathcal{D}$$

We thus arrive at

$K^{\mathrm{T}} \neq K$ (Irrational CFT)

$\mathcal{D} \times$ ($\mathbb{Z}_{N^k}$ shift sym.) $=$ ($\mathbb{Z}_{N^k}$ wind sym.) $\times \mathcal{D}$

$\mathcal{D} \times$ ($\mathbb{Z}_{N^k}$ wind sym.) $=$ ($\mathbb{Z}_{N^{k+1}}$ shift sym.) $\times \mathcal{D}$

Infinite number of topological lines !