Renormalization group and quantum error correction

Ryota Nasu[1]
Collaborators:Takaaki Kuwahara[1],Gota Tanaka[2],Asato Tsuchiya[1]

[1]Shizuoka Univ. [2]Doshisha Univ.

arXiv: 2312.XXXXX

KEK Theory Workshop 2023
29 November - 1 December 2023

1.Backgrounds

Bulk operator reconstruction in AdS/CFT

- Bulk operators in the causal wedge corresponding to a boundary sub-region can be reconstructed from the CFT operator on the sub-region.

$$\phi(x) = \int dY\, K(x;Y)\mathcal{O}(Y) \tag{1}$$

- Denote the bulk operator ϕ reconstructed by sub-region $A \cup B$ by ϕ_{AB} and the boundary operator on the sub-region C by $\mathcal{O}_C$.
  From the locality of field theory,

$$[\phi_{AB}, \mathcal{O}_C] = [\phi_{AC}, \mathcal{O}_B] = [\phi_{BC}, \mathcal{O}_A] = 0. \tag{2}$$

  (Radial commutativity)

- Suppose $\phi = \phi_{AB} = \phi_{BC} = \phi_{AC}$. By Schur's lemma, $\phi \propto I$.
  $\Rightarrow$This is paradox.

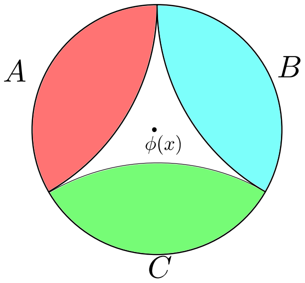


Figure: Bulk operator reconstruction. From[Harlow, 2017]

AdS/CFT and quantum error correction

3qutrit code

Combine three 3-level systems (qutrits) $\mathrm{span}\{|0\rangle, |1\rangle, |2\rangle\}$ and encode a qutrit as follows:

$$|\tilde{i}\rangle = (U_{12} \otimes 1_3)\left(|i\rangle \otimes \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |22\rangle)\right) \tag{3}$$

$$\begin{aligned} |\tilde{0}\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \quad |\tilde{1}\rangle = \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |\tilde{2}\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |021\rangle) \end{aligned} \tag{4}$$

Error correction condition(Knill-Laflamme conditon)

$$\langle\tilde{i}|X_a^\dagger Y_b|\tilde{j}\rangle \propto \delta_{ij}, \quad X, Y \text{ : Any operator acting on one qutrit.} \tag{5}$$

Operator reconstruction

$^\forall\tilde{O}$ acting on $|\tilde{i}\rangle$, we can define O_{12} acting on 1st and 2nd qutrits by $O_{12} := U_{12}O_1U_{12}^\dagger$ where $\langle\tilde{i}|\tilde{O}|\tilde{j}\rangle = {}_1\langle i|O_1|j\rangle_1$. Then, for $|\tilde{\psi}\rangle \in \mathrm{span}\{|\tilde{0}\rangle, |\tilde{1}\rangle, |\tilde{2}\rangle\}$,

$$\tilde{O}|\tilde{\psi}\rangle = O_{12}|\tilde{\psi}\rangle \quad (O_{12} \leftrightarrow \phi_{AB}). \tag{6}$$

$\Rightarrow \phi = \phi_{AB} = \phi_{BC} = \phi_{AC}$ does not hold.

$\Rightarrow$Solve the paradox.

Role of renormalization group in AdS/CFT

- Poincaré coordinate in AdS spacetime:

$$ds^2 = \frac{dz^2 - dx_0{}^2 + \sum_{i=1}^{d} dx_i{}^2}{z^2}. \quad (7)$$

- It is known that the z coordinate corresponds to energy scale for renormalization group.

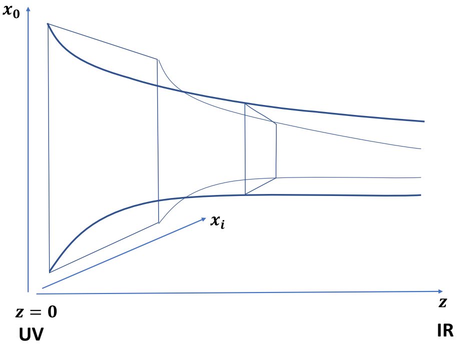
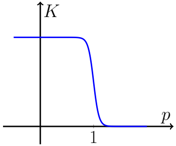


Figure: Poincaré coordinate

⇒From these backgrounds, we study a relationship between renormaliation group and quantum error correction.(c.f.[Furuya et al., 2022])

2.Renormalization group for wave functionals

Free scalar field theory



- In this talk, we consider scalar field theory and all quantities are dimensionless.
- In the free case, the Hamiltonian is given as follows:

$$H_\Lambda^{(0)} = \int_p K(p^2)\left[-\frac{1}{2}\frac{\delta}{\delta\varphi(p)}\frac{\delta}{\delta\varphi(-p)} + \frac{1}{2}\omega_{\Lambda,p}^2 K^{-2}(p^2)\varphi(p)\varphi(-p)\right] \tag{8}$$

where $\omega_{\Lambda,p} = \sqrt{p^2 + m^2/\Lambda^2}$, Λ is an effective energy scale and K is a cut-off function.

- The ground state wave functional is given by

$$\Psi_\Lambda^{(0)}[\varphi] = \mathcal{N}_\Lambda \exp\left[-\frac{1}{2}\int_p \varphi(p)\frac{\omega_{\Lambda,p}}{K_p}\varphi(-p)\right]. \tag{9}$$

- Define the creation and annihilation operators as follows:

$$a_\Lambda^{(0)}(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega_{\Lambda,p}}{K_p}}\varphi(p) + \sqrt{\frac{K_p}{\omega_{\Lambda,p}}}\frac{\delta}{\delta\varphi(-p)}\right), \quad a_\Lambda^{(0)\dagger}(p) = \frac{1}{\sqrt{2}}\left(\sqrt{\frac{\omega_{\Lambda,p}}{K_p}}\varphi(-p) - \sqrt{\frac{K_p}{\omega_{\Lambda,p}}}\frac{\delta}{\delta\varphi(p)}\right). \tag{10}$$

Unitary operator for renormalization group

- Represent the renormalization group flow of a ground state in scalar field theory by a unitary operator $U(\Lambda, \Lambda_{\text{UV}})$ as follows:

$$|\Psi\rangle_{\Lambda} = U(\Lambda, \Lambda_{\text{UV}}) |\Psi\rangle_{\text{UV}} , \tag{11}$$

  where Λ_{UV} denotes UV cut-off.

- The flow of creation/annihilation operators can be expressed using $U(\Lambda, \Lambda_{\text{UV}})$ as follows:

$$a_{\Lambda}(p) = U(\Lambda, \Lambda_{\text{UV}}) a_{\text{UV}}(p) U^{\dagger}(\Lambda, \Lambda_{\text{UV}}), \tag{12}$$

$$a^{\dagger}_{\Lambda}(p) = U(\Lambda, \Lambda_{\text{UV}}) a^{\dagger}_{\text{UV}}(p) U^{\dagger}(\Lambda, \Lambda_{\text{UV}}). \tag{13}$$

Concrete form of U and scaling of a and $a^\dagger$ in free case

- The concrete form of $U(\Lambda, \Lambda_{\mathsf{UV}})$ is given as follows:

$$U(\Lambda, \Lambda_{\mathsf{UV}}) = \exp\left[\int_{\Lambda}^{\Lambda_{\mathsf{UV}}} \frac{d\Lambda'}{\Lambda}\left\{-\frac{1}{4}\int_p \frac{-\Lambda\partial_\Lambda \omega_{\Lambda,p}}{\omega_{\Lambda,p}}\Big(a_\Lambda^\dagger(-p)a_\Lambda^\dagger(p) - a_\Lambda(p)a_\Lambda(-p)\Big)\right\}\right]. \tag{14}$$

- Define the following operators:

$$a^{(0)}_{+,\Lambda}(p) \coloneqq a^{(0)}_{\Lambda}(p) + a^{(0)\dagger}_{\Lambda}(-p), \tag{15}$$

$$a^{(0)}_{-,\Lambda}(p) \coloneqq a^{(0)}_{\Lambda}(p) - a^{(0)\dagger}_{\Lambda}(-p). \tag{16}$$

- The scaling of $a^{(0)}_{\pm,\Lambda}(p)$ is obtained as

$$a^{(0)}_{+,\Lambda}(p) = \sqrt{\frac{\omega_{\Lambda,p}}{\omega_{\mathsf{UV},p}}}\, a^{(0)}_{+,\mathsf{UV}}(p), \qquad a^{(0)}_{-,\Lambda}(p) = \sqrt{\frac{\omega_{\mathsf{UV},p}}{\omega_{\Lambda,p}}}\, a^{(0)}_{-,\mathsf{UV}}(p). \tag{17}$$

3.Encoding procedure

Constructing q-level states by coherent states (c.f.[Furuya et al., 2022])

- Consider coherent states

$$|f\rangle_\Lambda = \exp\left[\int_p \Big(f(p)a^\dagger_\Lambda(-p) - f^\dagger(-p)a_\Lambda(p)\Big)\right]|\Psi\rangle_\Lambda\,. \tag{18}$$

- Choose $f = rf_0$ where $r = 0, \cdots, q-1$ and f_0 is a real function such that $\int_p |f_0|^2$ is large.
  $\Rightarrow$ Construct q-level states as follows:

$$|rf_0\rangle_\Lambda = \exp\left[-r\int_p f_0(-p)\Big(a_\Lambda(p) - a^\dagger_\Lambda(-p)\Big)\right]|\Psi\rangle_\Lambda \tag{19}$$

$$= \exp\left[-r\int_p f_0(-p)a_{-,\Lambda}(p)\right]|\Psi\rangle_\Lambda\,. \tag{20}$$

- These states approximately satisfy orthonormal condition:

$${}_\Lambda\langle r'f_0|rf_0\rangle_\Lambda = \exp\left[-\frac{1}{2}(r-r')^2\int_p |f_0(p)|^2\right] \sim \delta_{rr'}\,. \tag{21}$$

  $\Rightarrow$These states can be regarded as orthonormal basis.

Encoding procedure

- Encode the q-level states as follows:

$$|rf_0\rangle_{\mathsf{UV}} = U^\dagger(\Lambda, \Lambda_{\mathsf{UV}})\, |rf_0\rangle_{\Lambda}\,, \tag{22}$$

where $U^\dagger(\Lambda, \Lambda_{\mathsf{UV}})$ is the Hermitian conjugate of the RG unitary operator.
$\Rightarrow$We represent the information at low energy scale in terms of d.o.f at high energy scale.

4.Quantum error correction

Error correction condition

- Consider an error operator $D_\Lambda[g]$ defined at low energy Λ as follows:

$$D_\Lambda[g] = \exp\left[\int_p g(-p)\Big(a_\Lambda(p) - a_\Lambda^\dagger(-p)\Big)\right] = \exp\left[\int_p g(-p)a_{-,\Lambda}(p)\right], \tag{23}$$

  where g is an arbitrary real function.

- Error correction condition (Knill-Laflamme condition) can be written as follows:

$${}_{\text{UV}}\langle r' f_0|\, D_\Lambda^\dagger[g] D_\Lambda[h]\, |r f_0\rangle_{\text{UV}} \sim \alpha[g,h]\delta_{rr'}, \tag{24}$$

  where $\alpha[g,h]$ is an Hermitian matrix on functional vector space.
  $\Rightarrow$ we will show that this condition is approximately satisfied in IR limit.

- To show this, it suffices to calculate

$${}_{\text{UV}}\langle r' f_0|\, D_\Lambda[g]\, |r f_0\rangle_{\text{UV}}\,, \tag{25}$$

  for any real functions g because $D_\Lambda^\dagger[g] D_\Lambda[h] = D_\Lambda[g-h]$.

Free case

- Calculating ${}_{\text{UV}}\langle r' f_0 | \, D_\Lambda[g] \, | r f_0 \rangle_{\text{UV}}$ we obtain

$$ {}_{\text{UV}}\langle r' f_0 | D_\Lambda[g] \, | r f_0 \rangle_{\text{UV}} = \exp\left[-\frac{1}{2} \int_p \left((r-r')^2 |f_0(p)|^2 - 2(r-r') \sqrt{\frac{\omega_{\text{UV},p}}{\omega_{\Lambda,p}}} g(-p) f_0(p) + \frac{\omega_{\text{UV},p}}{\omega_{\Lambda,p}} |g(p)|^2 \right) \right] \tag{26} $$

- Taking IR limit($\Lambda \ll m$), we obtain $\omega_{\Lambda,p} \gg \omega_{\text{UV},p}$ so that the second and third terms in the exponent vanish.
  $\Rightarrow$ when $\int_p |f_0(p)|^2$ is large enough, we obtain

$$ {}_{\text{UV}}\langle r' f_0 | \, D_\Lambda[g] \, | r f_0 \rangle_{\text{UV}} \sim \delta_{rr'}, \tag{27} $$

  in IR limit.
  $\Rightarrow$Error correction condition is satisfied.
  $\Rightarrow$The states $|r f_0\rangle_{\text{UV}}$ encoded by $U^\dagger$ are correctable from $D_\Lambda[g]$.

Interacting case 1

- Consider φ^4 interaction up to the first-order perturbation.
- The Hamiltonian is given by

$$H_\Lambda = H_\Lambda^{(0)} + \alpha H_{\mathsf{int},\Lambda}, \tag{28}$$

$$H_{\mathsf{int},\Lambda} = \frac{\delta m_\Lambda^2}{2}\int_p \varphi(p)\varphi(-p) + \frac{\lambda_\Lambda}{4!}\int_{p_1p_2p_3p_4} \varphi(p_1)\varphi(p_2)\varphi(p_3)\varphi(p_4)\tilde{\delta}\left(\sum_i p_i\right). \tag{29}$$

  α is an expansion parameter.
- Expand the ground state, creation and annihilation operators in α as follows:

$$|\Psi\rangle_\Lambda = |\Psi^{(0)}\rangle_\Lambda + \alpha\,|\Psi^{(1)}\rangle_\Lambda + \cdots\,, \tag{30}$$

$$a_\Lambda(p) = a_\Lambda^{(0)}(p) + \alpha a_\Lambda^{(1)}(p) + \cdots\,, \quad a_\Lambda^\dagger(p) = a_\Lambda^{(0)\dagger}(p) + \alpha a_\Lambda^{(1)\dagger}(p) + \cdots\,, \tag{31}$$

Interacting case 2

- We define $U(\Lambda, \Lambda_{\mathsf{UV}})$ as the renormalization group flow of the ground state.
- Encoding is performed as follows:

$$|rf_0\rangle_{\mathsf{UV}} = U^\dagger(\Lambda, \Lambda_{\mathsf{UV}})\,|rf_0\rangle_\Lambda\,. \tag{32}$$

- We consider an error operator as follows:

$$D_\Lambda[g] = \exp\left[\int_p g(-p)a_{-,\Lambda}(p)\right] = \exp\left[\int_p g(-p)(a^{(0)}_{-,\Lambda}(p) + \alpha a^{(1)}_{-,\Lambda}(p))\right]. \tag{33}$$

- We find that the error correction condition is satisfied in IR limit up to the first-order perturbation.

$${}_{\mathsf{UV}}\langle r' f_0|\, D_\Lambda[g]\,|rf_0\rangle_{\mathsf{UV}} \sim \delta_{rr'}. \tag{34}$$

⇒The states are correctable from $D_\Lambda[g]$, even if there is a interaction up to the first-order perturbation.

3.Conclusion

Conclusion

Conclusion

- The q-level states are constructed by coherent states.
- The encoding procedure is done by the inverse of the renormalization group unitary operator U.
- In the free case, the states are correctable from the error defined on IR.
- In the interacting case, up to the first-order perturbation of φ^4 interaction, the states are also correctable.

$$ {}_{\text{UV}}\langle r' f_0 | D^\dagger[g] D[h] | r f_0 \rangle_{\text{UV}} \sim \delta_{rr'} $$

Future work

- Extend this study to the non-perturbative theory.
- Relate this study to the bulk reconstruction.
- As we have seen, the encoding procedure is done by $U^\dagger$.
  $\Rightarrow$ It should be important to consider the inverse renormalization group.
  (c.f. [Berman et al., 2023, Cotler and Rezchikov, 2023])

References I

Berman, D. S., Klinger, M. S., and Stapleton, A. G. (2023).
Bayesian renormalization.
Mach. Learn. Sci. Tech., 4(4):045011.

Cotler, J. and Rezchikov, S. (2023).
Renormalizing Diffusion Models.

Furuya, K., Lashkari, N., and Moosa, M. (2022).
Renormalization group and approximate error correction.
Phys. Rev. D, 106(10):105007.

Harlow, D. (2017).
The Ryu–Takayanagi Formula from Quantum Error Correction.
Commun. Math. Phys., 354(3):865–912.

Backup Slides

Details of derivation of scaling in perturbation theory 1

Deriving the ground state up to first order perturbation of φ^4 interacting theory, we obtain

$$|\Psi\rangle_\Lambda = |\Psi^{(0)}\rangle_\Lambda + \alpha\,|\Psi^{(1)}\rangle_\Lambda\,, \tag{35}$$

where

$$|\Psi^{(1)}\rangle_\Lambda = A_\Lambda\,|\Psi^{(0)}\rangle_\Lambda \tag{36}$$

$$= \Bigg[-\frac{\lambda}{4!}\int_{k_1\cdots k_4}\frac{\tilde\delta(\sum_{i=1}^4 k_i)}{\omega_{\Lambda,1}+\omega_{\Lambda,2}+\omega_{\Lambda,3}+\omega_{\Lambda,4}}\prod_{i=1}^4\sqrt{\frac{K_i}{2\omega_{\Lambda,i}}}a_\Lambda^{(0)\dagger}(k_i) - \left(\frac{\delta m_\Lambda^2}{2}+\frac{\lambda}{4!}\int_{\bar p}\frac{6K_p}{2\omega_p}\right)\int_k\frac{K_k}{4\omega_{\Lambda,k}^2}a_\Lambda^{(0)\dagger}(k)a_\Lambda^{(0)\dagger}(-k)\Bigg]\,|\Psi^{(0)}\rangle\,. \tag{37}$$

Details of derivation of scaling in perturbation theory 2

We define $U(\Lambda, \Lambda_{\text{UV}})$, as the renormalization group flow of the ground state:

$$|\Psi\rangle_{\Lambda} = U(\Lambda, \Lambda_{\text{UV}}) |\Psi\rangle_{\text{UV}}, \tag{38}$$

We assume that $U(\Lambda, \Lambda_{\text{UV}})$ can be written as

$$U(\Lambda, \Lambda_{\text{UV}}) = T \exp\left[\int_{\Lambda}^{\Lambda_{\text{UV}}} \frac{d\Lambda'}{\Lambda'} X_{\Lambda'}\right]. \tag{39}$$

If we have known how the ground state flows by the analysing the renormalization group, we can calculate X_{Λ} from

$$-\Lambda\partial_{\Lambda} |\Psi\rangle_{\Lambda} = X_{\Lambda} |\Psi\rangle_{\Lambda}. \tag{40}$$

Details of derivation of scaling in perturbation theory 2

We can determine X_Λ perturbatively. Expanding the scaling equation for the ground state, we obtain

$$-\Lambda\partial_\Lambda |\Psi_0^{(0)}\rangle_\Lambda = X_\Lambda^{(0)} |\Psi_0^{(0)}\rangle_\Lambda \,, \tag{41}$$

$$-\Lambda\partial_\Lambda |\Psi_0^{(1)}\rangle_\Lambda = X_\Lambda^{(0)} |\Psi_0^{(1)}\rangle_\Lambda + X_\Lambda^{(1)} |\Psi_0^{(0)}\rangle_\Lambda \,. \tag{42}$$

From them, we obtain

$$-\Lambda\partial_\Lambda A_\Lambda = X_\Lambda^{(1)} + [X_\Lambda^{(0)}, A_\Lambda]. \tag{43}$$

Details of derivation of scaling in perturbation theory 3

We also expand the scaling equation for annihilation operators and obtain

$$-\Lambda\partial_\Lambda a_\Lambda^{(0)}(p) = \left[X_\Lambda^{(0)}, a_\Lambda^{(0)}(p)\right] \tag{44}$$

$$-\Lambda\partial_\Lambda a_\Lambda^{(1)}(p) = \left[X_\Lambda^{(1)}, a_\Lambda^{(0)}(p)\right] + \left[X_\Lambda^{(0)}, a_\Lambda^{(1)}(p)\right] \tag{45}$$

Same equations hold for $a^\dagger$.
Solving (45) by using (43), we see that the solution is

$$a_\Lambda^{(1)}(p) = -[a_\Lambda^{(0)}(p), A_\Lambda]. \tag{46}$$

Details of error correction in perturbation theory1

The error can be expressed as

$$D[g] = \exp\left[\int_p g(-p)a_{-,\Lambda}(p)\right] = \exp\left[\int_p g(-p)\Big(a^{(0)}_{-,\Lambda}(p) + \alpha a^{(1)}_{-,\Lambda}(-p)\Big)\right]. \tag{47}$$

And the quantity we evaluate is

$$_{\text{UV}}\langle r' f_0|\, D[g]\, |r f_0\rangle_{\text{UV}} \tag{48}$$

We calculate this perturbative.
First,

$$a_{-,\Lambda}(p) = a^{(0)}_{-,\Lambda}(p) + \alpha a^{(1)}_{-,\Lambda}\Big(p; a^{(0)}_{+,\Lambda}, a^{(0)}_{-,\Lambda}\Big) \tag{49}$$

$$= \sqrt{\frac{\omega_{\text{UV},p}}{\omega_{\Lambda,p}}} a^{(0)}_{+,\text{UV}}(p) + \alpha a^{(1)}_{-,\Lambda}\left(p; \sqrt{\frac{\omega_\Lambda}{\omega_{\text{UV}}}} a^{(0)}_{+,\text{UV}}, \sqrt{\frac{\omega_{\text{UV}}}{\omega_\Lambda}} a^{(0)}_{+,\text{UV}}\right). \tag{50}$$

This notation means that $a^{(1)}_{-,\Lambda}$ contains $a^{(0)}_{\pm,0}$.

Details of error correction in perturbation theory2

For the first term we use $a^{(0)}_{+,\mathsf{UV}} = a_{+,\mathsf{UV}} - \alpha a^{(1)}_{+,\mathsf{UV}}$ and for the α term, we can replace $a^{(0)}$ to a if we consider up to first order of perturbation.
Then,

$$a_{-,\Lambda}(p) = \sqrt{\frac{\omega_{\mathsf{UV},p}}{\omega_{\Lambda,p}}}a_{+,\mathsf{UV}}(p) + \alpha\left\{a^{(1)}_{-,\Lambda}\left(p; \sqrt{\frac{\omega_\Lambda}{\omega_{\mathsf{UV}}}}a_{+,\mathsf{UV}}, \sqrt{\frac{\omega_{\mathsf{UV}}}{\omega_\Lambda}}a_{+,\mathsf{UV}}\right) - \sqrt{\frac{\omega_{\mathsf{UV},p}}{\omega_{\Lambda,p}}}a^{(1)}_{+,\mathsf{UV}}(p; a_{+,\mathsf{UV}}, a_{+,\mathsf{UV}})\right\}. \tag{51}$$

Using the fact that coherent state is a eigenstate for annihilation operator:

$$a_\Lambda(p)\left|rf_0\right\rangle_\Lambda = rf_0(p)\left|rf_0\right\rangle_\Lambda \tag{52}$$

we can calculate error correction condition.