Analysis of entanglement entropy based on tensor renormalization group

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in collaboration with

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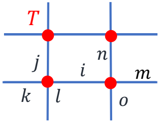
Introduction - Tensor network

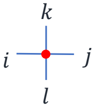
- Partition functions and expectation values of physical quantities can be expressed as a tensor network.

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} = \sum_{\dots,i,j,k,l,m,n,o,\dots} \dots T_{ijkl} \, T_{mnio} \dots =$$

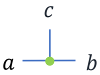


- Tensors network notation:

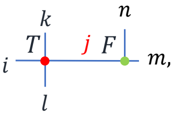
T_{ijkl} ⟹



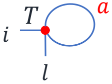
$, F_{abc}$ ⟹



$\sum_l T_{ijkl} F_{jmn}$ ⟹

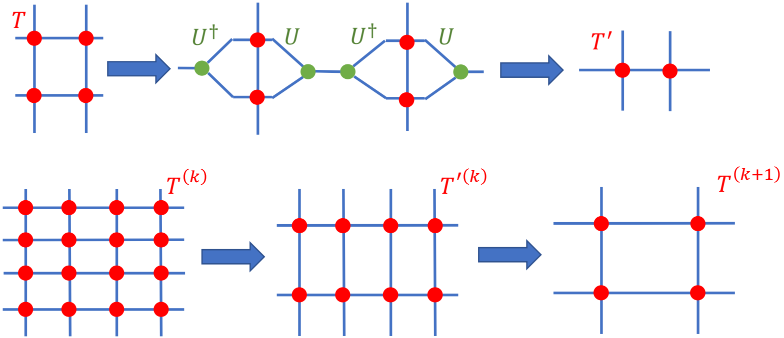


$\sum_a T_{iaal}$ ⟹



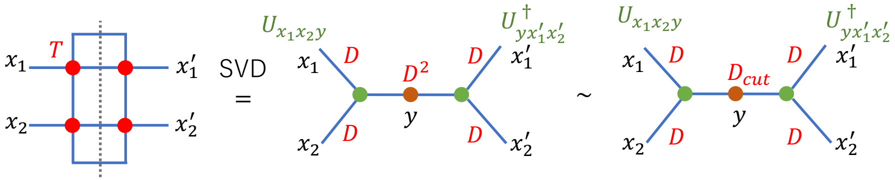
Introduction - Tensor renormalization group

- The tensor renormalization group [Levin-Nave, 2007] is a method for the truncation of tensor networks.
- We focus on the higher-order tensor renormalization group (HOTRG) method shown in the figure below,



Introduction Tensor renormalization group

- The isometry matrix can be obtained from the tensor T of the network using the singular value decomposition (SVD).



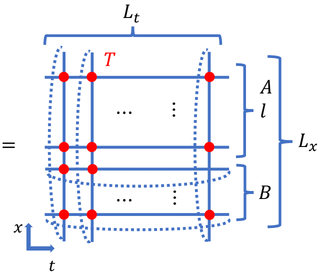
Entanglement entropy

- Definition of the entanglement entropy of subregion A:

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A \ , \ \rho_A = \mathrm{Tr}_B \rho$$

- The reduced density matrix ρ_A of the subregion A can be expressed as a tensor network.

$$\rho_A(l, L_x) \quad =$$

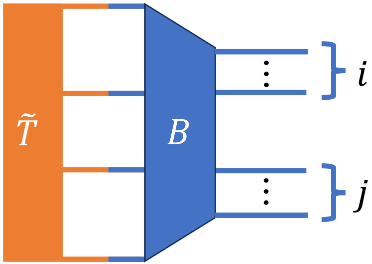


$\rightarrow$ We use the HOTRG method for evaluating the reduced density matrix.

Our approach to reduced density matrix [Hayazaki-Kadoh-Takeda-GT, arXiv:2312.xxxxx]

We develop a general method for evaluating the reduced density matrix of the subregion of any spatial size l based on the HOTRG. cf. [Luo-Kuramashi, 2023] for the case of $l = L_x/2$

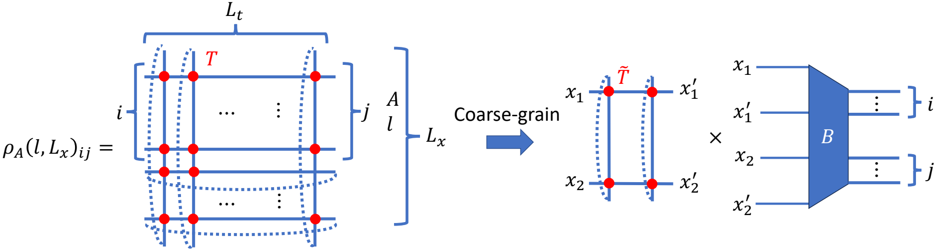
$$\rho_A(l, L_x)_{ij} \sim$$



For the subregion A of any size l

Detail of our approach

- The most important point of our method is that we decompose the reduced density matrix ρ_A into a contraction of coarse-grained tensor $\tilde{T}$ and the boundary factor B.

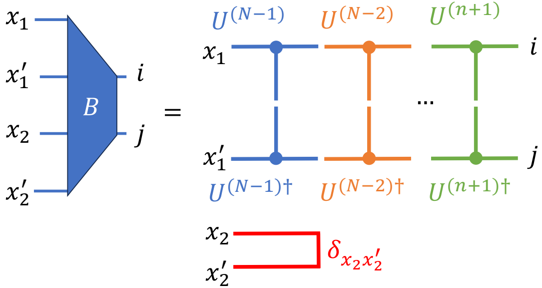


- The boundary factor B consists of isometry matrices $U^{(k)}$ and $U^{(k)\dagger}$ which perform the coarse-graining of tensor $T^{(k)}$ in x direction.

Detail of boundary factor B

- For simplicity, we consider the case that $L_x = 2^N$ and $l = 2^0, 2^1, 2^2, \ldots, 2^{N-1}$. In this case, the boundary factor $B_{x_1 x_1' x_2 x_2' ij}$ for $l = 2^n$ $(0 < n < N)$ is given by the product of a contraction of the isometry matrices and a Kronecker delta:

$B_{x_1x_1'x_2x_2'ij} =$



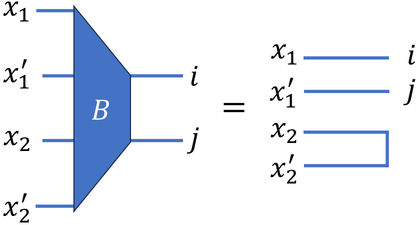
$U^{(k)}$: Isometry matrix performing k-th coarse-graining of the tensor network in the x direction

$$= \left(U^{(N-1)\dagger} U^{(N-2)\dagger} \ldots U^{(n+1)\dagger} U^{(n+1)} \ldots U^{(N-2)} U^{(N-1)}\right)_{x_1x_1'ij} \delta_{x_2x_2'}$$

Detail of boundary factor B

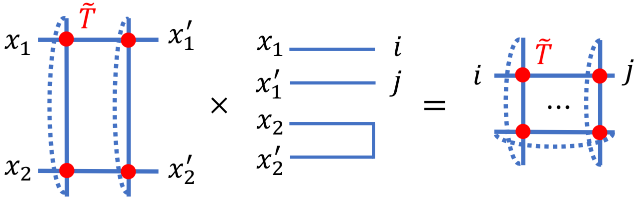
- In particular, the boundary factor B becomes trivial when $l = 2^{N-1} = L_x/2$

$B =$



$= \delta_{x_1 i}\delta_{x_1' j}\delta_{x_2 x_2'}$

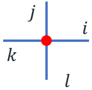
$\rho_A(l, L_x)_{ij} \sim$



Numerical results in the two-dimensional Ising model

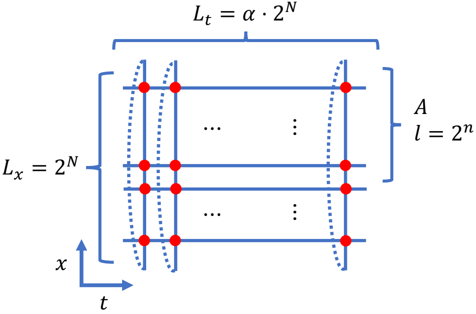
- We consider the two-dimensional Ising model

$$H = -\sum_{<x,y>} s_x s_y \,, \qquad Z = Tr\, e^{-\beta H} = \sum_{\cdots i,j,k,l,m,n,o,\cdots} \cdots T_{ijkl} T_{mnio} \cdots \,, \qquad T_{ijkl} =$$



- We set spatial length $L_x = 2^N$, temporal length $L_t = \alpha \cdot 2^N$ and temperature T to the critical temperature T_c.

- We take sufficiently large α and conformally map the plane into a cylinder of spatial length L_x.
- We evaluate the entanglement entropy S_A of subregion A of spatial size $l = 2^n$ in two cases:
  1. $L_x = 2^{11} = 2048$ and $\alpha = 32$
  2. fixed $x = 2^n/2^N$ and $\alpha = 8$



Numerical results in the two-dimensional Ising model

The entanglement entropy S_A of the theory on a cylinder is given by

$$S_A(n,N) = \frac{c}{3}\log\left(\frac{2^N}{\pi\gamma}\sin\left(\frac{2^n}{2^N}\pi\right)\right) + k$$

where γ is a UV regulator, c a central charge, and k a constant.
We extract c from the difference of the entanglement entropy in n using this eq.

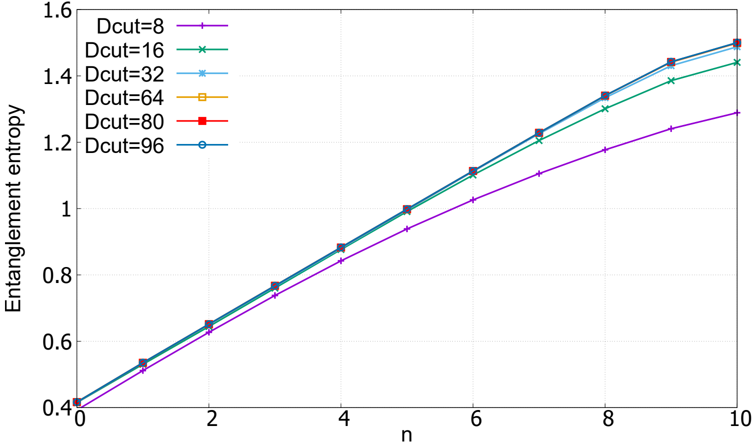
1. fixed $L_x = 2048$

$$c = (S(n+1,N) - S(n,N)) \cdot 3\left(\log\frac{\sin\left(2^{n+1-N}\pi\right)}{\sin\left(2^{n-N}\pi\right)}\right)^{-1}$$

2. fixed $x = 2^n/2^N$

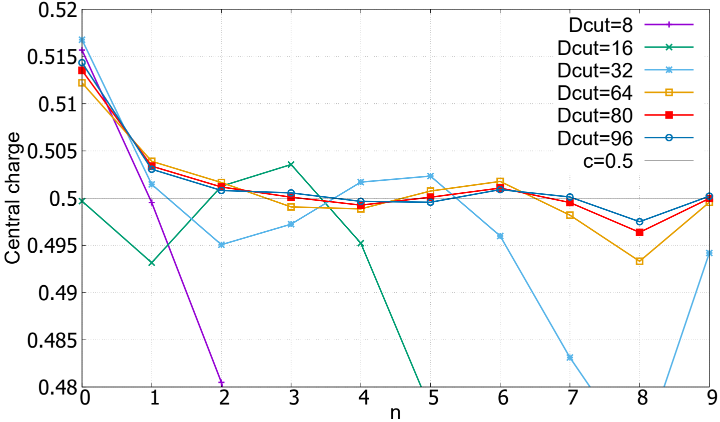
$$c = (S(n+1,N) - S(n,N)) \cdot \frac{3}{\log 2}$$

Result - Entanglement entropy with fixed $L_x = 2048$ and $\alpha = 32$



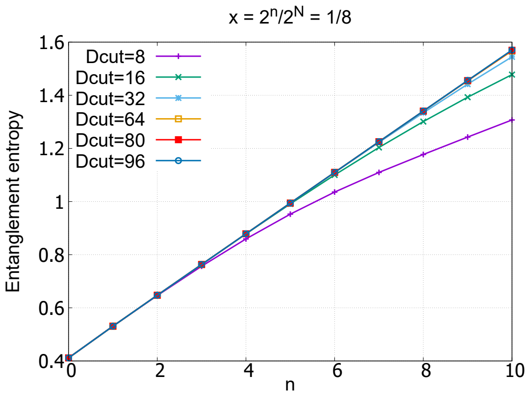
- The entanglement entropy exhibits the behavior consistent with a logarithmic function as expected.

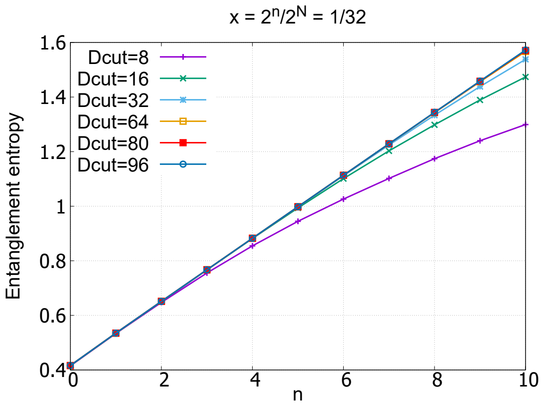
Result - Central charge with fixed $L_x = 2048$ and $\alpha = 32$



- With sufficiently large $D_{\rm cut}$, central charge shows the plateau behavior from $n = 3$ to 7.
- Averaging the data of $n = 3, 4, 5, 6$, and 7 with $D_{\rm cut} = 96$, we determine the central charge $c = 0.5002(7)$.

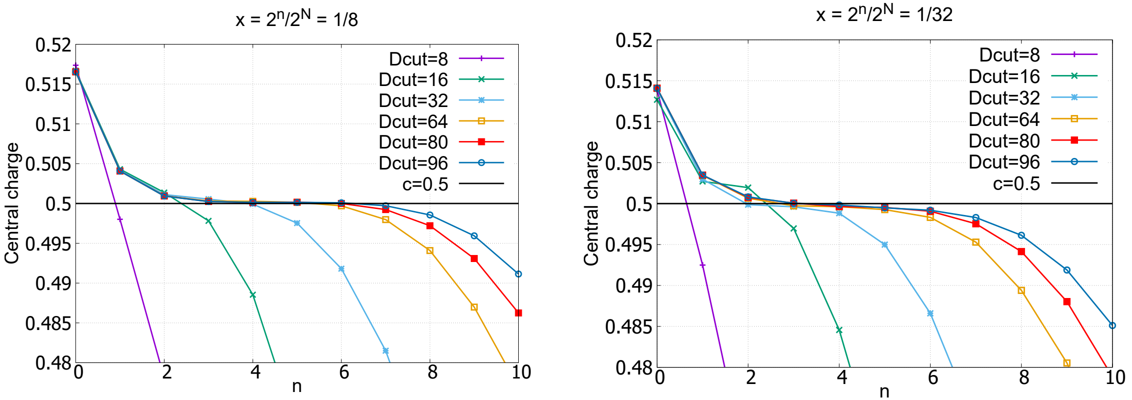
Result - Entanglement entropy with fixed $x = 2^n/2^N$ and $\alpha = 8$





- With fixed $x = 2^n/2^N$, the entanglement entropy shows the behavior consistent with a logarithmic function.

Result - Central charge with fixed $x = 2^n/2^N$ and $\alpha = 8$



- In the case of $x = 1/8$, the central charge c exhibits the plateau behavior around $c = 0.5$, while c no longer shows such behavior with $x = 1/32$ due to the extra truncation.
- Averaging the data of $n = 4, 5, 6$, with $D_{\mathrm{cut}} = 96$ and $x = 1/8$, we determine the central charge $c = 0.50013(3)$.

Conclusion and discussion

Conclusion

- We propose a general method based on the HOTRG for evaluating the entanglement entropy.
- Our method makes it possible to examine the entanglement entropy of a subregion of any spatial size.
- Applying our method to the two-dimensional Ising model yields results very close to the theoretical value.

Future direction

- Higher-dimensional quantum field theories
- Relation between holography and tensor network