A Recipe for bulk construction from a scalar CFT by a conformal flow

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work in progress with

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0. Motivation/Question

How can we map properties of QFT to a "geometry" ?

In this talk, to answer this question, we exclusively consider a scalar CFT with Euclidean signature, which is quantized by the path-integral.

We will provide a prototype of AdS/CFT correspondence without using string duality (gauge/gravity correspondence). Therefore results may differ from usual understanding. We however hope that our attempt may provide deeper understanding of the standard AdS/CFT correspondence.

We can easily extend our method to a general scalar (non-conformal) QFT, which may provide an answer to the above question.

1. Construction of a bulk space

Starting point We consider an O(N) scalar CFT in d-dimensions.

A non-singlet primary field satisfies $\langle 0|\hat{\varphi}^a(x)\hat{\varphi}^b(y)|0\rangle = \delta^{ab}\frac{C_0}{|x-y|^{2\Delta}}$

Smearing We smear the field by the flow equation as $\Delta < \frac{d}{2}$

$$(-\alpha\eta\partial_\eta^2 + \beta\partial_\eta)\hat{\phi}^a(x;\eta) = \Box_x\hat{\phi}^a(x;\eta), \quad \hat{\phi}^a(x;0) = \hat{\varphi}^a(x)$$

Conformal flow We take $\nu := 1 + \frac{\beta}{\alpha} = \frac{d}{2} - \Delta$ with $\eta = \frac{\alpha}{4}z^2$

and introduce a normalized field as $\hat{\sigma}^a(X) := \frac{\hat{\phi}^a(x;\eta)}{\sqrt{\langle 0|\hat{\phi}^2(x;\eta)|0\rangle}}$ $X := (x, z)$

z is an extra direction, which corresponds to an energy scale of CFT.

$$z = 0 \text{ (UV) and } z = \infty \text{ (IR)}$$

QFT(CFT) in d-dimension + energy scale ⟶ d+1 dimensional bulk space

Holography

Why do we call it a conformal flow ?

conformal symmetry on $\hat{\varphi}^a(x)$ generates a coordinate transformation on $\hat{\sigma}^a(X)$ as

$$\delta^{\rm conf}\hat{\varphi}^a(x) = -\delta x^\mu \partial_\mu \hat{\varphi}^a(x) - \frac{\Delta}{d}(\partial_\mu \delta x^\mu)\hat{\varphi}^a(x)$$

$$\delta x^\mu := a^\mu + w^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2x^\mu (b\cdot x)$$

↓

$$\delta^{\rm conf}\hat{\sigma}^a(X) = -\delta X^A \partial_A \hat{\sigma}^a(x) \qquad \delta X^\mu := \delta x^\mu + z^2 b^\mu \qquad \delta X^{d+1} := (\lambda - 2b\cdot x) z$$

This coordinate transformation is nothing but the AdS isometry.

Aoki-Balog-Onogi-Yokoyama, PTEP **2023**(2023) 013B03.

AdS/CFT correspondence ?

2. (Quantum) bulk space

Operators in the boundary and the bulk enjoy these symmetries as

$$\langle 0| \prod_{i=1}^{m} G^{i}_{A^i_1 \cdots A^i_{n_i}}(\tilde{X}_i) \prod_{j=1}^{s} O^{j}_{\mu^j_1 \cdots \mu^j_{l_j}}(\tilde{y}_j)|0\rangle = \prod_{i=1}^{m} \frac{\partial X_i^{B^i_1}}{\partial \tilde{X}_i^{A^i_1}} \cdots \frac{\partial X_i^{B^i_{n_i}}}{\partial \tilde{X}_i^{A^i_{n_i}}} \prod_{j=1}^{s} J(y_j)^{-\Delta_j} \frac{\partial y_j^{\nu^j_1}}{\partial \tilde{y}_j^{\nu^j_1}} \cdots \frac{\partial y_j^{\nu^j_{l_j}}}{\partial \tilde{y}_j^{\nu^j_{l_j}}}$$

$$\times \langle 0| \prod_{i=1}^{m} G^{i}_{B^i_1 \cdots B^i_{n_i}}(X_i) \prod_{j=1}^{s} O^{j}_{\nu^j_1 \cdots \nu^j_{l_j}}(y_j)|0\rangle$$

bulk operator (with an arbitrary spin) $G^{i}_{A^i_1 \cdots A^i_{n_i}}(X_i)$

coordinate transformation $X \to \tilde{X}$

boundary operator (with an arbitrary spin) $O^{j}_{\mu^j_1 \cdots \mu^j_{l_j}}(y_j)$

conformal transformation $y \to \tilde{y}$

Correlation functions including all quantum corrections are defined in the bulk.

Quantum bulk space

This is different from the standard AdS/CFT correspondence.

Example Bulk-boundary (scalar-scalar) propagator is given by

$$\langle 0|\hat{\Phi}(X)\hat{S}(y)|0\rangle_c = C_S \left(\frac{z}{(x-y)^2+z^2}\right)^{\Delta_S}$$

Δ_S: conformal dimension of $\hat{S}$

where $\hat{\Phi}(X)$ and $\hat{S}(y)$ are bulk and boundary scalar operators, respectively.

While a geometry of the bulk space is not defined so far, the above propagator satisfies a free Klein-Gordon equation in the AdS with the mass given by

$$m^2 = \frac{\Delta_S(\Delta_S - d)}{L_{\mathrm{AdS}}^2}$$

We add a small source at the boundary as

$$\Phi_J(X) := \langle 0|\hat{\Phi}(X)\exp\left[\int d^d y\, J(y)\hat{S}(y)\right]|0\rangle_c \simeq \int d^d y\, J(y)\langle 0|\hat{\Phi}(X)\hat{S}(y)|0\rangle_c + O(J^2)$$

In the $z \to 0$ limit, we obtain **GKP-Witten relation** as

$$\lim_{z\to 0}\Phi_J(X) = z^{d-\Delta_S}\left[\tilde{C}_s J(x) + O(z^2)\right] + z^{\Delta_S}\left[\langle 0|\hat{S}(x)|0\rangle_J + O(z^2)\right]$$

$$\langle 0|\hat{S}(x)|0\rangle_J := \langle 0|\hat{S}(x)\exp\left[\int d^d y\, J(y)\hat{S}(y)\right]|0\rangle \simeq C_S\int d^d y\,\frac{J(y)}{|x-y|^{2\Delta_S}} + O(J^2)$$

3. Geometry of the bulk space

How can we determine a geometrical structure of the bulk space ?

Possibilities

- a geometry which makes the (scalar) propagator solution to a free Klein-Gordon equation.
- a geometry which makes the (boundary) entanglement entropy equal to the minimal surface in the bulk.
- others

They are rather complicated. We instead consider a more direct method.

We determine a bulk geometry using Bures information metric.

Bures information metric

A distance between a density matrix ρ and $\rho + d\rho$ is defined as

$$d^2(\rho, \rho + d\rho) := \frac{1}{2}\operatorname{Tr}\left(d\rho\hat{G}\right)$$

where $\hat{G}$ satisfies $\quad \rho\hat{G} + \hat{G}\rho = d\rho$

State dependent density matrix

$|S\rangle$: some state in CFT

$$\rho_S(X) := \sum_{a=1}^{N} \hat{\sigma}_S^a(X)|S\rangle\langle S|\hat{\sigma}_S^a(X)$$

N entangled pairs (mixed state)

$$\operatorname{tr}\rho_S(X) = 1$$

$$\hat{\sigma}_S^a(X) := \frac{\hat{\sigma}^a(X)}{\sqrt{\langle S|\hat{\sigma}^2(X)|S\rangle}} = \frac{\hat{\phi}^a(x;\eta)}{\sqrt{\langle S|\hat{\phi}^2(x;\eta)|S\rangle}}$$

normalized for the state $|S\rangle$

since $\hat{G} = Nd\rho_S(X) = NdX^A\partial_A\rho_S(X)$, we obtain

$$\frac{1}{2}\operatorname{Tr}\left[d\rho_S(X)\hat{G}\right] = \frac{N}{2}\operatorname{Tr}\left[\partial_A\rho_S(X)\partial_B\rho_S(X)\right]dX^A dX^B$$

$\longrightarrow$ $$g_{AB}^S(X) := \ell^2\sum_{a=1}^{N}\langle S|\partial_A\sigma_S^a(X)\partial_B\sigma_S^a(X)|S\rangle$$

The metric is state dependent.

4. State dependent metric

$$g^S_{AB}(X) := \ell^2 \sum_{a=1}^{N} \langle S | \partial_A \sigma^a_S(X) \partial_B \sigma^a_S(X) | S \rangle$$

Bulk geometry is state dependent. The bulk geometry is not unique.

Vacuum case $|0\rangle$

$$\hat\sigma^a_0(X) = \hat\sigma^a(X) = \frac{\hat\phi^a(x;\eta)}{\sqrt{\langle 0|\hat\phi^2(x;\eta)|0\rangle}}$$

$$\longrightarrow \quad g^{\rm vac}_{AB}(X) = \ell^2 \frac{\Delta(d-\Delta)}{d+1}\frac{\delta_{AB}}{z^2}$$

AdS metric in the Poincare coordinate

A vacuum state in an arbitrary CFT $\longrightarrow$ AdS metric

AdS/CFT correspondence

Aoki-Yokoyama, PTEP **2018**(2018) 031B01.

What is a metric for excited states ?

Finite temperature (Thermo field double)

$$|\mathrm{TFD}\rangle := \frac{1}{\sqrt{Z_T}} \sum_n e^{-E_n/2T} |E_n\rangle \otimes |\widetilde{E_n}\rangle \qquad Z_T := \sum_n e^{-E_n/T}$$

If the $O(N)$ model is free, the metric becomes the asymptotic AdS.

In the UV region ($z \to 0$), the metric is a classical solution to $f(R)$ gravity.

Aoki-Shimada-Balog-Kawana, arXiv:2308.01076[hep-th]

Scalar state

$|S\rangle = |\Phi\rangle$: scalar state with a conformal dimension Δ_Φ

In general, the metric becomes the asymptotic AdS.

In the case of free $O(N)$ model with $\Delta_\phi = 2\Delta$, we have

$$g^{\Phi}_{AB}(X) = g^{\mathrm{vac}}_{AB}(X) + \frac{1}{N}\delta g_{AB}(X) + O(1/N^2)$$ Asymptotic AdS with $1/N$ correction

Aoki-Balog-Shimada, work in progress

5. Summary

CFT → bulk (quantum) space with symmetry

conformal flow

↓ state dependent normalization

state dependent metric/geometry

vacuum -> AdS

excited state -> asymptotic AdS

metric becomes classical in the large N limit

Is a concept of the state-dependent geometry compatible with the conventional AdS/CFT correspondence ?

We are also working on the metric for excited states from the standard point of view.

Stay tuned !