WKB analysis for affine Toda field theoriess

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Introduction

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Introduction

Liouville integrability

An integrable model (IM) is a Hamiltonian system with

number of degrees of freedom = number of conserved charges

- The integrable model has (infinite) numbers of conserved charges.
- The equations of motion for an integrable field theory can be rewritten into Lax pairs, which lead to two **linear problems**.
- It is possible to diagonalize the linear problems with affine Lie algebra structures, where the diagonal elements turn to be classical conserved currents. [Drinfeld, Sokolov (1984)]

Motivation

The quantum integrable models

- The scattering S-matrix (satisfied by TBA Equations) is exactly solvable. However, its conserved charges are difficult to calculate (2d effective CFT).
- Especially in affine Toda field theory, few of them were known.

The ODE/IM correspondence [Dorey-Tateo 9812211]

- It is a relation between the spectral analysis of the ordinary differential equation and the "functional relations" in quantum IM.
- The simplest one is between $[\epsilon^2\partial_z^2 + V(z) - E]\psi(z,\epsilon) = 0$ and the Sine-Gordon model.

Motivation

Maybe we can obtain the quantum conserved charges from the classical ones!

Inspiration

The classical conserved currents for the $A_1^{(1)}$ Toda field theory

$$I_2(z) = \frac{T(z)}{2},$$
$$I_4(z) = \frac{\partial_z^2 T(z) - T^2(z)}{8},$$
$$I_6(z) = \frac{1}{32}\left(-5T'(z)^2 - 6T(z)T''(z) + T^{(4)}(z) + 2T(z)^3\right),$$

The WKB solutions for $(\epsilon^2\partial_z^2 + \epsilon^2 u_2(z) - p(z))\psi(z,\epsilon) = 0$ with WKB ansatz $\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon))$ are

$$P_0(z) = \sqrt{p(z)},$$
$$P_1(z) = -\frac{1}{2}\partial_z \ln P_0,$$
$$P_2(z) = \frac{P_0''}{16P_0^2} + \frac{u_2(z)}{2P_0} + \partial_z(\frac{3P_0'}{16P_0^2}),$$
$$P_3(z) = -\partial_z(-\frac{u_2(z)}{4P_0^2} + \frac{3P_0'^2}{16P_0^4} - \frac{P_0''}{8P_0^3}),$$

Affine Toda field equations

Affine Toda field equations

The action of $\hat{\mathfrak{g}}$ affine Toda field theory in $2d$ complex plane:

$$S = \int d^2 z \Big\{ \frac{1}{2}\partial_z \phi \cdot \bar{\partial}_{\bar{z}} \phi + \left(\frac{m^2}{\beta}\right)\Big[\sum_{i=1}^{r} \exp\left(\beta \alpha_i \cdot \phi\right) + \exp\left(\beta \alpha_0 \cdot \phi\right)\Big]\Big\}.$$

Its equation of motion: the $\hat{\mathfrak{g}}$ affine Toda field equation is

$$\bar{\partial}_{\bar{z}} \partial_z \phi(z, \bar{z}) - \left(\frac{m^2}{\beta}\right)\Big[\sum_{i=1}^{r} \alpha_i \exp\left(\beta \alpha_i \cdot \phi\right) + \alpha_0 \exp\left(\beta \alpha_0 \cdot \phi\right)\Big] = 0.$$

$$\phi(z, \bar{z}) = \sum_{i=1}^{r} \alpha_i^{\vee} \phi_i(z, \bar{z}),$$

$\alpha_i(\alpha_i^{\vee})$: roots(coroots) of $\hat{\mathfrak{g}}$,

β : a coupling constant,

m : a mass parameter.

Affine Toda field equations

The affine Toda field equations can be separated into Lax pairs:

$$\mathcal{L} = \partial_z + \beta \sum_{i=1}^{r} \partial_z \phi_i(z, \bar{z}) H_i + m\lambda\Lambda,$$

$$\bar{\mathcal{L}} = \partial_{\bar{z}} + e^{-\beta \sum_{i=1}^{r} \phi_i H_i} (m\lambda^{-1}\bar{\Lambda})\, e^{\beta \sum_{i=1}^{r} \phi_i H_i}.$$

E_{α_i}, $E_{-\alpha_i}$: ladder operators, $H_i = \alpha_i^\vee \cdot H$: Cartan subalgebras

λ: a spectral parameter,

$\Lambda = \sum_{i=0}^{r} E_{\alpha_i}$ and $\bar{\Lambda} = \sum_{i=0}^{r} E_{-\alpha_i}$

The flatness condition giving the equation of motion

$$[\mathcal{L}, \bar{\mathcal{L}}] = 0$$

is the integrability condition of the linear problem

$$\mathcal{L}\Psi = \bar{\mathcal{L}}\Psi = 0$$

Affine Toda field equations

Take the conformal transformation ($\rho^\vee$ is the co-Weyl vector)

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z}), \quad \phi \to \hat{\phi} = \phi - \rho^\vee \log(\partial_z w \partial_{\bar{z}} \bar{w}),$$

then the affine Toda field equations will be modified into

$$\partial_{\bar{z}}\partial_z \phi(z,\bar{z}) - [\sum_{i=1}^{r} \alpha_i \exp\left(\alpha_i \cdot \phi\right) + p(z)\bar{p}(\bar{z})\alpha_0 \exp\left(\alpha_0 \cdot \phi\right)] = 0$$

with $p(z) = (\partial_z w)^h, \quad \bar{p}(\bar{z}) = (\partial_{\bar{z}} \bar{w})^h$. The modified Lax operators are

$$\mathcal{L}_m = \partial_z + \sum_{i=1}^{r} \partial_z \phi_i(z,\bar{z}) H_i + \lambda(\sum_{i=1}^{r} E_{\alpha_i} + p(z) E_{\alpha_0}),$$

$$\bar{\mathcal{L}}_m = \partial_{\bar{z}} + \lambda^{-1} e^{-\phi_i H_i}(\bar{p}(\bar{z}) E_{\alpha_0} + \sum_{i=1}^{r} E_{-\alpha_i}) e^{\phi_i H_i}.$$

The diagonalization approach

The diagonalization approach

It is possible to diagonalize the linear problem and the diagonal elements are classical conserved currents [Drinfeld, Sokolov (1984)].

Let us focus on the holomorphic part $\mathcal{L}_m$ ($[\mathcal{L}_m, \bar{\mathcal{L}}_m] = 0$).
We replace the spectral parameter λ with Planck constant $\epsilon = \lambda^{-1}$.

$$\epsilon\mathcal{L}_m = \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}.$$

One can view it as a covariant derivative with connection:

$$A(z) = \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}$$

Then the gauge transformation is given by

$$\mathbf{Gau}_T[A(z)] = T^{-1}(z)A(z)T(z) + \epsilon T^{-1}(z)\partial_z T(z).$$

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The diagonalization approach

The transformation matrix T can be decomposed into

$$T(z) = T_d T_{d-1} \dots T_3 T_2 T_1.$$

d is the representation dimension and $T_i(z)$ are $d \times d$ matrices satisfying

$$T_i(z)_{ab} = \begin{cases} 1, & \text{if} \quad a = b, \\ g_{i,b}(z,\epsilon), & \text{if} \quad a = i, \quad b \neq i, \quad 1 \leq b \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition means we diagonalize the connection row by row from the bottom to the top. For instance

$$T_d = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ g_{d,1} & g_{d,2} & \cdots & g_{d,d-1} & 1 \end{pmatrix},$$

The diagonalization approach

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The connection after the first gauge transformation:

$$A'(z) = \begin{pmatrix} & & \ddots & & \\ & & & & \\ \mathbf{Gau}_{T_d}[A]_{d,1} & \mathbf{Gau}_{T_d}[A]_{d,2} & \cdots & \mathbf{Gau}_{T_d}[A]_{d,d-1} & \mathbf{Gau}_{T_d}[A]_{d,d} \end{pmatrix},$$

For each step of the gauge transformation $\mathbf{Gau}_{T_i}$, we fix $g_{i,b}(z)$ such that the connection $A'(z)$ satisfies (the red parts)

$$A'_{ij} = 0, \quad 1 \leq j \leq d, \quad j \neq i.$$

The final diagonalized connection $A_{\mathrm{diag}}(z)$ is given by

$$A_{\mathrm{diag}}(z) = \mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2} \ldots \mathbf{Gau}_{T_{d-2}} \circ \mathbf{Gau}_{T_{d-1}} \circ \mathbf{Gau}_{T_d}[A(z)].$$

The diagonalization of $A_1^{(1)}$

The diagonal elements of $A_1^{(1)}$ can be summarized as ($d(*)$: total derivatives)

$$A_{\text{diag}}(z) = \begin{pmatrix} -f(z,-\epsilon) + d(*) & 0 \\ 0 & f(z,\epsilon) \end{pmatrix},$$

$f(z,\epsilon)$ in $A_1^{(1)}$ satisfies the Riccati equation

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0.$$

$u_2(z) = \phi'(z)^2 - \phi''(z)$ is the classical energy-momentum tensor.
$f(z,\epsilon)$ can also be obtained from

$$[\epsilon^2 \partial_z^2 - \epsilon^2 u_2(z) - p(z)]\psi(z,\epsilon) = 0$$

with the WKB ansatz $\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int dz\, f(z,\epsilon))$.

One can solve $f(z,\epsilon)$ perturbatively after expanding $f = \sum_{n=0}^{\infty} f_n \epsilon^n$

Generalized to other affine Lie algebras

The ODEs satisfied by $\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int dz\, f(z,\epsilon))$

$$A_r^{(1)}:\ (-\epsilon)^h(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1) \cdots (\partial_z + \partial_z\phi_r)\psi(z,\epsilon) = p(z)\psi(z,\epsilon)$$

$$A_{2r-1}^{(2)}:\ \epsilon^{(2r-1)}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - \partial_z\phi_r + \partial_z\phi_{r-1})(\partial_z + \partial_z\phi_r - \partial_z\phi_{r-1}) \cdots (\partial_z + \partial_z\phi_1)\psi - 2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0$$

$$B_r^{(1)}:\ \epsilon^{2r}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - 2\partial_z\phi_r + \partial_z\phi_{r-1})\partial_z(\partial_z + 2\partial_z\phi_r - \partial_z\phi_{r-1}) \cdots (\partial_z + \partial_z\phi_1)\psi - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0$$

$$D_{r+1}^{(2)}:\ \epsilon^{(2r+2)}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - 2\partial_z\phi_r + \partial_z\phi_{r-1})\partial_z(\partial_z + 2\partial_z\phi_r - \partial_z\phi_{r-1}) \cdots (\partial_z + \partial_z\phi_1)\psi - 4p(z)\partial_z^{-1}p(z)\psi = 0$$

$$D_r^{(1)}:\ \epsilon^{(2r-2)}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - \partial_z\phi_r - \partial_z\phi_{r-1} + \partial_z\phi_{r-2})\partial_z^{-1} (\partial_z + \partial_z\phi_r + \partial_z\phi_{r-1} - \partial_z\phi_{r-2})\cdots(\partial_z + \partial_z\phi_1)\psi - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0$$

These (pseudo)-ODEs have been also found in

[Dorey, Dunning, Masoero, Suzuki, Tateo (2007); Ito, Locke (2015)].

Conserved current vs. WKB solution

Conserved current vs. WKB solution

The classical conserved currents for the modifed $A_1^{(1)}$ Toda field theory

$$f_0(z) = \sqrt{p(z)},$$
$$f_1(z) = -\frac{1}{2}\partial_z \ln f_0,$$
$$f_2(z) = \frac{f_0''}{16 f_0^2} + \frac{u_2(z)}{2 f_0} + \partial_z\left(\frac{3 f_0'}{16 f_0^2}\right),$$

The appearance of $p(z)$: conformal transformation $z \to w(z)$

$$dw = \sqrt{p(z)} dz, \quad \hat{u}_2(w(z)) = \frac{1}{p(z)}\left[u_2(z) + \frac{4pp'' - 5p'^2}{16p^2}\right]$$

After the conformal transformation,

$$\hat{f}_0(w) = 1, \quad \hat{f}_2(w) = \frac{\hat{u}_2(w)}{2}, \quad \hat{f}_4(w) = \frac{\partial_w^2 \hat{u}_2(w) - \hat{u}_2^2(w)}{8}.$$

Ongoing work

Quantum conserved charges via ODE/IM correspondence

- Set $p(z) = (z^{2\alpha} - 1)$ and $\phi(z) \to l \log(z)$, then, the integral along the loop: $\infty \cdot e^{+i0}$ to $\infty \cdot e^{-i0}$ around $z = 1$.

$$Q_1 = \int_C dz \left(\frac{\left(l + \frac{1}{2}\right)^2}{2z^2\sqrt{(z^{2\alpha} - 1)}} + \frac{2\alpha(2\alpha - 1)z^{2\alpha-2}}{48(z^{2\alpha} - 1)^{\frac{3}{2}}} - \frac{1}{8z^2\sqrt{(z^{2\alpha} - 1)}} \right)$$
$$= [(l + \frac{1}{2})^2 - \frac{1}{24}(4\alpha + 4)] \cdot \Gamma(\dots)$$

- Quasi-momentum: $P = \frac{l+\frac{1}{2}}{2\alpha+2}$, Coupling constant: $\beta^2 = \frac{1}{\alpha+1}$

$$Q_1 \sim \frac{P^2}{\beta^2} - \frac{1}{24} = I_1,$$

  is the first conserved charge (effective central charge) for the quantum sine-Gordon model (effective CFT).

- Similar calculations for higher orders or ranks are in progress.

Summary

Summary

- The diagonal elements of linear problems are the WKB solutions to a set of (pseudo) ordinary differential equations.
- There is a relation between the conserved currents and the WKB solutions via the conformal transformation.
- The corresponding quantum conserved charges are under calculation.

Thank you for watching.