ON STABILITY OF FERMIONIC SUPERCONDUCTING CURRENT IN COSMIC STRING

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Strong CP Problem

- ✓ Experimentally, **QCD** is known to preserve **CP** symmetry very well.
- ✓ **CP** violating transitions in the **SM** are caused by **CP** violation in the weak interaction (i.e. by the CKM phase).

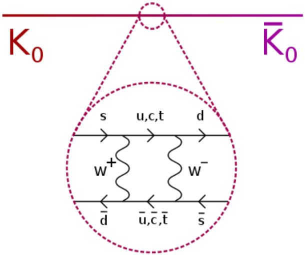


Figure from : `https://en.wikipedia.org/wiki/Kaon`

Strong CP Problem

CP conservation is not automatically guaranteed in **QCD** .

- ✓ **QCD** has its own **CP**-voilating parameter : $\boldsymbol{\theta}$

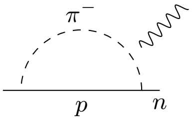
$$\mathcal{L}_{\rm QCD} = -\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} + \boxed{\frac{g_s^2}{32\pi^2}\theta G^a_{\mu\nu}\tilde{G}^{a\mu\nu}} + \sum_i^{N_f}\bar{q}_i(i\not{D} - m)q_i$$

[positive quark masses : $m > 0$]

- ✓ $\boldsymbol{\theta}$-term is not invariant under **P** and **CP** transformation

$$\frac{g_s^2}{32\pi^2}\theta G^a_{\mu\nu}\tilde{G}^{a\mu\nu} \xrightarrow{\textbf{CP}} -\frac{g_s^2}{32\pi^2}\theta G^a_{\mu\nu}\tilde{G}^{a\mu\nu}$$

- ✓ $\boldsymbol{\theta}$-term is highly constrained experimentally.



Null observation of the neutron **EDM**:
$d_n/e < 10^{-26}$ cm @ 90%CL
[PRL 124, 081803 (2020)].

$d_n/e \sim 10^{-15}\theta$ cm ['79 Crewther, Veccia, Veneziano, Witten]
$\rightarrow \theta < 10^{-11}$ why so small ? = strong CP problem !

Peccei-Quinn Mechanism and Axion

How to solve the strong **CP** problem?

If massless colored fermions exist, the strong **CP** problem goes away!

['77, Peccei-Quinn]

Massless colored **PQ** fermion : $\psi_{L,R} = P_{L,R}\psi$

$$\mathcal{L} = \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R \ , \quad (D_\mu = \partial_\mu - i g_s G^a_\mu t^a) \ .$$

A chiral U(1) rotation,

$$\psi_L \to e^{i\alpha}\psi_L \ , \quad \psi_R \to \psi_R \ ,$$

shifts $\theta \to \theta + \alpha$ through the chiral anomaly.

θ can be set to 0 !

[Symmetry under the chiral U(1) rotation = **PQ** symmetry]

No such a massless colored fermions in reality... How to reconcile ?

Peccei-Quinn Mechanism and Axion

Make the **PQ** symmetry broken spontaneously !

✓ A complex field ϕ rotates under the PQ symmetry :

$$\phi \to e^{i\phi}\phi \ , \quad \psi_L \to e^{-i\alpha}\psi_L \ , \quad \psi_R \to \psi_R$$

✓ The chiral U(1) symmetry is spontaneously broken by $\langle\phi\rangle = v$.

The colored fermions obtain mass through:

$$\mathcal{L} = \mathcal{L}(\phi) + \bar{\psi}_L i\not{D}\psi_L + \bar{\psi}_R i\not{D}\psi_R + \boxed{y\phi\bar{\psi}_R\psi_L + h.c.}$$

$\mathcal{L}(\phi) = \partial_\mu\phi^*\partial^\mu\phi - V(\phi)$ $\leftarrow$ symmetric under the U(1) rotation

Colored fermions obtain mass,

$$m_\psi = yv$$

We can make the new colored fermions arbitrarily heavy!

$\cdots$ Any low-energy implications?

Peccei-Quinn Mechanism and Axion

Spontaneous breaking is associated with a Goldstone mode : Axion !

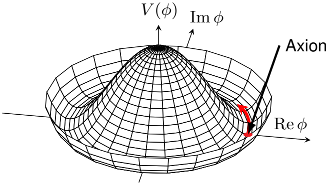
['78, Weinberg, '78 Wilczek]

Axion = phase direction of ϕ :

$$\phi(x)|_{\rm axion} = \frac{f_a}{\sqrt{2}} \exp\left[i\frac{a(x)}{f_a}\right] , \quad f_a = \sqrt{2} \times v \text{ (axion decay constant)}$$

Axion couples to QCD and QED through :

$$\mathcal{L}_{\rm eff} = \frac{g_s^2}{32\pi^2}\frac{a}{f_a} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} + \frac{C_e e^2}{32\pi^2}\frac{a}{f_a} F^a_{\mu\nu}\tilde{F}^{a\mu\nu}$$



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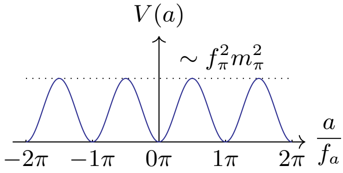


Axion obtains non-trivial potential due to **QCD** dynamics through the chiral anomaly

Peccei-Quinn Mechanism and Axion

✓ The lower limit on the axion decay constant:

$$f_a \gtrsim 10^{8-9}\,\mathrm{GeV}\,,\quad \text{(supernovae/red-giant cooling)}$$

https://cajohare.github.io/AxionLimits/docs/ap.html

✓ Axion is very light:

$$m_a = \mathcal{O}(1)\,\mu\mathrm{eV} \times \left(\frac{10^{12}\,\mathrm{GeV}}{f_a}\right)$$

✓ Axion is a good candidate for dark matter:

$$\Omega_a h^2 = 0.18 \times \theta_{\rm init}^2 \left(\frac{f_a}{10^{12}\,\mathrm{GeV}}\right)^{1.19} \left(\frac{\Lambda_{\rm QCD}}{400\,\mathrm{MeV}}\right)\,,$$

from the coherent oscillation of the axion (c.f. $\Omega_{\rm DM} h^2 \simeq 0.12$).

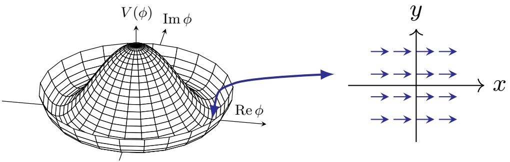
[e.g. 1301.1123 Kawasaki, Nakayama]

The PQ mechanism not only solves the strong CP problem but also provides a good candidate for dark matter!

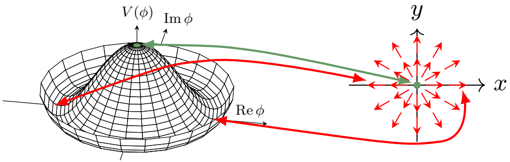
Cosmic String

U(1) symmetry breaking is associated with cosmic strings !

✓ Trivial vacuum configuration of the broken phase
= phase in ϕ is aligned to the same direction [e.g. $(\mathrm{Re}\,\phi, \mathrm{Im}\,\phi) = (v, 0)$]



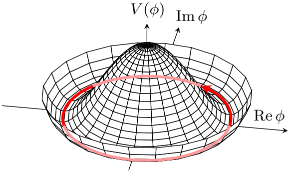
✓ Non-trivial vacuum configuration due to the finite causality length
= phase of ϕ depends on the spatial direction [e.g. $(\cos\varphi, \sin\varphi)$]

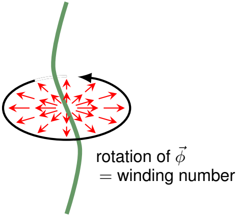


U(1) symmetry is restored at the core of the configuration!
$\rightarrow$ energy concentration at the core = topological defect

Cosmic String

Energy concentrations connected in one dimension = **Cosmic String**





✓ The string tension (=mass per unit length)

$$\mu_{\rm str} \sim \pi f_a^2 \log \frac{f_a}{\ell_{\rm corr}} \ , \quad \ell_{\rm corr} \sim \text{distance of the stirngs}$$

✓ Long strings are stable due to the winding of the ϕ phase (= topological charge, $\pi_1[U(1)] = \mathbb{Z}$)

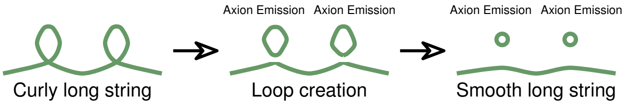
Cosmological Evolution of Axionic Cosmic String

Assume the **PQ** symmetry breaking takes place after inflation at $T \sim f_a$.

- ✓ On average, $\mathcal{O}(10-100)$ number of cosmic strings are generated per Hubble volume, $V \sim H^{-3}$
- ✓ Long strings keep producing lots of string loops
- ✓ Loops (= 0 net topological charge) disappear by emitting axions [lifetime $\sim H^{-1}$ at the production time]
- ✓ Energy density of the string network keeps being subdominant

$$\rho_{\rm str} \sim \mu_{\rm str}^2 H^2 \sim \frac{f_a^2}{M_{\rm Pl}^2} \times T^4 \ll \rho_{\rm tot} \sim T^4 \ ,$$

with the correlation length between strings, $\ell_{\rm corr} \sim 0.3 H^{-1}$



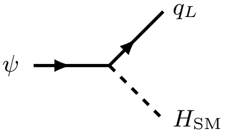
Fermion Zero Mode

- ✓ The colored fermions in the **PQ** mechanism become heavy ($\psi = yv$) via

$$\mathcal{L} = \mathcal{L}(\phi) + \phi + \bar{\psi}_L i \not{D} \psi_L + \bar{\psi}_R i \not{D} \psi_R + \boxed{y \phi \bar{\psi}_R \psi_L + h.c.}\ .$$

- ✓ The massive fermions decay into the SM quarks via,

$$\mathcal{O}_{\rm D} = \boxed{y_D H_{\rm SM} \bar{\psi}_R q_L} + h.c.,$$



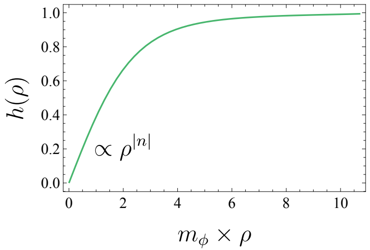
with the decay rate,

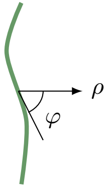
$$\Gamma_D = \frac{|y_D|^2}{16\pi} m_\psi$$

Fermion Zero Mode

✓ **PQ** symmetry is restored at the core of the string !

profile function $\phi(x) = vh(\rho)e^{in\varphi}$





✓ Fermions do not have masses at the string core.

The existence of fermionic zero modes propagating along the string at the speed of light!

['81 Weinberg, '81 Jakiew&Rossi]

Fermion Zero Mode

Dirac equation around cosmic string along the z-axis,

$$\left[i\gamma^\mu\partial_\mu - m_\psi h(\rho)\left(e^{in\varphi}P_L + e^{-in\varphi}P_R\right)\right]\psi = 0$$

The transverse configuration:

$$i\gamma^1\left(\partial_1 + i(i\gamma^1\gamma^2)\partial_2\right)\psi_L = m_\psi h(\rho)e^{-i\varphi}\psi_R\ ,$$
$$i\gamma^1\left(\partial_1 + i(i\gamma^1\gamma^2)\partial_2\right)\psi_R = m_\psi h(\rho)e^{i\varphi}\psi_L\ .$$

Noting $\partial_1 \pm i\partial_2 = e^{\pm i\varphi}(\partial_\rho \pm i\rho^{-1}\partial_\varphi)$, φ–independent solution :

$$\psi^0(x,y) = \mathcal{N}\eta\,\boxed{\exp\left(-\int_0^\rho m_\psi h(\rho')d\rho'\right)}\ ,\quad \eta = (0,1,i,0)^T$$

2D Chirality : $\gamma_5^{(xy)} = i\gamma^1\gamma^2,\quad \gamma_5^{(zt)} = \gamma^0\gamma^3$

$$\gamma_5^{(xy)}\psi_L^0 = -\psi_L^0\ ,\quad \gamma_5^{(xy)}\psi_R^0 = +\psi_R^0\ ,\quad \gamma_5^{(zt)}\psi^0 = +\psi^0$$

Zero mode is localized around the string with $\sim e^{-m_\psi\rho}$

Fermion Zero Mode

"Massless" propagation:

$$\psi^0(t,x,y,z) = \alpha(t,z) \times \psi^0(x,y)$$

The longitudinal part of Dirac equation:

$$(\gamma^0\partial_0 + \gamma^3\partial_3)\alpha(t,z)\eta = 0\ , \quad \gamma_5^{(zt)}\eta = +\eta\ , \quad \rightarrow \quad (\partial_0 + \partial_3)\alpha(t,z) = 0$$

✓ The zero-mode propagation is at the speed of light!

$$\alpha(t,z) = \alpha(t-z)$$

✓ Only right-movers exist [no $\alpha(t+z)$ mode]!

✓ Anti-particles are also right-movers:

$$\text{Antiparticle}: \psi^{0c} = i\gamma^2\psi^{0*} \rightarrow \gamma_5^{(zt)}\psi^{0c} = \psi^{0c}$$

✓ Left-movers appear along the string with $n = -1$

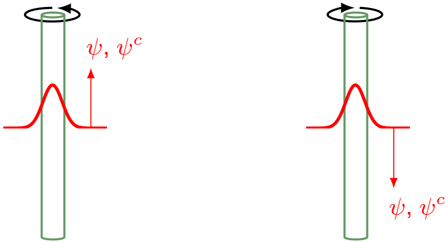
The fermion zero mode propagates at the speed of light but is one-way!

Fermion Zero Mode

The fermion zero mode propagates at the speed of light but is one-way!

String with $n = 1$

String with $n = -1$



Fermion Zero modes are localized around the string

Chiral Superconductivity

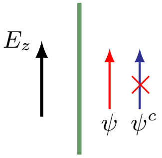
✓ **PQ** fermions are also charged under U(1)$_Y$ of the **SM** and hence **QED**

∵ **PQ** fermions couple to the **SM** quarks via,

$$\mathcal{O}_{\rm D} = y_D H_{\rm SM}\bar{\psi}_R q_L + h.c.,$$

Zero modes of **PQ** fermions on the string can carry **QED** and **QCD** current!

Apply an electric field E_z along the string in z-direction with $n = 1$



Particle is accelerated in $z > 0$ direction

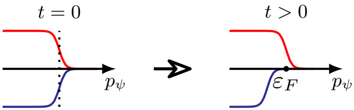
Antiparticle is accelerated in $z < 0$ direction

Only right-movers exist

= Only particles get accelerated !

Chiral Superconductivity

✓ Particles obtain Fermi momentum $\varepsilon_F = q_\psi E_Z t$ ['85, Witten]



✓ **QED** current on the string ['85, Witten]

$$J = \frac{1}{2\pi} N_c q_\psi \varepsilon_F = \frac{1}{2\pi} N_c q_\psi^2 E_z t$$

✓ **QED** current also has **QED** charge ['85, Callan&Harvey]

$$\frac{Q}{L_{\text{string}}} = J = \frac{1}{2\pi} N_c q_\psi^2 E_z t$$

✓ **QED** current/charge remain even after E_z turned off

A current keeps flowing along the string = superconductivity!

[Strings also obtain **QCD** current similarly]

Vorton : Stable Remnant?

In the early Universe, the electric field is 0 on average.

- ✓ Local/temporal electric field may induce superconductive current.
- ✓ Loop productions in that region/time produce **QED** /**QCD** charged loops!

What happens to the charged loops?

A string loop with **QED** charge Q with a length $L_{\rm loop}$:

$$Q = \frac{1}{2\pi} N_c q_\psi \varepsilon_F L_{\rm loop} \,, \qquad E_Q = \frac{1}{4\pi} N_c \varepsilon_F^2 L_{\rm loop} = \frac{\pi Q^2}{N_c q_\psi^2 L_{\rm loop}}$$

Total string energy :

$$E(L_{\rm loop}) \sim \mu_{\rm str} L_{\rm loop} + \frac{\pi Q^2}{N_c q_\psi^2 L_{\rm loop}}$$

Loop length is stabilized at

$$L_{\rm loop}^{(\rm vorton)} \sim \frac{Q}{q_\psi v} \gg \frac{1}{v} \to$$ Stable Loop $=$ Vorton ['88 Davis& Shellard]

Vorton : Stable Remnant?

Vortons may survive until today?

['21 H. Fukuda, A. V. Manohar, H. Murayama O. Telem]

- ✓ Charge leakage from the string through plasma scattering and string oscillation becomes irrelevant for

$$T < T_{\rm leak} \sim y_D^{-2} \times 10^3 \,\text{GeV for } f_a \sim 10^{10} \,\text{GeV}$$

- ✓ Loops formed at $T_{\rm leak}$ statistically obtain a charge

$$Q \sim \sqrt{L_{\rm loop}^{\rm (init)} T_{\rm leak}} \,, \quad L_{\rm loop}^{\rm (init)} \sim H^{-1}(T_{\rm leak})$$

$$\to Q \sim 10^3 \times (10^8 \,\text{GeV}/T_{\rm leak})^{1/2}$$

- ✓ $L_{\rm loop}$ shrinks down to $L_{\rm loop}^{\rm (vorton)}$ by emitting axions
- ✓ Stable vorton looks like a heavy atom, $M_V \sim Q f_a$.

Vorton may be discoverable?

Stability of Zero Mode

In the vacuum, **PQ** fermions decay via $\psi \to q_L + H_{SM}$...

✓ Why do zero modes not decay ? $\because$ Zero modes are "massless"

The string loses $(E, p_z) = (E, E)$ if a zero mode decays

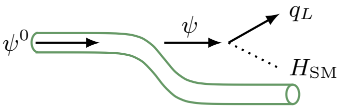
$\to$ invariant mass of the final state : $s = 0$. No such a final state!

✓ The stability of the zero mode is valid only for the straight string.

['21 MI, S. Kobayashi, Y. Nakayama, S. Shirai]

Zero modes can hit the inner wall of the string and pop out!

Escape & Decay



$p_\psi^\perp \gg m_\psi$

Stability of Zero Mode

✓ Maximum momentum in Vorton = Fermi-Momentum

$$\varepsilon_F \sim Q/L_{\rm loop}^{\rm (vorton)} \sim v$$

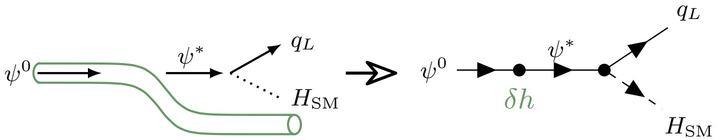
✓ Curvature radius

$$L_{\rm loop}^{\rm (vorton)} \sim Q/v \gg v^{-1} \to p_\psi^\perp \sim \frac{\varepsilon_F}{m_\phi L_{\rm loop}^{\rm (vorton)}} \ll m_\psi$$

✓ Tunneling & Decay is relevant !

Curve on the String
→ Perturbation on the profile function

$$h(\rho) \to h(\rho) + \delta h(\rho, \varphi, z)$$



Stability of Zero Mode

✓ Annihilation operators :

$$\hat{b}^0(E) \text{ zero mode }, \quad \hat{a}_q \text{ SM quark }, \quad \hat{a}_H \text{ SM Higgs}$$

✓ Ground State with $n = 1$ string : $|0\rangle$

✓ Decay Amplitude :

$$\hat{T} = \langle 0|\hat{a}_q \hat{a}_H T e^{i\int d^4x[\mathcal{O}_M + \mathcal{O}_D]} \hat{b}^0(E)^\dagger |0\rangle$$
$$\mathcal{O}_M = m_\psi \delta h(\rho, \varphi, z)\bar{\psi}(e^{i\varphi}P_L + e^{-i\varphi}P_R)\psi$$
$$\mathcal{O}_{\mathrm{D}} = y_D H_{\mathrm{SM}} \bar{\psi}_R q_L + h.c.,$$

✓ Born Approximation, we keep only $\mathcal{O}(\delta h)$-term ($|\delta h|^2 \ll |\delta h|$).

Stability of Zero Mode

Decay rate of a zero mode with energy E.

✓ For a long string with perturbative modulation,

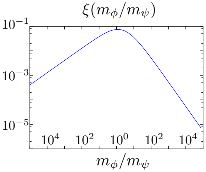
$$\Gamma_{\text{pert}}(E) \sim \mathcal{O}\left(10^{-1}\right) \times \mathcal{C}_{\text{pert}} \boxed{\frac{|y_D|^2 E}{m_\phi R}} \times \xi^2(m_\phi/m_\psi)$$

R : curvature radius of string

$\mathcal{C}_{\text{pert}}$: a factor associated with the shape of perturbation

[$C_{\text{pert}} \to \mathcal{O}(1)$ for mildly perturbative curve]

$\xi(m_\phi/m_\psi) =$ overlap between zero-mode wave function and the potential wall, $dh(\rho)/d\rho$



$\xi(m_\phi/m_\psi)$ modulation independent

Stability of Zero Mode

✓ For a closed loop like a circle, although no more perturbative, we can put lower limit,

$$\Gamma_{\rm loop}(E) > \Gamma_{\rm pert}(E)|_{\mathcal{C}_{\rm pert}=\mathcal{O}(1)}$$

Zero mode decay $\rightarrow$ Vorton charge and size decrease
$\rightarrow \varepsilon_F$ is always of $\mathcal{O}(v)$

✓ Charge Leakage Rate

$$-\frac{\dot{Q}}{Q} > \Gamma_{\rm loop}(\epsilon_F) \sim \frac{\Gamma_D \varepsilon_F}{m_\phi m_\psi}\frac{v}{Q}\xi^2 > \frac{\Gamma_D \varepsilon_F}{m_\phi m_\psi}\frac{v}{Q_{\rm init}}\xi^2$$

$\rightarrow$ Vorton does not survive cosmological timescale

$$-\frac{\dot{Q}}{Q} > 10^{-3} \times \left(\frac{T_{\rm leak}}{10^8\,\text{GeV}}\right)^{1/2}\left(\frac{\varepsilon_F v}{m_\phi m_\psi}\right)\xi^2 \times \Gamma_D$$

Summary

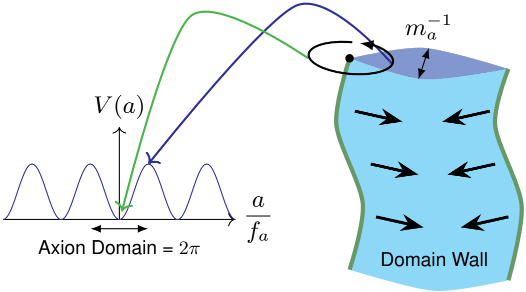
- ✓ Cosmic string appearing in the PQ mechanism have fermionic zero modes
- ✓ Propagation direction of zero modes is one-way.
- ✓ On the string, chiral superconducting current can flow
- ✓ The chiral superconducting current has non-vanishing **QED** and **QCD** charges
- ✓ Non-vanishing **QED** and **QCD** charges may stabilize the string loop forming Vorton
- ✓ By taking into account the finite width and curvature of the string, the zero mode leaks from the string
  $\rightarrow$ Vorton do not survive in a cosmological timescale.
- ✓ Cosmological evolution of chiral superconducting strings, including charge leakage effects, is still open to study

Backup Slides

Cosmological Evolution of Axionic Cosmic String

Below $T \sim \mathcal{O}(1)$ GeV, the axion potential is generated leading to energy contrast around cosmic strings.

✓ Strings are attached by domain walls!



✓ For a model with $N_{\rm DW} = 1$, the strings are pulled with each other by the domain wall tension and shredded into pieces.

The string-domain wall network disappear immediately.

Perturbative Modulation

String along z-direction. Modulation in the y-direction

$$(x, y, z) = (0, f(z), z)$$

The modulation of the profile function,

$$h(\rho) \to h(\rho) + \delta h(\rho, \varphi, z) \ , \quad \delta h(\rho, \varphi, z) = \frac{y}{\rho}\frac{dh(\rho)}{d\rho} \times f(z)$$

Perturbative modulation,

$$|f| \sim \frac{\varepsilon}{m_\phi} \ , \quad \varepsilon \ll 1.$$

Fermion Quantization around String

✓ Mode expansion of the fermion

$$\hat{\psi} = \frac{1}{\sqrt{L_{\rm str}}} \sum_{n>0} \left(e^{-iE_n(t-z)} u(\rho)\, \hat{b}^0_n + e^{iE_n(t-z)} v(\rho)\, \hat{d}^{0\,\dagger}_n \right)$$

$$+ \sum \text{(bounded massive modes)} + \sum \text{(unbounded modes)}$$

$E_n = 2\pi n / L_{\rm str}$.

✓ Normalization :

$$u(\rho) = \mathcal{N} \eta \exp\left(-\int_0^\rho m_\psi h(\rho') d\rho' \right) , \quad v(\rho) = i\gamma_2 u(\rho)^*$$

$$\int dx dy |u(\rho)|^2 = 1$$

✓ Quantization :

$$\{\hat{b}^0_n, \hat{b}^{0\dagger}_{n'}\} = \delta_{nn'} , \quad \{\hat{d}^0_n, \hat{d}^{0\dagger}_{n'}\} = \delta_{nn'} .$$

Modulation of long string

✓ Perturbative modulation on a long string:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n e^{-i\frac{2\pi n}{L} z} ,$$

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} dz e^{i\frac{2\pi n}{L} z} f(z) ,$$

with $c_{-n} = c_n^*$.

✓ Master Formula of the decay rate :

$$\Gamma(E) \simeq \frac{1}{24} |y_D|^2 \xi^2 \left(\frac{m_\phi}{m_\psi} \right) E^3 \sum_{n>0}^{k_n<2E} |c_n|^2 \mathcal{F}(k_n/E)$$

$$\mathcal{F}(x) = x^2 (1 - x/2)$$

Sinuous Modulation

- ✓ Sinuous modulation :

$$f(z) = \frac{\varepsilon}{m_\phi} \sin\left(\frac{2\pi z}{L}\right) , \quad (\varepsilon \ll 1)$$

- ✓ Fourier coefficients :

$$c_1 = i\frac{\varepsilon}{2m_\phi} , \quad c_n = 0 , \quad (n > 1)$$

- ✓ Decay rate :

$$\Gamma(E) \simeq \frac{\pi^2}{24}\frac{\varepsilon^2 |y_D|^2 E}{m_\phi^2 L^2}\left(1 - \frac{\pi}{EL}\right)\xi^2\left(\frac{m_\phi}{m_\psi}\right) .$$

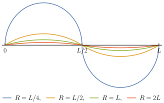
- ✓ Curvature radius :

$$R = \frac{L^2}{(2\pi)^2 \varepsilon m_\phi} .$$

- ✓ Rewritten decay rate :

$$\Gamma(E) \simeq \frac{1}{96}\varepsilon \boxed{\frac{|y_D|^2 E}{m_\phi R}} \xi^2 \left(\frac{m_\phi}{m_\psi}\right) .$$

Piecewise Circle Modulation



✓ Piecewise Circle Modulation :

$$f(z) = \begin{cases} \sqrt{R^2 - (z - L/4)^2} - \sqrt{R^2 - L^2/16}\,, & (0 < z < L/2)\,, \\ -\sqrt{R^2 - (z - 3L/4)^2} + \sqrt{R^2 - L^2/16}\,, & (L/2 < z < L)\,, \end{cases}$$

✓ Order of modulation :

$$f(z = L/4) \simeq \left(\frac{L^2}{32R}\right) \rightarrow \varepsilon = \frac{L^2 m_\phi}{32R}$$

✓ Fourier coefficients :

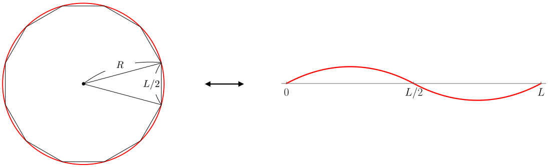
$$c_{2n+1} = \frac{iL^2}{2\pi^3(2n+1)^3 R}, \quad c_{2n} = 0\,,$$

✓ Decay rate :

$$\Gamma(E) \simeq \frac{1}{72}\varepsilon \boxed{\frac{|y_D|^2 E}{m_\phi R}} \xi^2 \left(\frac{m_\phi}{m_\psi}\right) .$$

Zero Mode Decay on Circle

Decay rate on circle $>$ decay rate on piecewise circle modulation.



The smaller L is taken, the better perturbative expansion.

The smaller L is taken, more the rate is underestimated.

$\rightarrow$ We put a lower limit on the zero mode decay rate using L which is just barely within the range of the perturbation expansion :

$$L = \sqrt{\frac{R}{m_\phi}} \leftrightarrow \varepsilon \sim 1 \,.$$