Minkowski functionals for isotropic random fields in the Euclidean space and the sphere

Kuriki, Satoshi
Inst. Statist. Math. (ISM) and SOKENDAI

Statistical Analysis of Random Fields in Cosmology
KEK-Cosmo 2024, Mon 4 March 2024

Joint work with T. Matsubara and C. Hikage

Contents of talk

I. MFs for isotropic random fields: The Euclidean space case

Matsubara & K (2021), Matsubara, Hikage & K (2022),

K and Matsubara (2023)

II. MFs for isotropic random fields: The sphere case

I. MFs for isotropic random fields: The Euclidean space case

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II. MFs for isotropic random fields: The sphere case

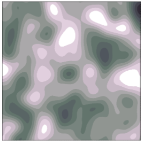
Excursion set for a smooth isotropic random field

- Isotropic random field $X(t)$, $t \in T \subset \mathbb{R}^n$:
  $\forall T'$(finite set) $\subset T$,

$$\{X(t)\}_{t \in T'} \overset{d}{=} \{X(Pt+b)\}_{t \in T'}, \ \forall (P,b) \in O(n) \times \mathbb{R}^n$$

- Excursion set is the sup-level set of a function $X(t)$:

$$T_v = \{t \in T \mid X(t) \geq v\} = X^{-1}([v,\infty))$$

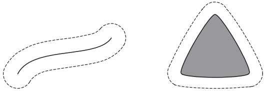


Left: Isotropic random field, Right: Its excursion set

Minkowski functional (MF) and Lipschitz-Killing curvature

- Let $M \subset \mathbb{R}^n$ be a closed set. Tube about M with radius ρ:

$$\mathrm{Tube}(M,\rho) = \{x \in \mathbb{R}^n \mid \mathrm{dist}(x, M) \le \rho\}$$



- Definitions of Minkowski functional $\mathcal{V}_j(\cdot)$ and Lipschitz-Killing curvature $\mathcal{L}_j(\cdot)$:
  For small $\rho > 0$, and $\omega_d = \pi^{d/2}/\Gamma(d/2+1)$,

$$\mathrm{Vol}_n(\mathrm{Tube}(M,\rho)) = \sum_{j=0}^{n} \omega_{n-j}\rho^{n-j}\mathcal{L}_j(M) = \sum_{j=0}^{n} \rho^j \binom{n}{j} \mathcal{V}_j(M)$$

  ($\mathcal{L}_j(\cdot)$ is defined independently of the ambient space)

When the dimension n is 2

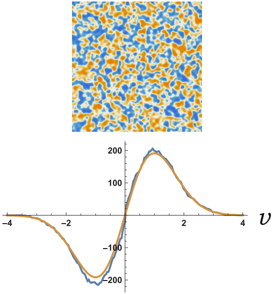
- When $n = 2$,

$$\mathcal{L}_2(M) = \mathcal{V}_0(M) = \int_M \mathrm{d}x = \mathrm{Area}(M)$$
$$\mathcal{L}_1(M) = \mathcal{V}_1(M) = \frac{1}{2}\int_{\partial M} \mathrm{d}x = \frac{1}{2}\,\mathrm{Length}(\partial M)$$
$$\mathcal{L}_0(M) = \frac{1}{\pi}\mathcal{V}_2(M) = \chi(M) \quad \text{(Euler characteristics)}$$

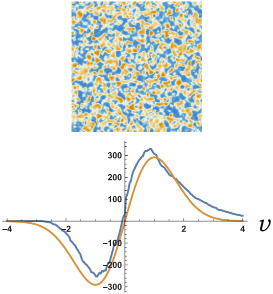
- For general n and i, $\mathcal{L}_i(M)$ is represented as an integral of the curvature measure over ∂M.

MF of the excursion set T_v as a goodness-of-fit statistic

- The Minkowski functional $\mathcal{V}_j(T_v)$ of the excursion set T_v can be used as a statistic for testing goodness-of-fit
(e.g., applications in cosmology)



Gaussian



non-Gaussian

— observed Euler characteristic $\chi(T_v)$

— $\mathbb{E}[\chi(T_v)]$ under the assumption of Gaussianity

- Our purpose: To obtain the formulas $\mathbb{E}[\chi(T_v)]$ (and $\mathbb{E}[\mathcal{L}_k(T_v)]$) under non-Gaussian distributions

Kinematic Formula for isotropic random fields

Fact (Kinematic Formula in $\mathbb{R}^n$)

For $A, B \subset \mathbb{R}^n$ and for $gB = \{Px + b \,|\, x \in B\}$ (g: rigid motion),

$$\int \chi(A \cap gB)\mathrm{d}g \propto \sum_{d=0}^{n} \mathcal{L}_d(A)\, \mathcal{L}_{n-d}(B).$$

- Let $A := T$, $gB := \{t \in \mathbb{R}^n \,|\, X(t) \geq x\}$, then $A \cap gB = T_x$.

Proposition (Kinematic Formula for isotropic random fields)

When $X(t)$, $t \in T$, is a smooth isotropic random field,

$$\mathbb{E}[\chi(T_x)] = \mathbb{E}[\mathcal{L}_0(T_x)] = \sum_{d=0}^{n} \mathcal{L}_d(T)\, \Xi_d(x),$$

where $\Xi_d(x)$ is the Euler characteristic density.

$\mathbb{E}[\mathcal{L}_k(T_x)]$ for $k \geq 1$

- Expected MF except for $k \geq 1$:

$$\mathbb{E}[\mathcal{L}_k(T_x)] = \sum_{d=0}^{n-k} \frac{\Gamma(\frac{k+d+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{d+1}{2})} \mathcal{L}_{k+d}(T)\, \Xi_d(x), \quad k = 1, \ldots, n$$

- We only need to know the Euler characteristic density $\Xi_d(x)$.

EC density for the Gaussian random field

Proposition (Tomita, 1986, *PTP*, and many)

Suppose that $\mathrm{Var}(\nabla X(t)) = \gamma I$*. When* $X(t)$ *is Gaussian,*

$$\Xi_d(x) = (\gamma/2\pi)^{d/2}\phi(x)H_{d-1}(x)$$

where $\phi(x)$*: pdf of* $\mathcal{N}(0,1)$*,* $H_j(x)$*: Hermite polynomial*

- Therefore,

$$\mathbb{E}[\chi(T_x)] = \sum_{d=0}^{n} \mathcal{L}_d(T)\,(\gamma/2\pi)^{d/2}\phi(x)H_{d-1}(x)$$

- When $X(t)$ is not Gaussian?

Weakly non-Gaussian case

- The non-Gaussianity is characterized by k-point correlation functions (k-th cumulant). In the applications to cosmology,

$$\mathrm{cum}(X(t_1),\ldots,X(t_k)) = O(\nu^{k-2}), \quad \nu \ll 1$$

- e.g., Central limit random fields by Chamandy, et al. (2008):

$$X(t) = \tfrac{1}{\sqrt{N}} \textstyle\sum_{i=1}^N Z_{(i)}(t), \quad \nu = 1/\sqrt{N}.$$

  $Z_{(i)}$: zero mean i.i.d. isotropic non-Gaussian random field.
- The k-point correlation function for the isotropic random field:

$$\begin{aligned} &\mathrm{cum}(X(t_1),\ldots,X(t_k)) \\ &= \nu^{k-2}\kappa^{(k)}(\tfrac{1}{2}\|t_1-t_2\|^2, \tfrac{1}{2}\|t_1-t_3\|^2, \ldots, \tfrac{1}{2}\|t_{k-1}-t_k\|^2) \end{aligned}$$

  (A function of pairwise distances between $t_1,\ldots,t_k$)

Asymptotic expansion of the Euler density I

Theorem

Under regularity conditions, as $\nu \to 0$ uniformly in x,

$$\Xi_{d,\nu}(x) = \left(\frac{\gamma}{2\pi}\right)^{d/2} \phi(x) \Big(H_{d-1}(x) + \nu \Delta_{1,d}(x) + \nu^2 \Delta_{2,d}(x) \Big) + o(\nu^{-2})$$

where

$$\Delta_{1,d}(x) = \tfrac{1}{2\gamma^2} \kappa_{11} (d)_2 H_{d-2}(x) - \tfrac{1}{2\gamma} \kappa_1 d H_d(x) + \tfrac{1}{6} \kappa_0 H_{d+2}(x)$$

$$\kappa_0 = \kappa^{(3)}(0,0,0),\ \kappa_1 = \tfrac{\mathrm{d}\kappa^{(3)}(x,0,0)}{\mathrm{d}x}\big|_{x=0},\ \kappa_{11} = \tfrac{\mathrm{d}^2 \kappa^{(3)}(x,y,0)}{\mathrm{d}x\mathrm{d}y}\big|_{x=y=0}$$

Asymptotic expansion of the Euler density II

and

$$\begin{aligned}\Delta_{2,d}(x) =&\Big(-\tfrac{1}{6\gamma^3}(3\widetilde{\kappa}^a_{111}+\widetilde{\kappa}^d_{111})+\tfrac{1}{8\gamma^4}(d-7)\kappa^2_{11}\Big)(d)_3 H_{d-3}(x)\\ &+\Big(\tfrac{1}{8\gamma^2}(\widetilde{\kappa}^{aa}_{11}(d-2)+4\widetilde{\kappa}^a_{11}(d-1)\\ &-\tfrac{1}{4\gamma^3}\kappa_1\kappa_{11}(d-1)(d-4)\Big)dH_{d-1}(x)\\ &+\Big(-\tfrac{1}{4\gamma}\widetilde{\kappa}_1+\tfrac{1}{24\gamma^2}(3\kappa_1^2(d-2)+2\kappa_0\kappa_{11}(d-1))\Big)dH_{d+1}(x)\\ &+\Big(\tfrac{1}{24}\widetilde{\kappa}_0-\tfrac{1}{12\gamma}\kappa_0\kappa_1 d\Big)H_{d+3}(x)+\tfrac{1}{72}\kappa_0^2 H_{d+5}(x)\end{aligned}$$

$\widetilde{\kappa}_0=\kappa^{(4)}(0,0,0,0,0,0)$, $\widetilde{\kappa}_1=\frac{\mathrm{d}\kappa^{(4)}(x_1,0,0,0,0,0)}{\mathrm{d}x_1}|_{x_1=0}$, $\ldots$

Sketch of the proof

- Kac-Rice formula (Morse theory) for the EC density:

$$\Xi_d(v) = \mathbb{E}[\chi(T_v)] = \mathbb{E}[\mathbb{1}(X(t) \geq v)\det(-\nabla^2 X(t))\delta(\nabla X(t))] \qquad (*)$$

  $\Xi_d(v)$ is independent of t (because X is isotropic).

- Obtain the Edgeworth expansion of the pdf (or mgf) of

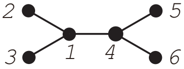
$$(X(t), \nabla X(t), \nabla^2 X(t)) \in \mathbb{R}^{1+d+d(d+1)/2}$$

  and evaluate $\Xi_d(v)$ by $(*)$.

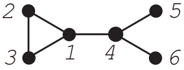
(Non)Detectable non-Gaussianity

- Diagram of the derivatives:

$$\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{13}}\frac{\partial}{\partial x_{14}}\frac{\partial}{\partial x_{45}}\frac{\partial}{\partial x_{46}}\kappa^{(6)}(x_{12},\ldots,x_{56})\Big|_{x=0}$$



$$\frac{\partial}{\partial x_{12}}\frac{\partial}{\partial x_{13}}\frac{\partial}{\partial x_{14}}\frac{\partial}{\partial x_{23}}\frac{\partial}{\partial x_{45}}\frac{\partial}{\partial x_{46}}\kappa^{(6)}(x_{12},\ldots,x_{56})\Big|_{x=0}$$

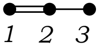


Theorem

The derivatives with loops do not appear in the formula for $\Xi_d(v)$.

- In particular, the second derivatives don't appear

$$\left(\frac{\partial}{\partial x_{12}}\right)^2\frac{\partial}{\partial x_{23}}\kappa^{(3)}(x_{12},x_{13},x_{23})\Big|_{x=0}$$



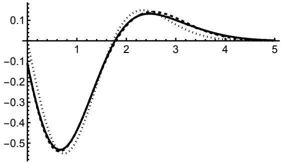
Chi-square random field

- A weakly non-Gaussian random field when d.f. is large:

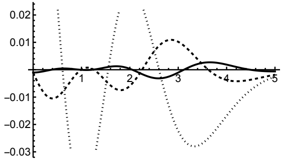
$$Y(t) = Y_N(t) = \frac{1}{\sqrt{2N}} \sum_{i=1}^{N} \bigl(Z_{(i)}(t)^2 - 1\bigr), \quad t \in T \subset \mathbb{R}^4,$$

$Z_{(i)}(t)$: zero mean Gaussian s.t. $\mathbb{E}[Z_{(i)}(s) Z_{(i)}(t)] = e^{-\frac{1}{4}\|s-t\|^2}$, $i = 1, 2, \ldots$ i.i.d.

- The EC density for $Y(t)$ is known:

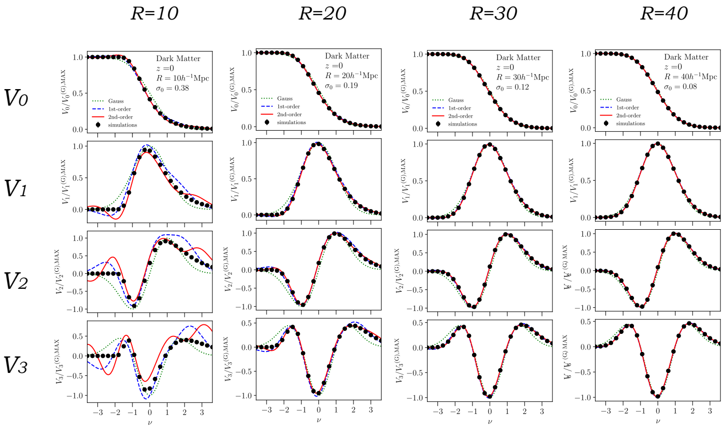


EC density, $N = 100$ (−·− true, $\cdots$ 0th approx, − −1st approx, —2nd approx)



Difference from the true, $N = 100$ ($\cdots$0th approx, − −1st approx, —2nd approx)

Comparison with simulator (Matsubara, Hikage & K, 2022)



Simulator (dot) and expansion formulas (solid line)

R: Radius of smoothing kernel (h^{-1}Mpc)

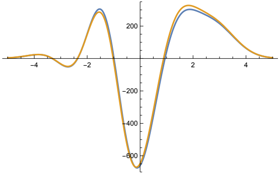
periodical boundary condition ($\partial T = \emptyset$)

Boundary correction

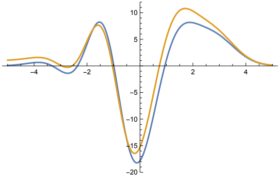
- When the dimension is $n = 3$, and $\partial T \neq \emptyset$,

$$\mathbb{E}[\chi(T_v)] = \mathrm{Vol}(T)\Xi_3(v) + \underbrace{\mathcal{L}_2(T)\Xi_2(v) + \mathcal{L}_1(T)\Xi_1(v) + \chi(T)\Xi_0(v)}_{\text{boundary correction (contribution of } \partial T\text{)}}$$

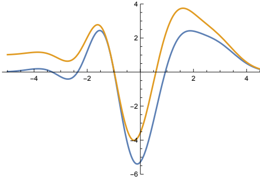
- We use the same parameters for $R = 20$. T is a cubic.



$T = (1000\text{Mpc})^3$



$T = (300\text{Mpc})^3$



$T = (200\text{Mpc})^3$

Orange: $\mathbb{E}[\chi(T_x)]$ with boundary corrections

Blue: without boundary corrections

I. MFs for isotropic random fields: The Euclidean space case

Matsubara & K (2021), Matsubara, Hikage & K (2022),
K and Matsubara (2023)

II. MFs for isotropic random fields: The sphere case

Orthogonally invariant random field on the sphere

- In the second half of the talk, we deal with a random field $X(\cdot)$ on the n-dim sphere $\mathbb{S}_R^n$ with radius R.
- Instead of the isotropy, assume that $X(t)$ is orthogonally invariant: $\forall T'$(finite set) $\subset T$,

$$\big\{X(t)\big\}_{t\in T'} \stackrel{d}{=} \big\{X(Pt)\big\}_{t\in T'},\ \forall P \in O(n)$$

- Excursion set is the sup-level set of a function $X(t)$:

$$T_v = \{t \in T \mid X(t) \geq v\} = X^{-1}([v,\infty))$$

- We consider the MFs of T_v. How does R affect?

Covariance and k-point correlation functions

- Distance: For $s, t \in \mathbb{S}^n_R$,

$$\mathrm{dist}_R(s,t) = \text{the great circle distance between } s \text{ and } t$$

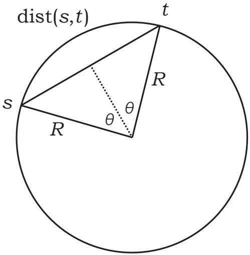
- Covariance:

$$\begin{aligned}\mathrm{Cov}(X(s), X(t)) &= \text{A function of } \mathrm{dist}_R(s,t) \\ &= \rho\big(\tfrac{1}{2}\mathrm{dist}_R(s,t)^2\big), \quad \text{say,}\end{aligned}$$

- 3-point correlation function:

$$\begin{aligned}&\mathrm{cum}(X(s), X(t), X(u)) \\ &= \kappa^{(3)}\big(\tfrac{1}{2}\mathrm{dist}_R(s,t)^2, \tfrac{1}{2}\mathrm{dist}_R(s,u)^2, \tfrac{1}{2}\mathrm{dist}_R(t,u)^2\big)\end{aligned}$$

Great circle distance



$$\tfrac{1}{2}\|s-t\| = R\sin\theta = R\sin(\tfrac{1}{2R}\mathrm{dist}(s,t))$$

or

$$\tfrac{1}{2}\mathrm{dist}(s,t)^2 = x + \tfrac{1}{6R^2}x^2 + \cdots$$

where

$$x = \tfrac{1}{2}\|s-t\|^2$$

Extended random field $\widetilde{X}(\cdot)$

- Construct an isotropic random field $\widetilde{X}(\cdot)$ on $\mathbb{R}^{n+1}$ such that its restriction on $\mathbb{S}_R^n$ satisfies

$$(\widetilde{X}|_{\mathbb{S}_R^n}, \nabla\widetilde{X}|_{\mathbb{S}_R^n}, \nabla^2\widetilde{X}|_{\mathbb{S}_R^n})(t) \stackrel{d}{=} (X, \nabla X, \nabla^2 X)(t), \quad t \in \mathbb{S}_R^n$$

  where ∇ is the gradient on the tangent space $T_t\mathbb{S}_R^n$.
- Noting that

$$\mathrm{cum}(X(t_1), X(t_2), \ldots) = \kappa^{(k)}\left(\tfrac{1}{2}\mathrm{dist}_R(t_1,t_2)^2, \ldots\right), \quad t_i \in \mathbb{S}_R^n$$

  we define

$$\mathrm{cum}\left(\widetilde{X}(t_1), \widetilde{X}(t_2), \ldots\right) = \kappa^{(k)}\left(x_{12} + \tfrac{1}{6R^2}x_{12}^2 + \cdots, \ldots\right)$$

  where

$$x_{ij} = \tfrac{1}{2}\|t_i - t_j\|^2, \quad t_i \in \mathbb{R}^{n+1}$$

k-point correlations and there derivatives

- We apply the theorem for the Euclidean space to $\widetilde{X}$ with the index set $T \subset \mathbb{S}_R^n \subset \mathbb{R}^{n+1}$.
- We evaluate the parameters γ, κ_0, $\kappa_1, \ldots$, $\widetilde{\kappa}_0, \ldots$ that construct $\Xi_{d,\nu}(x)$ as

$$
\begin{aligned}
\underset{\text{sphere}}{\gamma} &= \frac{\mathrm{d}}{\mathrm{d}x}\rho(x + \tfrac{1}{6R^2}x^2 + \cdots)\Big|_{x=0} = \rho'(0) = \underset{\text{Euclid}}{\gamma} \\
\underset{\text{sphere}}{\kappa_1} &= \frac{\partial}{\partial x_{12}}\kappa^{(3)}(x_{12} + \tfrac{1}{6R^2}x_{12}^2 + \cdots, 0, 0)\Big|_{x=0} \\
&= \frac{\partial}{\partial x_{12}}\kappa^{(3)}(x_{12}, 0, 0)\Big|_{x=0} = \underset{\text{Euclid}}{\kappa_1}
\end{aligned}
$$

etc.

- Because the second derivatives do not appear, all of the parameters are equivalent for the Euclidean space case and the sphere case.

Expected MFs: the spherical case

Theorem

The expected MFs for the weakly non-Gaussian orthogonally invariant random field $X(t)$, $t \in T$, on the sphere $\mathbb{S}^n_R$ are

$$\mathbb{E}[\chi(T_x)] = \mathbb{E}[\mathcal{L}_0(T_x)] = \sum_{d=0}^{n} \mathcal{L}_d(T)\, \Xi_{d,\nu}(x),$$

$$\mathbb{E}[\mathcal{L}_k(T_x)] = \sum_{d=0}^{n-k} \frac{\Gamma(\frac{k+d+1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{k+1}{2})\Gamma(\frac{d+1}{2})} \mathcal{L}_{k+d}(T)\, \Xi_{d,\nu}(x), \quad k = 1, \ldots, n$$

The expression for the EC density $\Xi_{d,\nu}(x)$ is the same as in the Euclidean space case.

Remark

These formulas look exactly the same as in the Euclidean case. However, the T is a subset of the sphere and $\mathcal{L}_d(T)$ depends on R, and the definitions of ρ and $\kappa^{(k)}$ constructing $\Xi_{d,\nu}$ are different.

Corollary: $X(\cdot)$ is Gaussian and $T = \mathbb{S}_R^n$ (the whole sphere)

- Recall that

$$\mathcal{L}_j(\mathbb{S}_R^n) = \begin{cases} 2R^j \binom{n}{j} \frac{\Omega_{n+1}}{\Omega_{n-j+1}} & (n-j : \text{even}), \\ 0 & (\text{otherwise}), \end{cases} \qquad \Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

Corollary

When $X(\cdot)$ is Gaussian and $T = \mathbb{S}_R^n$ (the whole sphere),

$$\mathbb{E}[\chi(T_x)] = \frac{(\gamma R^2)^{n/2}\Omega_{n+1}}{(2\pi)^{n/2}} \phi(x) H_n(x; (\gamma R^2)^{-1}),$$

where $$H_n(x;\delta) = \sum_{k=0}^{[n/2]} \delta^{k/2} \frac{n!}{2^k k!(n-2k)!} H_{n-2k-1}(x)$$

(Cheng and Xiao (2016); for $n = 2$, Schmalzing and Górski (1998))

An implication: When the random field is observed on the celestial sphere $\mathbb{S}_1^2$

- Suppose that $n = 2$ and the random field on the sphere $\mathbb{S}_R^2$ is observed on the celestial sphere $\mathbb{S}_1^2$:

$$\overline{X}(t) = X(Rt), \quad t \in \overline{T} = R^{-1}T \subset \mathbb{S}_1^n \ \text{(unit sphere)}$$

- Then, $\mathcal{L}_j(T) = R^j \mathcal{L}_j(\overline{T})$, $\gamma = \mathrm{Var}(\partial X(t)/\partial t^1) = \overline{\gamma} R^{-1}$ with $\overline{\gamma} = \mathrm{Var}(\partial \overline{X}(t)/\partial t^1)$,

$$\mathbb{E}[\chi(\overline{T}_x)] = \mathbb{E}[\chi(T_x)] = \sum_{j=0}^{n} (\gamma/2\pi)^{j/2} H_{j-1}(x) \mathcal{L}_j(T)$$
$$= \sum_{j=0}^{n} (\overline{\gamma}/2\pi)^{j/2} H_{j-1}(x) \mathcal{L}_j(\overline{T})$$

No information on R can be extracted from the MFs

- This is the case even when $X(\cdot)$ is weakly non-Gaussian.

Summary

What we have done:

- Expected MFs formulas are provided when the random field $X(\cdot)$ is isotropic (or, orthogonally invariant) and weakly non-Gaussian in the Euclidean space (or, on the sphere, respectively).

Take home messages:

1. The boundary effect of the MF is often substantial and cannot be ignored. The boundary correction is easy and should be incorporated always.
2. The expected MF for the sphere (=space of constant curvature) is almost same as the Euclidean space case.
3. The MFs of the celestial sphere data do not contain the information on the curvature R^{-1}.

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Assumption

(i) $t \mapsto X_N(t)$ *is of* C^2 *a.s.*

(ii) *Recall that* $t = (t^j)_{1 \leq j \leq n}$. *There exists*

$$\frac{\partial^8 \mathbb{E}\big[X_N(t_1)X_N(t_2)X_N(t_3)X_N(t_4)\big]}{\partial t_1^{i_1} \partial t_1^{j_1} \cdots \partial t_4^{i_4} \partial t_4^{j_4}} \quad \text{around } t_1 = t_2 = t_3 = t_4$$

(iii) *For* t *fixed,* $(X_N(t), \nabla X_N(t), \nabla^2 X_N(t))$ *has a density* p_N. p_N *is bounded for some* N. *It has a moment of order* $\binom{n+2}{2} + 1$.

Key identities on the Hermite polynomial

- $A \sim \mathrm{GOE}(n)$, that is, $A = (a_{ij}) \in \mathrm{Sym}(n)$, $a_{ii} \sim \mathcal{N}(0,2)$, $a_{ij} \sim \mathcal{N}(0,1)$ $(i < j)$. Then, $\mathbb{E}[e^{\mathrm{tr}(\Theta A)}] = e^{\mathrm{tr}(\Theta^2)}$
- For $B = (b_{ij}) \in \mathrm{Sym}(n)$, define a matrix differential operator

$$(D_B)_{ij} = (1/2)(1+\delta_{ij})(\partial/\partial b_{ij}) \quad (i \le j)$$

Lemma

Let $A \sim \mathrm{GOE}(n)$. *Let* $m = \sum_{i=1}^{\ell} c_i$. *Then,*

$$\begin{aligned}(-1/2)^{m-\ell}(n)_m H_{n-m}(x) &= \mathbb{E}\big[\mathrm{tr}(D_A^{c_1})\cdots\mathrm{tr}(D_A^{c_\ell})\det(xI_n + A)\big] \\ &= \det(xI + D_\Theta)\big(e^{\mathrm{tr}(\Theta^2)}\mathrm{tr}(\Theta^{c_1})\cdots\mathrm{tr}(\Theta^{c_\ell})\big)\Big|_{\Theta=0}\end{aligned}$$

In particular, when $\ell = m = 0$,

$$H_n(x) = \mathbb{E}[\det(xI_n + A)] = \det(xI_n + D_\Theta)e^{\mathrm{tr}(\Theta^2)}\Big|_{\Theta=0}$$