Dispersive analysis of the isospin-breaking corrections to $e^+e^- \to \pi^+\pi^-$ and $\pi^+\pi^- \to \pi^+\pi^-$

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Seventh Plenary Workshop of the Muon g-2 Theory Initiative

September 10th 2024, KEK, Tsukuba, Japan

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Introduction

Interference: RC to the forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$

Isospin-breaking corrections for $\pi\pi$ scattering

Dispersive approach to FSR in $e^+e^- \to \pi^+\pi^-$

Summary / Outlook

Work in collaboration with
Gilberto Colangelo, Martina Cottini, Martin Hoferichter and Joachim Monnard

Outline

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RC to HVP contribution to $(g-2)_\mu$

Contribution	Value $\times 10^{-11}$
QED	116 584 718.931(104)
Electroweak	153.6(1.0)
HVP (e^+e^-, LO + NLO + NNLO)	6845(40)
HLbL (phenomenology + lattice + NLO)	92(18)
Total SM Value	116 591 810(43)
Experiment	116 592 059(22)
Difference: $\Delta a_\mu \equiv a_\mu^{\text{exp}} - a_\mu^{\text{SM}}$	249(48)

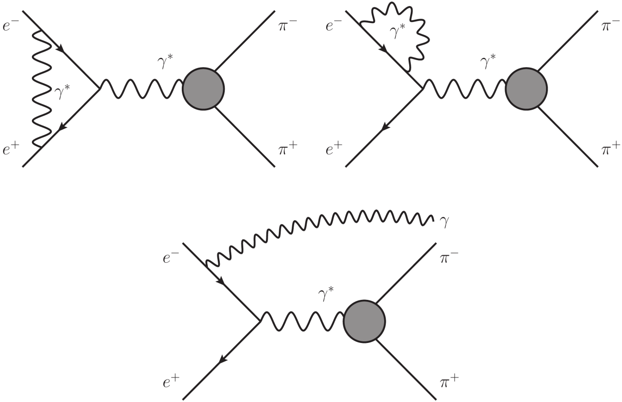
- HVP dominant source of theory uncertainty

  $\Delta a_\mu^{\text{HVP}} / a_\mu^{\text{HVP}} \sim 0.6\%$
- 2π channel provides 70% of the HVP contribution [Talk from T. Leplumey]

  $\hookrightarrow$ RC in $e^+e^- \to \pi^+\pi^-$ must be under control
- RC evaluation based on models so far

  $\hookrightarrow$ a **dispersive** approach could lead to **model-independent** results

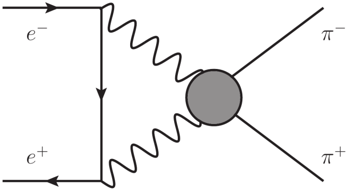
- initial state radiation:



can be calculated in QED in terms of $F_\pi^V(s)$

Radiative corrections to $e^+e^- \to \pi^+\pi^-$

- interference terms



- require hadronic matrix elements beyond $F_\pi^V(s)$
- so far estimated using sQED$\times F_\pi^V(s)$ or (generalized) VMD models

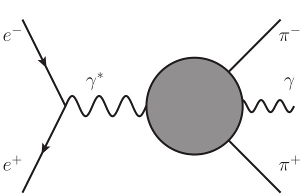
[Arbuzov, Kopylova, Seilkhanova (2020), Ignatov, Lee (2022)]

$\hookrightarrow$ talk by Yannick Ulrich, Strong 2020 report

- pion-pole contribution analyzed dispersively, this talk [Colangelo, Hoferichter, Monnard, JRE (2022)]

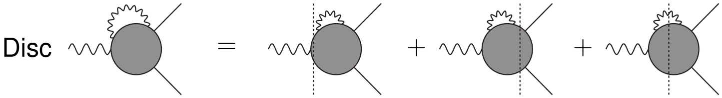
Radiative corrections to $e^+e^- \to \pi^+\pi^-$

- final state radiation:



- also requires hadronic matrix elements beyond $F_\pi^V(s)$
- known in ChPT to one loop [Kubis, Meißner (2001)]

  $\hookrightarrow$ dispersive determination, this talk



- neglecting intermediate states beyond 2π, unitarity reads

$$\begin{aligned}\text{Im}\, F_V^{\pi,\alpha}(s) = &\int d\phi_2\, F_V^{\pi}(s) \times T_{\pi\pi}^{\alpha}(s,t)^* \\ &+ \int d\phi_2\, F_V^{\pi,\alpha}(s) \times T_{\pi\pi}(s,t)^* \\ &+ \int d\phi_3\, F_V^{\pi,\gamma}(s,t) \times T_{\pi\pi}^{\gamma}(s,t')^*\end{aligned}$$

- need $T_{\pi\pi}^{\alpha}$ as well as $F_{\pi}^{V,\gamma}$ and $T_{\pi\pi}^{\gamma}$ as input

  $\hookrightarrow$ dispersive approach to RC to $\pi\pi$ scattering

- The DR for $F_{\pi}^{V,\alpha}(s)$ takes the form of an integral equation

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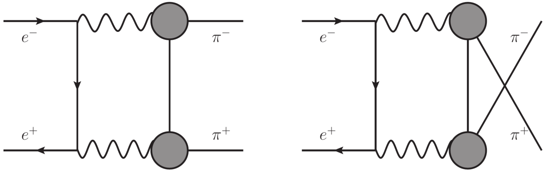
Interference terms and the forward-backward asymmetry

- interference terms: pion-pole contribution



- do not contribute to the total cross section

  can be tested in the forward-backward asymmetry [CMD-3 results, talk from Ivan Logashenko]

$$A_{\text{FB}}(z) = \frac{\frac{d\sigma}{dz}(z) - \frac{d\sigma}{dz}(-z)}{\frac{d\sigma}{dz}(z) + \frac{d\sigma}{dz}(-z)}, \quad z = \cos\theta,$$

  non-vanishing from RC, C-odd terms

- box diagram contributes together to ISR-FSR soft radiation

$$\left.\frac{d\sigma}{dz}\right|_{\substack{C\text{-odd}\\ \text{soft}}} = \frac{d\sigma_0}{dz}\left[\delta_{\text{soft}}(m_\gamma^2, \Delta) + \delta_{\text{virt}}(m_\gamma^2)\right]$$

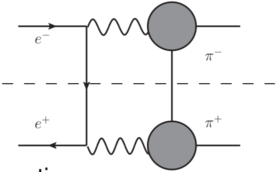
Forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$

- δ_{soft} computed analytically in QED

$$\delta_{\text{soft}} = \frac{2\alpha}{\pi}\left\{ \log\frac{m_\gamma^2}{4\Delta^2}\log\frac{1+\beta z}{1-\beta z} + \log(1-\beta^2)\log\frac{1+\beta z}{1-\beta z} + \cdots \right\},$$

[Arbuzov et al. (2020), Ignatov, Lee (2022), Colangelo, Hoferichter, Monnard, JRE (2022)]

- δ_{virt} computed dispersively
  - ▷ start from a fixed-s dispersion relation



  ↪ for scalar particles D_0 function

  - ▷ for real pions: dispersive representation of $F_\pi^V(s)$

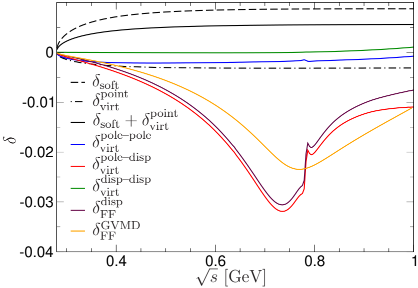
$$\frac{F_\pi^V(s)}{s} = \frac{1}{s} + \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im}\, F_\pi^V(s')}{s'(s'-s)} \to \frac{1}{s-m_\gamma^2} - \frac{1}{\pi}\int_{4m_\pi^2}^{\infty} ds' \frac{\text{Im}\, F_\pi^V(s')}{s'} \frac{1}{s-s'}$$

  ↪ the VFF corrections can be interpreted as a propagator

Forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$: results

- δ_{virt} decomposed in pole-pole, pole-disp and disp-disp contributions
- pole-pole and pole-disp IR divergent

  ↪ cancel against the real emission



- disp-pole term dominates: infrared enhancement [Colangelo, Hoferichter, Monnard, JRE (2022)]

  ↪ significant corrections beyond sQED$\times F_\pi^V(s)$
- similar results from GVMD models [Ignatov, Lee (2022)]

Forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$: results

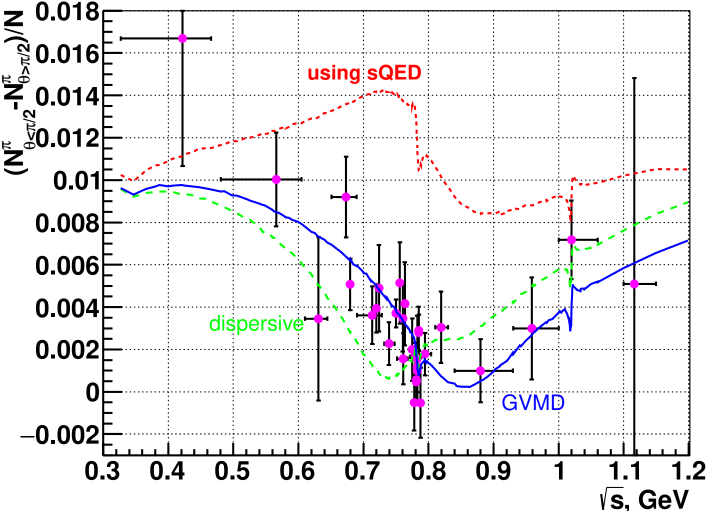
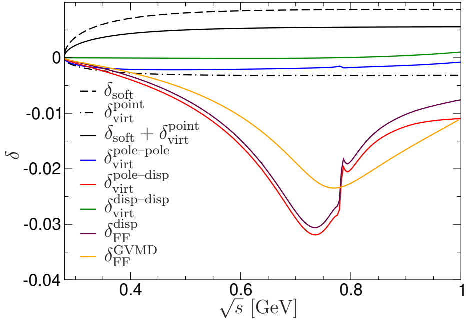


figure taken from Ignatov et al. (CMD3) Collaboration, Phys.Rev.D 109 (2024) 11, 112002

Forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$: results



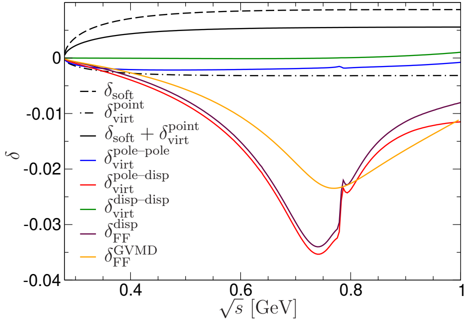
- recently implemented in BabaYaga

[Budassi, Carloni Calame, Ghilardi, Gurgone, Montagna, Moretti, Nicrosini, Piccinini, Ucci (2024)]

$\hookrightarrow$ uncovering scheme ambiguity of endpoint singularity in pole-disp imag. part

- talk by Martina Cottini [Cottini, Holz, Ulrich (2024)]

Forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$: results



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- numerical impact small

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Pion-pion scattering in the isospin limit

- starting point: Roy-equation solution for $\pi\pi$ scattering below $s_1 \sim 1$ GeV

[Ananthanarayan, Colangelo, Gasser, Leutwyler (2001), Garcia-Martin, Kaminski, Pelaez, JRE (2011)]

- $\pi\pi$ invariant amplitude $\quad A(s,t,u) = A(s,t,u)_{SP} + A(s,t,u)_d$

- A_{SP} contribution of S and P waves below s_1

$$A(s,t,u)_{SP} = 32\pi \left\{ \frac{1}{3} W^0(s) + \frac{3}{2}(s-u) W^1(t) + \frac{1}{2} W^2(t) + (t \leftrightarrow u) \right\}$$

$\triangleright$ $W^I(s)$ only RHC, DR in terms of the S and P partial waves t_J^I

$$W^0(s) = \frac{a_0^0\, s}{4M_\pi^2} + \frac{s(s-4M_\pi^2)}{\pi} \int_{4M_\pi^2}^{s_1} ds' \, \frac{\mathrm{Im}\, t_0^0(s')}{s'(s'-4M_\pi^2)(s'-s)} \,,$$

- A_d is the "background amplitude", higher partial waves and higher energies

$\hookrightarrow$ for $s < s_1$ small and smooth, polynomial

- construct isospin amplitudes T^0, T^1 and T^2

Pion-pion scattering away from the isospin limit

- three different isospin-breaking effects

  1. strong isospin breaking: effects proportional $(m_u - m_d)$
  2. effects proportional to $M_{\pi^+} - M_{\pi^0}$
  3. further photon exchanges

- each of them can be **considered separately** from the other two

Strong isospin-breaking effects

- at low energies chiral symmetry imposes $\mathcal{O}\left((m_u - m_d)^2\right)$
  $\hookrightarrow$ small shift in M_{π^0} [Gasser, Leutwyler (84)]

- higher energies, generate $\pi^0 - \eta$ and $\rho - \omega$ mixing

- $\pi^0 - \eta$ not relevant for F_π^V: can be estimated phenomenologically
  rescattering effects can be estimated from $\eta \to 3\pi$ [Colangelo, Lanz, Leutwyler, Passemar (2018)]

- $\rho - \omega$ mixing contribution allows for a high-precision description of F_π^V
  [Colangelo, Hoferichter, Kubis, Stoffer (2022)]

  1. ω meson described with a narrow-width approximation
  2. $\rho - \omega$ interference through a single parameters ϵ_ω
  3. ρ and ω coupling to radiative channels induces a non-negligible phase

Roy equations away from the isospin limit

- first, switch from the isospin to the charge basis

$\hookrightarrow T^0, T^1, T^2 \Rightarrow T^c, T^n, T^X$

$$T^c := T(\pi^+\pi^- \to \pi^+\pi^-),\ T^x := T(\pi^+\pi^- \to \pi^0\pi^0),\ T^n := T(\pi^0\pi^0 \to \pi^0\pi^0)$$

- adapt unitarity relation

$$\begin{aligned}
\mathrm{Im} t_{n,S}(s) &= \sigma_0(s)|t_{n,S}(s)|^2 + 2\sigma(s)|t_{x,S}(s)|^2 \\
\mathrm{Im} t_{x,S}(s) &= \sigma_0(s)t_{n,S}(s)t^*_{x,S}(s) + 2\sigma(s)t_{x,S}(s)t^*_{c,S}(s) \\
\mathrm{Im} t_{c,S}(s) &= \sigma_0(s)|t_{x,S}(s)|^2 + 2\sigma(s)|t_{c,S}(s)|^2
\end{aligned}$$

where

$$\sigma(s) = \sqrt{1 - \frac{4M^2_{\pi^+}}{s}}, \quad \sigma_0(s) = \sqrt{1 - \frac{4M^2_{\pi^0}}{s}}$$

$\hookrightarrow$ encode the effect of $\Delta_\pi = M^2_{\pi^+} - M^2_{\pi^0}$

- assume that the *input* above s_1 does not change for $\Delta_\pi = M^2_{\pi^+} - M^2_{\pi^0} \neq 0$

  ▷ concentrate in T_{SP}, S and P waves below $s_1 =\sim 1$ GeV

- express W^I in terms of the imaginary parts of the physical channels

$$T^n_{SP}(s,t,u) = 32\pi \left(W^{00}_{n,S}(s) + W^{+-}_{n,S}(s) + (s \leftrightarrow t) + (s \leftrightarrow u) \right)$$

where

$$W^{00}_{n,S}(s) = \frac{a^{00}_n\, s}{4M^2_{\pi^0}} + \frac{s(s-4M^2_{\pi^0})}{\pi} \int_{4M^2_{\pi^0}}^{s_1} ds' \frac{\mathrm{Im} t^{00}_{n,S}(s')}{s'(s'-4M^2_{\pi^0})(s'-s)}$$

$$W^{+-}_{n,S}(s) = \frac{s(s-4M^2_{\pi^0})}{\pi} \int_{4M^2_{\pi}}^{s_1} ds' \frac{\mathrm{Im} t^{+-}_{n,S}(s')}{s'(s'-4M^2_{\pi^0})(s'-s)}$$

with

$$\mathrm{Im} t^{00}_{n,S}(s) = \sigma_0(s)|t_{n,S}(s)|^2, \quad \mathrm{Im} t^{+-}_{n,S}(s) = 2\sigma(s)|t_{x,S}(s)|^2$$

▷ similar for the other channels

Strategy:

1. analytical projection into S and P partial waves

$$t_{n,S}(s) = a_n^{00} + \int_{4M_{\pi^0}^2}^{s_1} ds'\, K_n(s,s')\, \mathrm{Im}\, t_{n,S}^{00}(s') + \int_{4M_\pi^2}^{s_1} ds'\, K_n(s,s')\, \mathrm{Im}\, t_{n,S}^{+-}(s') + d_{n,S}(s)\,,$$

▷ $K_n(s,s')$ analytically known

▷ $d_{n,S}(s)$, background integral contribution

2. compute the scattering lengths in ChPT with Δ_π [Knecht, Nehme (2002)]

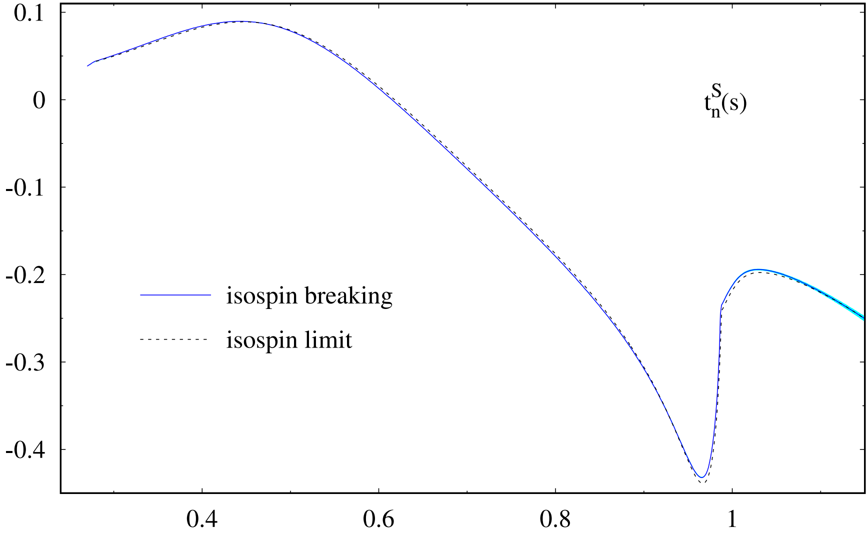
3. take the isospin limit Roy-equation solution $\delta^0_{n,S}, \delta^0_{c,S}, \delta^0_{c,P}, \cdots$

   and parameterize Δ_π effects as a polynomial

$$\delta^{IB}_{n,S}(s) = \delta^0_{n,S}(s)\,[1 + \Delta_\pi\,(a_n + b_n s + \cdots)]$$

4. solve the coupled system of integral equations with $a_n, b_n, \cdots$ fitting parameters

Roy equations and Δ_π: results



[Colangelo, Cottini, Monnard, JRE, to be submitted for publication]

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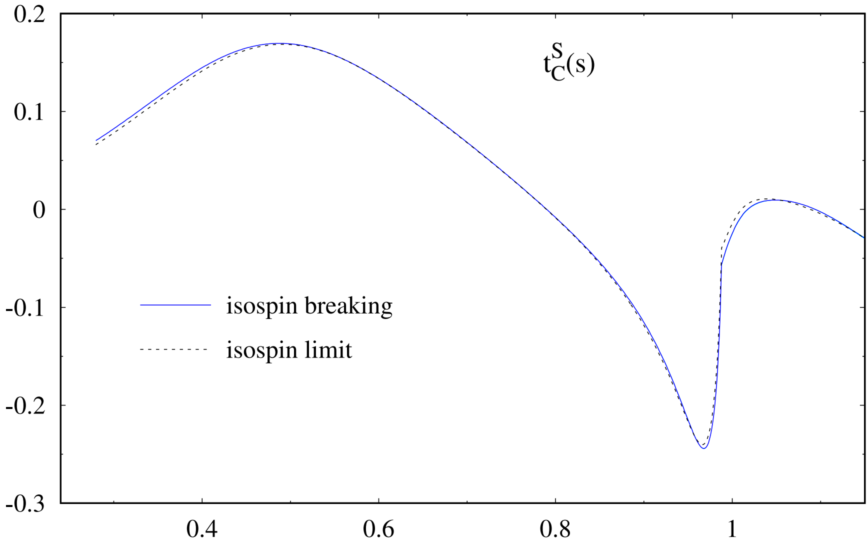

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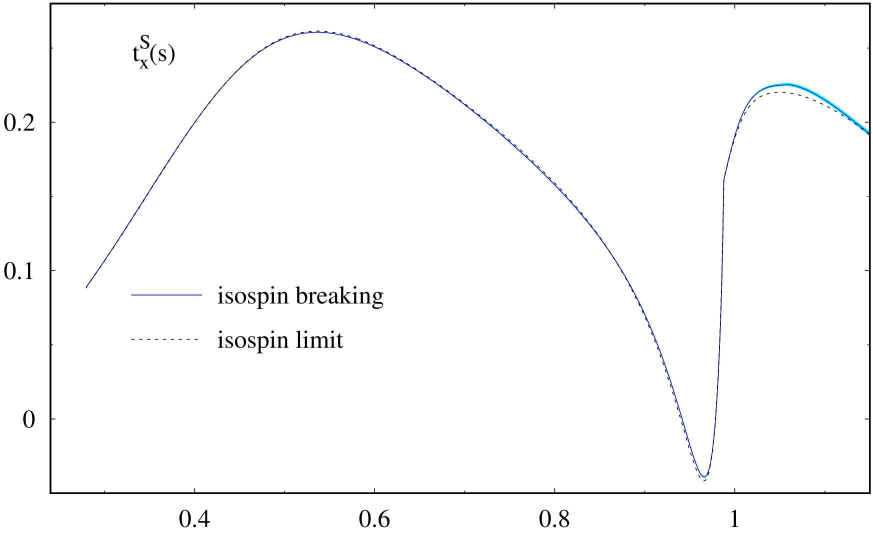

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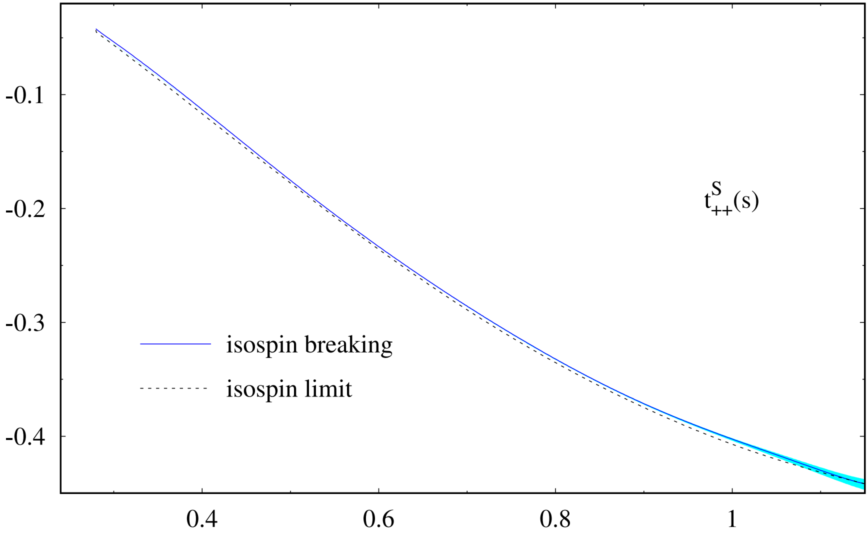

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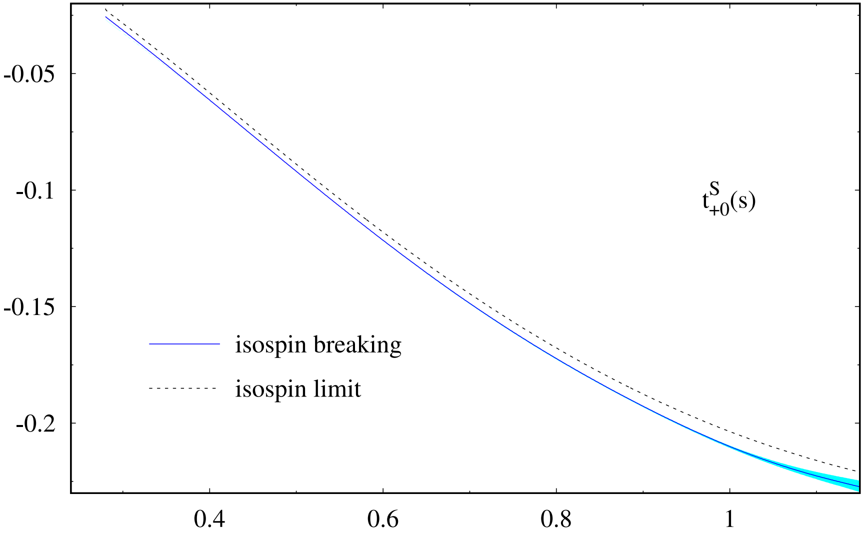

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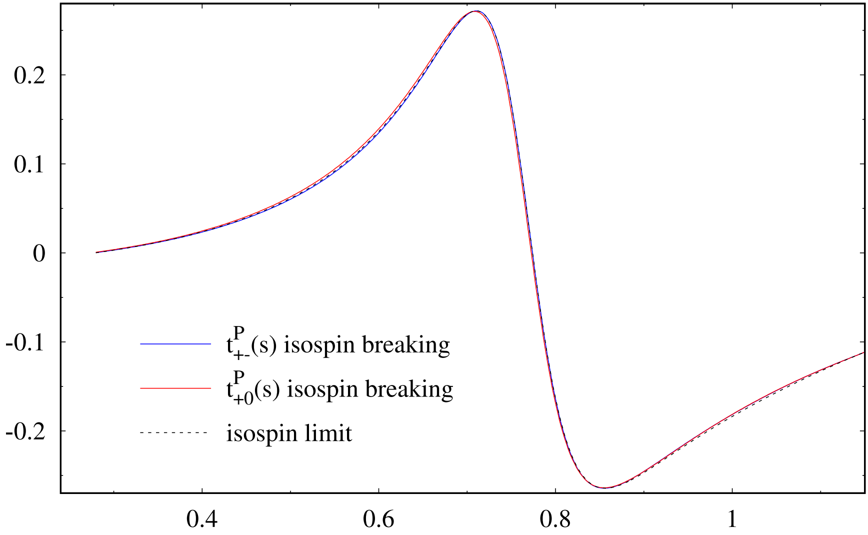

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Isospin-breaking corrections for the $\rho(770)$: Δ_π

- isospin breaking in F_π^V ingredient for interpretation of τ data

  $\hookrightarrow$ talk by Martina Cottini

- only pole parameters provide a model-independent result

$$\sqrt{s_\rho} = M_\rho - i\,\frac{\Gamma_\rho}{2}$$

  $\hookrightarrow$ Breit-Wigner or Gounaris-Sakurai parameters reaction-dependent

- Roy equations provide model-independent access to the complex plane

$$\sqrt{s_{\rho^0}} = 763.29 - i\,71.65\text{ MeV}, \quad \sqrt{s_{\rho^\pm}} = 762.30 - i\,71.89\text{ MeV},$$

$\hookrightarrow$ $M_{\rho^0} - M_{\rho^\pm}\Big|_{\Delta_\pi} \sim 1$ MeV

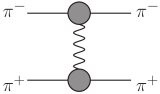
- only part of isospin breaking

  $\hookrightarrow$ radiative corrections have to be included

Roy equations and photon-exchange effects

- photon-exchange diagrams are not included in Roy equations
- modify Roy-equation solutions (T_0^i) to include $\mathcal{O}(\alpha)$ effects
- we start with the Born term

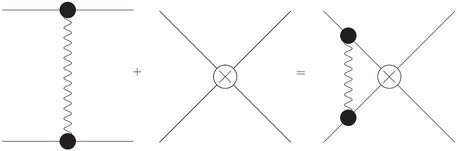
$$T_B(t,s,u) :=$$



$$= 4\pi\alpha\frac{s-u}{t}F_\pi^V(t)^2$$

contribution to $T_B^C(s,t,u) = T_B(t,s,u) + T_B(s,t,u)$

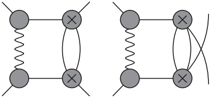
- adding T_B^C to T^C affects unitarity relations for all amplitudes



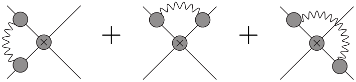
$\hookrightarrow$ we are generating further $\mathcal{O}(\alpha)$ corrections: **iterative procedure**

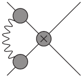
Roy equations and photon-exchange effects: first iteration

- remark: through this procedure we are not generating box diagrams



- compute them through double-spectral representation

$T_D^C(s,t,u) :=$  $+$ flipped diags.

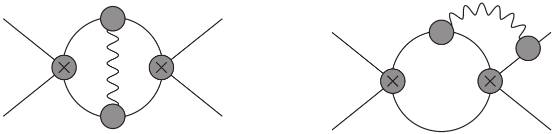
$T_D^X(s,t,u) :=$ 

- include them as starting point for further iterations

$$
\begin{aligned}
T^C(s,t,u) &= T_0^C(s,t,u) + T_B^C(s,t,u) + T_D^C(s,t,u) \\
T^X(s,t,u) &= T_0^X(s,t,u) + T_D^X(s,t,u) \\
T^n(s,t,u) &= T_0^n(s,t,u)
\end{aligned}
$$

Roy equations and photon-exchange effects: further iterations

- for the second iteration we have the diagrams



- they have to be cut in all possible ways:

  ↪ contributions from subamplitudes with real photons: more later

- after N-iterations:

$$
\begin{aligned}
T^C(s,t,u) &= T_0^C(s,t,u) + T_B^C(s,t,u) + T_D^C(s,t,u) + \sum_{k=2}^{N} R_k^c(s,t,u) \\
T^X(s,t,u) &= T_0^X(s,t,u) + T_D^X(s,t,u) + \sum_{k=2}^{N} R_k^x(s,t,u) \\
T^n(s,t,u) &= T_0^n(s,t,u) + \sum_{k=2}^{N} R_k^n(s,t,u)
\end{aligned}
$$

- each iteration k is $\mathcal{O}(p^{2k})$ in the chiral expansion

Roy equations and photon-exchange effects: comments

- the evaluation of R^i_{k+1}, with $k \geq 1$ is done as follows:

  1. project the R^i_k amplitudes onto partial waves
  2. insert these into the unitarity relations combined with the projections of T^i_0
  3. add the contribution of subdiagrams with real photons
  4. solve the corresponding dispersion relation

- subtraction constants can be fixed by matching to ChPT

  ▷ ChPT $\pi\pi$ amplitude with RC known to one loop [Knecht, Urech (1997), Knecht, Nehme (2002)]

- work in progress: **preliminary** results J. Monnard thesis, (2021)

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$$\text{Im}\, F_V^{\pi,\alpha}(s) = \int d\phi_2\, F_V^{\pi}(s) \times T_{\pi\pi}^{\alpha}(s,t)^* + \int d\phi_2\, F_V^{\pi,\alpha}(s) \times T_{\pi\pi}(s,t)^* + \int d\phi_3\, F_V^{\pi,\gamma}(s,t) \times T_{\pi\pi}^{\gamma}(s,t')^*$$

- after this long digression we have obtained **preliminary** results for $T_{\pi\pi}^{\alpha}$
- for $F_V^{\pi,\gamma}(s,t)$ and $T_{\pi\pi}^{\gamma}(s,t')$

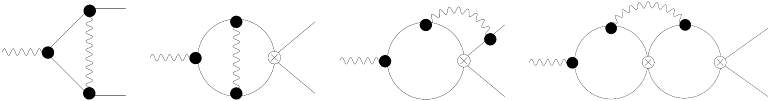
- pion-pole contribution + $\gamma\gamma \to \pi\pi$ input

  $\hookrightarrow$ all subamplitudes known: $F_V^{\pi,\gamma}(s,t)$ and $T_{\pi\pi}^{\gamma}(s,t')$ computed

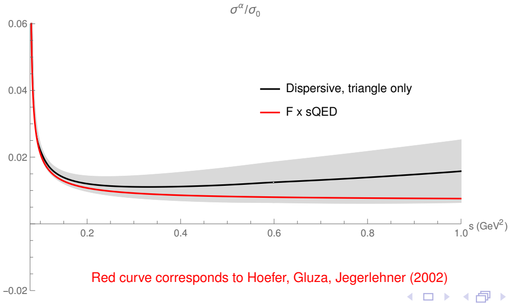
Evaluation of $F_\pi^{V,\alpha}$

- work in progress:
  1. controlled matching to ChPT of all (sub)amplitudes
  2 improved estimate of uncertainties

- having evaluated all the following diagram



- the results for $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$ look as follows: **preliminary** J. Monnard thesis (2021)

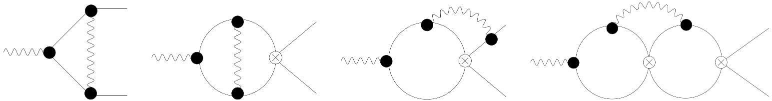


Red curve corresponds to Hoefer, Gluza, Jegerlehner (2002)

Evaluation of $F_\pi^{V,\alpha}$

- work in progress:
  1. controlled matching to ChPT of all (sub)amplitudes
  2 improved estimate of uncertainties

- having evaluated all the following diagram



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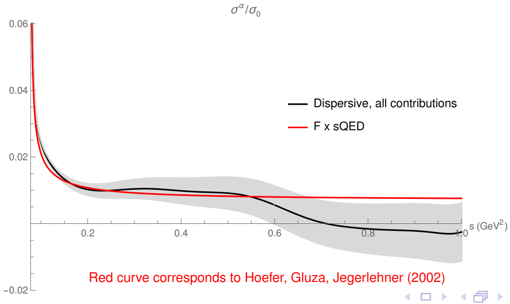

Red curve corresponds to Hoefer, Gluza, Jegerlehner (2002)

Impact on a_μ^{HVP}

- ideally one would use the calculated RC directly in the data analysis
- to get an idea of the impact we did the following: [thanks to M. Hoferichter and P. Stoffer]

  1. remove RC from the measured $\sigma(e^+e^- \to \pi^+\pi^-(\gamma))$
  2. fit with the dispersive representation for F_π^V
  3. insert back the RC

- the impact on a_μ^{HVP} (comparison with result obtained by removing RC)

$$10^{11}\,\Delta a_\mu^{\text{HVP}} = \begin{cases} 10.2 \pm 0.5 \pm 5 & \text{sQED} \times F_\pi^V(s) \\ 10.5 \pm 0.5 & \text{triangle} \\ 13.2 \pm 0.5 & \text{full} \end{cases}$$

preliminary, J. Monnard thesis (2021)

Outline

Introduction

Interference: RC to the forward-backward asymmetry in $e^+e^- \to \pi^+\pi^-$

Isospin-breaking corrections for $\pi\pi$ scattering

Dispersive approach to FSR in $e^+e^- \to \pi^+\pi^-$

Summary / Outlook

Summary / Outlook

- dispersive (pion-pole) determination of the interference terms to $e^+e^- \to \pi^+\pi^-$ and its contribution to the forward-backward asymmetry [Colangelo, Hoferichter, Monnard, JRE (2022)]
- dispersive calculation of pion-mass difference effects to $\pi^+\pi^-$

  [Colangelo, Cottini, Monnard, JRE (to be submitted)]
- formalism for evaluating dispersively RC to the $\pi\pi$ scattering and F_π^V considering only 2π intermediate states [Colangelo, Cottini, Monnard, JRE (in progress)]
- our **preliminary** evaluation of the corrections to F_π^V shows no unexpectedly large effects [J. Monnard, PhD thesis, (2021)]
- our **preliminary** estimate of the impact on a_μ^{HVP} also shows moderate effects

  [J. Monnard, PhD thesis, (2021)]
- the final goal is to provide a ready-to-use code which can be implemented in MC and used in data analysis

Spare slides

$\gamma\pi^- \to (3\pi)^-$

- One-loop ChPT calculation [Kaiser (2010)]
- Experimental results [COMPASS (2012)]
- Dispersive result for the pion pole + resonances [Colangelo, Cottini, Monnard, JRE (in progress)]